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PRODUCT FORMULAE FOR NIELSEN NUMBERS OF FIBRE MAPS

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This work simplifies proofs of a recent publication by You and gives simple sufficient conditions for Brown's product formula for the Nielsen number of a fibre map, as well as new product formulae in this context. Product formulae are also given relating absolute and relative Nielsen numbers, together with corresponding results for Reidemeister numbers.

Introduction. Let $p\colon E \to B$ be a fibration in which E, B and all fibres are compact connected ANR's, and let $f\colon E \to E$ be a fibre preserving map inducing self maps $\bar{f}$ on B and f_b on the fibre F_b over some fixed point b in the base. Since Brown [**1**] introduced his multiplicative formula $N(f) = N(f_b)N(\bar{f})$ for the Nielsen number $N(f)$ of f, various attempts have been made both to improve his results (cf. [**4**], [**5**]) and to generalize his formula (cf. [**6**], [**14**], [**18**]). In a recent paper which supercedes most of what precedes it in both of the aspects mentioned above, You [**20**] gives, among other things, necessary and sufficient conditions for Brown's formula together with a new result relating the Nielsen numbers of f, $\bar{f}$ and a relative Nielsen number $N_K(f_b)$ of f_b. Here K is the kernel of the inclusion induced homomorphism $\Pi_1 F_b \to \Pi_1 E$.

In this work we consider the second of these two results and use it (1) as a focus to give what we feel are more eccessible proofs of the results in [**20**]; (2) as a springboard to give new product theorems for fibre maps. Our results here include conditions under which $N(f) = N_k(f_b)N(\bar{f})$ and also conditions under which $N_K(f_b) = N(f_b)$. By combining these we thus obtain new sufficient conditions for Brown's formula. These conditions are simpler to verify than You's. We investigate the hypotheses of You's theorems giving conditions under which they hold. In the process we develop product formulae relating relative and absolute Nielsen numbers, together with corresponding results for Reidemeister numbers.

The unifying tool in this work is a certain exact sequence associated with a self morphism of a short exact sequence of groups. This result is a kind of non-abelian snake lemma and is a special case of a theorem (cf. [**9**]) originally proved in connection with localization of orbit sets. All the product theorems mentioned above ultimately derive from this sequence.

We also make use of the concept of nilpotent homomorphism, which gives some of the new results mentioned above and simplifies further proofs in this area.

Although our main interest is topological, we introduce our sequence in section one, at the algebraic level in the category of groups. This separation of algebraic from topological considerations allows a simple formulation of the concepts involved. The role of nilpotent and eventually commutative homomorphisms is discussed here. In section two we introduce consistent with the relative (or H) Reidemeister number $R_H(f)$, of a self map f of a compact connected ANR X. This number, being an easy generalization of the absolute case, is seen to be independent of various choices and to be greater than the number of H-Nielsen fixed point classes of f. For computational purposes $R_H(f)$ is also compared with Reidemeister numbers of homomorphisms induced on homology. In section three the H-Nielsen number is defined following **[16]** and **[20]**. The bijections of section two are then seen to preserve this number. The Jiang subgroup is introduced at this point in order to give a condition which allows product formulae comparing $R_H(f)$ or $N_H(f)$ with the ordinary Reidemeister or Nielsen number. This condition was inspired by **[12]**. Section four gives the central theorem and our main results. Our method of proof needs a result **[10]** on locally equiconnected spaces in order to ensure we can choose translation functions to be base point preserving. This is the key to our simplification.

At times our methods overlap with You's. At such places we either sketch proofs or omit them, referring the reader to **[20]**. Jiang **[12]** has redone much of You's work in the context of covering spaces, so our results also overlap with Jiang's. Details and acknowledgements are given in the text. Special thanks are due to R. F. Brown whose help and encouragement interested me in this subject, and also to S. Wilson, P. J. Higgins, M. A. Armstrong and other friends at Durham for much help with exposition.

1. The algebra of Reidemeister operations. Let G be a (*not necessarily abelian*) group and $f\colon G \to G$ a homomorphism. We write composition in G additively.

DEFINITION 1.1. The *Reidemeister operation of f on G* is the left action of G on itself given by

$$(g_1, g_2) \to g_1 + g_2 - f(g_1).$$

Let $1 - f\colon G \to G$ denote the *function* defined by $(1 - f)(g) = g - f(g)$; then by a slight abuse we write the set of orbits of the operation as

$\mathrm{Coker}(1 - f)$ with elements $[g]$ for $g \in G$. To motivate this somewhat bizarre notation, we observe that if $j\colon G \to \mathrm{Coker}(1 - f)$ has $j(g) = [g]$, then $j(g_1) = j(g_2)$ if and only if there is a $g \in G$ with $g_1 = g + g_2 - f(g)$, and there is then an exact sequence (with the obvious base points)

$$(1.2) \qquad 0 \to \mathrm{Fix}\, f \to G \xrightarrow{1-f} G \xrightarrow{j} \mathrm{Coker}(1 - f) \to 0,$$

of groups and based sets, where $\mathrm{Fix}\, f$ is the subgroup of G consisting of those g for which $g = f(g)$. We warn the reader that since $1 - f$ need not be a homomorphism, $\mathrm{Coker}(1 - f)$ need not be the quotient of G by a subgroup.

The order $\#\mathrm{Coker}(1 - f)$ of the orbit set is called the *Reidemeister number* of f and is written $R(f)$. Dispite the fact that $1 - f$ need not be a homomorphism, we still have, from the properties of the action in 1.1:

PROPOSITION 1.3. *The function* $1 - f$ *in* 1.2 *is injective if and only if* $\mathrm{Fix}\, f$ *is trivial, and is surjective if and only if* $R(f) = 1$. *If* G *is abelian,* $1 - f$ *is a homomorphism and* $\mathrm{Coker}(1 - f)$ *has a canonical group structure in which* j *is a homomorphism.* □

For abelian G the group structure on $\mathrm{Coker}(1 - f)$ is well known. The next Lemma is given by Jiang **[12]** for G the fundamental group of a space.

LEMMA 1.4 (*Jiang*). *For all* $g_1, g_2 \in G$, $[g_1 + g_2] = [g_2 + f(g_1)]$. *In particular* $[g] = [f(g)]$ *for all* $g \in G$.

Proof. $[g_1 + g_2] = [-g_1 + (g_1 + g_2) - f(-g_1)] = [g_2 + f(g_1)]$. □

We say that $f\colon G \to G$ is *nilpotent* if for some positive integer n, $f^n\colon G \to G$ is the trivial homomorphism.

PROPOSITION 1.5. *If* f *is nilpotent, then* $\mathrm{Fix}\, f = 0$. $R(f) = 1$ *and the function* $(1 - f)\colon G \to G$ *is a bijection.*

Proof. Let $g \in \mathrm{Fix}\, f$, then $g = f(g) = f^n(g) = 0$. Let $g \in G$, then by 1.4, $[g] = [f(g)] = [f^n(g)] = [0]$. The result follows from 1.3. □

We consider next the naturally of Reidemeister operations. Suppose we are given a commutative diagram

$$(1.6)\qquad \begin{array}{ccc} G & \xrightarrow{f} & G \\ q\downarrow & & \downarrow q \\ \overline{G} & \xrightarrow{\bar{f}} & \overline{G} \end{array}$$

of groups and homomorphisms. Then q restricts to a homomorphism q: $\operatorname{Fix} f \to \operatorname{Fix} \bar{f}$ also denoted by q: further, q induces a function q_*: $\operatorname{Coker}(1 - f) \to \operatorname{Coker}(1 - \bar{f})$ in the obvious way.

In addition to the above, let the sequence

$$(1.7)\qquad 0 \to H \xrightarrow{j} G \xrightarrow{q} \overline{G} \to 0$$

be exact, then f restricts to a homomorphism $f|H$: $H \to H$ and we have

THEOREM 1.8. *In the above situation there is an exact sequence*

$$0 \to \operatorname{Fix}(f|H) \to \operatorname{Fix} f \xrightarrow{q} \operatorname{Fix} \bar{f} \xrightarrow{\delta} \operatorname{Coker}(1 - f|H)$$
$$\xrightarrow{j_*} \operatorname{Coker}(1 - f) \xrightarrow{q_*} \operatorname{Coker}(1 - \bar{f}) \to 0$$

of groups and based sets in which δ is given by

$$\delta(\bar{g}) = [g - f(g)] \quad \textit{where } q(g) = \bar{g}:$$

Furthermore, if G is abelian, then the sequence can be regarded as an exact sequence of groups.

Proof. The result can be proved by arranging sequences of the form of 1.2 for $f|H$, f and $\bar{f}$ on a grid and using those of type 1.7 to connect them. The first and last three terms fall out easily. That the boundary is well defined is an easy consequence of the definition of the orbit set $\operatorname{Coker}(1 - f)$. Alternatively, the theorem is a direct application of [**9**; Theorem 2.5] from which it was inspired. □

The function q_*: $\operatorname{Coker}(1 - f) \to \operatorname{Coker}(1 - \bar{f})$ in 1.8 is surjective so

$$R(f) \geq R(\bar{f}).$$

It would clearly be interesting to have a product formula involving $R(f|H)$, $R(f)$ and $R(\bar{f})$ (see 1.15). Let $g \in G$, then the Reidemeister operation of f on G induces an operation (on the left) of H on $H + g$ by restriction. let $\operatorname{Orb}(H + g)$ denote the orbits of this operation with elements $[h+g]$ for $h \in H$, then the inclusion of H into G induces a function j_*: $\mathrm{Orb}(H+g) \to \mathrm{Coker}(1-f)$ and an easy generalization of 1.8 shows

$$(1.9) \qquad j_*(\mathrm{Orb}(H+g)) = q_*^{-1}(q_*([g])).$$

(This type of exactness is treated in [**9**].)

Unfortunately, if G is not abelian, then $\mathrm{Orb}(H)$ and $\mathrm{Orb}(H+g)$ can have different cardinality as can be seen by taking $G = S_3$, the symmetric group on 3 letters, H to be the cyclic subgroup of order 3 and f to be the identity. (I am grateful to S. Wilson for this example.)

The following elementary observation from 1.8 forms the basis for our product formulae.

Observation 1.10. If $E(f)$ is a finite subset of $\mathrm{Coker}(1-f)$ in 1.6 with the property that the order $\#\{q_*^{-1}(q_*[g]) \cap E(f)\}$ is independent of $[g] \in E(f)$, then

$$\#E(f) = \left(\#\{q_*^{-1}(q_*[g]) \cap E(f)\}\right)(\#q_*(E(f))).$$

PROPOSITION 1.11. *If in* 1.8 *the restriction* $f|H$ *of* f *to* H *is nilpotent then* q_*: $\mathrm{Coker}(1-f) \to \mathrm{Coker}(1-\bar{f})$ *is bijective so* $R(f) = R(\bar{f})$. *In particular, this is so if* q *in* 1.6 *is an isomorphism.*

Proof. Using 1.9, we need only show that for all $h \in H$, and all $g \in G$ that $[h+g] = [g]$ in $\mathrm{Coker}(1-f)$. Now

$$\begin{aligned}[h+g] &= [f^n(h+g)] = \left[(f|H)^n(h) + f^n(g)\right] \\ &= [f^n(g)] = [g]. \end{aligned}$$

□

DEFINITION 1.12. Let f: $G \to G$ be a homomorphism. Then f is said to be eventually commutative ([**13**]) if for some positive integer n, $f^n(G)$ is abelian.

The easy proof of the following proposition is left to the reader.

PROPOSITION 1.13. *The homomorphism* f: $G \to G$ *is eventually commutative if and only if the restriction* $f|G'$ *of* f *to the commutator subgroup,* G' *of* G, *is nilpotent.* □

We are now ready to compare the Reidemeister number of a homomorphism f: $G \to G$ with the Reidemeister number of $H_1(f)$: $H_1(G) \to H_1(G)$ where $H_1 = H_1^{Gp}$ is the first integral homology functor from groups to abelian groups. For any group G the sequence

$$0 \to G' \to G \xrightarrow{\theta} H_1(G) \to 0 \tag{1.14}$$

is exact (see for example [**17**]).

COROLLARY 1.15. *If $f\colon G \to G$ is eventually commutative, then* $\operatorname{Coker}(1 - f)$ *has a canonical group structure and* $R(f) = R(H_1(f)) = \#\operatorname{Coker}(1 - H_1(f))$. *Furthermore if f is as in* 1.6, *then*

$$[\operatorname{Fix} \bar{f}\colon q \operatorname{Fix} f]\, R(f) = R(f|H) R(\bar{f}).$$

Proof. For the first part we apply 1.11 and 1.13 to the obvious morphism of 1.14 induced by f. For the second part we observe that if f is eventually commutative, then so are $\bar{f}$ and $f\,|H$ and the sequence of 1.8 is an exact sequence of groups. The result follows easily. □

We conclude this section with a condition inspired by [**5**] which gives $[\operatorname{Fix} \bar{f}\colon q \operatorname{Fix} f] = 1$.

DEFINITION 1.16. The short exact sequence (1.7) of groups is said to have a *normal splitting* if there is a section $\sigma\colon \bar{G} \to G$ of q with $\sigma(\bar{G}) \lhd G$. A homomorphism $f\colon G \to G$ is said to *preserve* this normal splitting if f induces a morphism of (1.7) with $f(\sigma(\bar{G})) \subset \sigma(\bar{G})$.

PROPOSITION 1.17. *If* (1.7) *has a normal splitting which is preserved by* $f\colon G \to G$, *then* $[\operatorname{Fix} \bar{f}\colon q \operatorname{Fix} f] = 1$.

Proof. Let $\bar{g} \in \operatorname{Fix} \bar{f}$, then $f(\sigma(\bar{g})) = \sigma(\bar{g}')$, for some $\bar{g}' \in \bar{G}$. Now

$$\bar{g}' = q\sigma(\bar{g}') = qf(\sigma(\bar{g})) = \bar{f}q\sigma(\bar{g}) = \bar{f}(\bar{g}) = \bar{g}$$

so $f(\sigma(\bar{g})) = \sigma(\bar{g})$ and $\bar{g} \in q \operatorname{Fix} f$. □

We remark that the proof of 1.17 defines a section to $q_*\colon \operatorname{Fix} f \to \operatorname{Fix} \bar{f}$.

2. Estimation of H-Reidemeister numbers. Let X be a compact connected ANR, ΠX the fundamental groupoid of X (cf. for example [**19**]) and $\Pi_1(X, x)$ the fundamental group of X at x, i.e. the vertex group of ΠX at $x \in X$. Composition in ΠX will be written as addition, $\lambda + \mu$ meaning first λ then μ. We say that H is a *normal subgroupoid* of ΠX

written $H \lhd \Pi X$ if H is a subgroupoid of ΠX, with a vertex group $H = H(x)$ for each $x \in X$, such that for each path $\omega: x \to y$, we have $\omega + H(y) - \omega = H(x)$. We call $\operatorname{Ob} \Pi X$, the objects of ΠX, the trivial normal subgroupoid of ΠX. The reader should be warned that the condition of being a normal subgroupoid is slightly more general than requiring that $H(x)$ is a normal subgroup of $\Pi_1 X$ for each $x \in X$.

A map $f: X \to X$ is said to preserve $H \lhd \Pi X$ if for each $x \in X$. $f_*: \Pi_1(X, x) \to \Pi_1(X, f(x))$ restricts to a homomorphism $f_*|H: H(x) \to H(f(x))$. Choose $x \in X$ as a base point and $\omega: x \to f(x)$ an element of ΠX. (We shall frequently not distinguish between a *path* $\omega: x \to f(x)$ and its *path class* in ΠX.)

We start this section by defining the H-Reidemeister number of f and showing it is independent of various choices made. Some of the bijections involved here are seen to be special cases of ones found in [**20**]. These simplifications are used later in our approach.

DEFINITION 2.1. Given X, H, f, x and ω as above, an *H-Reidemeister operation* of f on X is the Reidemeister operation of the induced homomorphism

$$f^{\omega}_{*/H}: \Pi_1(X, x)/H \to \Pi_1(X, x)/H$$

where $f^{\omega}_{*/H}$ is given by $f^{\omega}_{*/H}(H + \lambda) = H + \omega + f_*(\lambda) - \omega$. The *$H$-Reidemeister number* of f, written $R_H(f)$, is the Reidemeister number of $f^{\omega}_{*/H}$, i.e. the order of the orbit set $\operatorname{Coker}(1 - f^{\omega}_{*/H})$ of the H-Reidemeister operation of f on X described above. This definition is equivalent to one given in [**12**].

To show $R_H(f)$ is well defined, we exhibit two bijections, one associated with the choice of path class μ from x to $f(x)$ in ΠX, the other associated with change of base point $x \in X$. Proofs are left to the reader.

LEMMA 2.2. *For $\omega, \mu: x \to f(x)$ in ΠX, there is a bijection*

$$r = r_{\omega,\mu}: \operatorname{Coker}\left(1 - f^{\omega}_{*/H}\right) \to \operatorname{Coker}\left(1 - f^{\mu}_{*/H}\right)$$

given by $r[H + \alpha] = [H + \alpha + \omega - \mu]$. Hence, $R_H(f)$ is independent of ω: $x \to f(x)$ in ΠX. □

LEMMA 2.3. *For $\omega: x \to f(x)$ as above, $x' \in X$ and $u: x \to x'$ there is a bijection*

$$u_* = u_{*,\omega}: \operatorname{Coker}\left(1 - f^{\omega}_{*/H}\right) \to \operatorname{Coker}\left(1 - f^{-u+\omega+f_*(u)}_{*/H}\right)$$

given by $u_*[H + \alpha] = [H - u + \alpha + u]$. Hence $R_H(f)$ is independent of the base point $x \in X$. □

Let $\Phi(f) = \{x \in X \mid f(x) = x\}$ denote the set of fixed points of f: $X \to X$, and let $x \in \Phi(f)$. By abuse of notation, we denote the constant path class O_x by x. Composing u_* and r in 2.3 and 2.2 we have:

LEMMA 2.4. *Given ω: $x \to f(x)$ in ΠX and $x' \in \Phi(f)$, then for any u: $x \to x'$ in ΠX there is a bijection*

$$u_*^f\colon \operatorname{Coker}\left(1 - f_{*/H}^{\omega}\right) \to \operatorname{Coker}\left(1 - f_{*/H}^{x'}\right)$$

given by $u_*^f[H + \alpha] = [H - u + \alpha + \omega + f_*(u)]$. In particular if ω is the constant path x at x then $u_*^f[H + \alpha] = [H - u + \alpha + f_*(u)]$. □

Following McCord **[16]** and You **[20]** we define an equivalence relation on $\Phi(f)$.

Let $x, y \in \Phi(f)$, then x is said to be *H-Nielsen-equivalent* to y. If there is a path λ: $x \to y$ in ΠX with $\lambda - f(\lambda) \in H(x)$. We write $\Phi'_H(f)$ for the set of H-Nielsen equivalence classes with elements $\mathbf{F} = \mathbf{F}_H$. If H is trivial, we write $\Phi'(f)$ for $\Phi'_H(f)$, the usual Nielsen classes of f. It is clear that each H-Nielsen class is a union of (ordinary) Nielsen classes and it follows ([**1**]) that $\Phi'_H(f)$ is finite.

Let ω: $x \to f(x)$ in ΠX be given and c: $x \to x'$ be a path where $x' \in \mathbf{F}$ in $\Phi'_H(f)$. Define $\rho(F) \in \operatorname{Coker}(1 - f_{*/H}^{\omega})$ to be $[H + c - f_*(c) - \omega]$. We see that the definition of $\rho = \rho(\omega, c)$ involves choices.

LEMMA 2.5 (*You*). *The relation ρ is an injective function*; *moreover*, *for μ and u as in* 2.2, 2.3 *and* 2.4, *we have $r \circ \rho = \rho$. $u_* \circ \rho = \rho$ and $u_*^f \circ \rho = \rho$.*

Proof. (See also [**20**: 1.1 and 1.2].) To see that ρ is well defined, let c': $x \to x'$ be another path. We need a $\delta \in H(x)$ such that $H + c' - f_*(c') - \omega = H + \delta + c - f_*(c) - \omega - f_{*/H}^{\omega}(\delta)$. The path $\delta = c' - c$ does the trick. If x'' is another representative of the class of x', and d: $x' \to x''$ has $d - f_*(d) \in H(x')$ then

$$\begin{aligned} H + c + d - f_*(c + d) - \omega &= H + c + (d + f_*(d)) - f_*(c) - \omega \\ &= H + c - f_*(c) - \omega \end{aligned}$$

since H is a normal subgroupoid of ΠX.

To see that ρ is injective, let $x' \in \mathbf{F}$, $x'' \in \mathbf{F}'$ and $\rho(\mathbf{F}) = \rho(\mathbf{F}')$, then for any c: $x \to x'$, and c': $x \to x''$ there is a $\delta \in H(x)$ with $H + c - f_*(c) - \omega = H + \delta + c' - f_*(c') - f_*(\delta) - \omega$. Now $-c + \delta + c' - f_*(-c + \delta + c') \in H$ by the normality of H in ΠX. This shows that $\mathbf{F} = \mathbf{F}'$.

If ω and μ are as in 2.2, and $x' \in \mathbf{F}$, and $c\colon x \to x'$ are given then

$$r \circ \rho(\mathbf{F}) = r[H + c - f_*(c) - \omega] = [H + c - f_*(c) - \mu] = \rho(\mathbf{F}).$$

Similarly $u_* \circ \rho = \rho$ etc.. □

Given X, H, f, x and $\omega\colon x \to f(x)$ as in this section we say f is *nilpotent mod H*; respectively, f is *eventually commutative mod H* if $f_*^\omega\colon \Pi_1(X, x)/H \to \Pi_1(X, x)/H$ is nilpotent, respectively eventually commutative. If H is trivial, we simply drop "mod H" and say f is nilpotent, etc. It is clear that nilpotent mod H, respectively eventually commutative mod H, is more general than nilpotent, respectively eventually commutative, as can be seen by taking $H = \Pi X$. It is also easy to see that nilpotent mod H and eventually commutative mod H are independent of x and ω. From 1.5 we have:

COROLLARY 2.6. *If f is nilpotent* mod H, *then*

$$\mathrm{Fix}\, f_{*/H}^\omega = 0 \quad \textit{and} \quad R_H(f) = 1. \qquad \square$$

Let $H_1 = H_1^S$ denote the singular homology functor from topological spaces to abelian groups. It is well known that H_1^S and $H_1^{Gp} \cdot \Pi_1$ are naturally equivalent as functors: thus, if $\theta\colon \Pi_1(X, x) \to H_1(X)$ is the Hurecwiz homomorphism we have induced homomorphisms

$$H_1(f)/\theta(H)\colon H_1^S(X)/\theta(H) \to H_1^S(X)/\theta(H)$$

and

$$H_1(f_{*/H}^\omega)\colon H_1^{Gp}(\Pi_1(X, x)/H) \to H_1^{Gp}(\Pi_1(X, x)/H).$$

PROPOSITION 2.7. *There is a natural isomorphism*

$$\mathrm{Coker}(1 - H_1(f)/\theta(H)) = \mathrm{Coker}\big(1 - H_1(f_{*/H}^\omega)\big). \qquad \square$$

The first part of the next corollary is essentially the first part of [**12**; Theorem 3.2.8].

COROLLARY 2.8. *For X, f and H as in this section $R_H(f) \geq R(H_1(f_{*/H}^\omega))$. If f is eventually commutative* mod H, *then by* 1.15 *equality holds and* Coker$(1 - f_{*/H}^\omega)$ *has the structure of an abelian group which is natural in the obvious sense.* □

Our methods allow us to prove that if the induced homomorphism from $H/(H \cap (\Pi_1 X)')$ to itself is nilpotent, then $R(H_1(f_{*/H}^\omega)) = R(H_1(f))$. Clearly, this is the case if $H \subset (\Pi_1 X)'$, the commutator subgroup of $\Pi_1 X$.

Let X and $\bar{X}$ be topological spaces, $H \lhd \Pi X$, $\bar{H} \lhd \Pi \bar{X}$ normal subgroupoids, $f\colon X \to X$, $\bar{f}\colon \bar{X} \to \bar{X}$ and $h\colon X \to \bar{X}$ maps with $f_*(H) \subset H$, $\bar{f}_*(\bar{H}) \subset \bar{H}$ and $h_*(H) \subset \bar{H}$ such that the diagram

$$(2.9) \qquad \begin{array}{ccc} X & \xrightarrow{f} & X \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle h} \\ \bar{X} & \xrightarrow{\bar{f}} & \bar{X} \end{array}$$

is commutative. Let $x \in X$, $\omega\colon x \to f(x)$, $\bar{x} = h(x)$, and $\bar{\omega} = h(\omega)$. We write $h_* = h_*^{\omega}$ for both the induced homomorphism $\Pi_1(X, x)/H \to \Pi_1(\bar{X}, \bar{x})/\bar{H}$, and the induced function

$$\operatorname{Coker}\left(1 - f^{\omega}_{*/H}\right) \to \operatorname{Coker}\left(1 - \bar{f}^{\bar{\omega}}_{*/\bar{H}}\right).$$

PROPOSITION 2.10. *If* $h_*\colon \Pi_1(X, x)/H \to \Pi_1(\bar{X}, \bar{x})/\bar{H}$ *is surjective and* $f_*|\operatorname{Ker} h_*$ *is nilpotent, then*

$$h_*\colon \operatorname{Coker}\left(1 - f^{\omega}_{*/H}\right) \to \operatorname{Coker}\left(1 - \bar{f}^{\bar{\omega}}_{*/\bar{H}}\right)$$

is bijective and $R_H(f) = R_{\bar{H}}(\bar{f})$. *In particular this is so if* $h_*\colon \Pi_1(X, x)/H \to \Pi_1(\bar{X}, \bar{x})/\bar{H}$ *is an isomorphism.* □

REMARK 2.11. If $x' \in \Phi(f)$, then $\bar{x}' = h(x') \in \Phi(\bar{f})$ and for any $u\colon x \to x'$, there is a commutative diagram

$$(2.11) \qquad \begin{array}{ccc} \operatorname{Coker}\left(1 - f^{\omega}_{*/H}\right) & \xrightarrow{h_*} & \operatorname{Coker}\left(1 - \bar{f}^{\bar{\omega}}_{*/\bar{H}}\right) \\ {\scriptstyle u^f_*}\downarrow & & {\scriptstyle \bar{u}^{\bar{f}}_*}\downarrow \\ \operatorname{Coker}\left(1 - f^{x'}_{*/H}\right) & \xrightarrow{h^{x'}_*} & \operatorname{Coker}\left(1 - \bar{f}^{\bar{x}'}_{*/\bar{H}}\right) \end{array}$$

in which $u^f_* \rho(F) = [H]$ where $x' \in \mathbf{F} \in \Phi'_H(f)$. This diagram is useful in that it induces a bijection

$$\left(u^f_*\right)_*\colon h_*^{-1}\left(h_*[H + u - f_*(u)]\right) \cong \operatorname{Ker} h^{x'}_*$$

under which $[H + u - f_*(u)]$ is taken to $[H]$.

3. Nielsen numbers. H-Nielsen numbers and their relationship. If $\mathbf{F} \subset \Phi(f)$ is such that there is an open U in X with $U \cap \Phi(f) = \mathbf{F}$ and $\delta U \cap \Phi(f)$ empty, where δU denotes the boundary of U, then one can define the *fixed point index* $\operatorname{ind}(\mathbf{F})$ of $\mathbf{F}$ to be $i(f, U)$ for any such U, where i here is the usual fix point index on compact ANR's (cf. [3]). Any

Nielsen class $\mathbf{F} \in \Phi'(f)$ has this property, so any H-Nielsen class does; and the index of an H-Nielsen class in $\Phi(f)$ is well defined. Let $\mathbf{E}_H(f)$ denote the subset of $\Phi'_H(f)$ of these fixed point classes whose index is non-zero. These are called the *essential fixed point classes*. The cardinality $\#\mathbf{E}_H(f)$ of $\mathbf{E}_H(f)$ is called the *H-Nielsen number* and is written $N_H(f)$. Note that $N_H(f) \leq N(f)$, the usual Nielsen number, and thus that $N_H(f)$ is finite. Note also that $N_H(f) \leq R_H(f)$ by 2.5.

Now $\rho(\omega)(\Phi'_H(f))$ is a subset of $\operatorname{Coker}(1 - f^{\omega}_{*/H})$ for any $\omega\colon x \to f(x)$ in ΠX, so we can define the index of elements $[H + x]$ in $\operatorname{Coker}(1 - f^{\omega}_{*/H})$ as follows

$$(3.1)\quad i([H+\alpha]) = \begin{cases} \operatorname{ind}(\mathbf{F}) & \text{if } \rho(\omega)(\mathbf{F}) = [H+\alpha], \mathbf{F} \in \Phi'_H(f), \\ 0 & \text{otherwise.} \end{cases}$$

PROPOSITION 3.2. *The functions ρ, r, u_* and u^f_* of section two are index preserving.*

Proof. This essentially boils down to the compatibility of ρ with r, u_*, and u^f_* (see 2.5). □

Let $G\colon f \simeq g\colon X \to X$, where $f_*(H) \subset H$. Let $G(x)$ denote the path class given by $G(x)(t) = G(x,t)$.

PROPOSITION 3.3. *There is an index preserving bijection*

$$G_{\#}\colon \operatorname{Coker}\left(1 - f^{\omega}_{*/H}\right) \to \operatorname{Coker}\left(1 - g^{\omega + G(x)}_{*/H}\right)$$

given by $G_{\#}[H+\alpha] = [H+\alpha]$; hence, $R_H(f) = R_H(g)$ and $N_H(f) = N_H(g)$.

Proof. Let $q\colon \tilde{X}_H \to X$ be the covering space of X associated with $H(x)$. Then $\tilde{X}_H$ has elements $H + \lambda$ for $\lambda\colon x \to y$ in ΠX. Let $\operatorname{Lift}_H f$ denote the set of continuous lifts of f to $\tilde{X}_H$. If $\omega\colon x \to f(x)$ is in ΠX, then there is a unique lift $\tilde{f}_\omega$ of f determined by $\tilde{f}_\omega(H) = H + \omega$. Thus a choice of ω gives a "base point" to $\operatorname{Lift}_H f$. Identifying $\Pi_1(X,x)/H$ with the group of deck transformations of q, we see that the set $\{(H+\alpha)\circ \tilde{f}_\omega | H + \alpha \in \Pi_1(X,x)/H\}$ exhausts $\operatorname{Lift}_H f$ and that the set $\operatorname{Lift}'_H f$ of conjugacy classes (under conjugation by elements of $\Pi_1(X,x)/H$) is in bijective correspondence with $\operatorname{Coker}(1 - f^{\omega}_{*/H})$. This bijection takes the class of $(H+\alpha)\circ\tilde{f}_\omega$ to $[H+\alpha]$. We note that if $\mathbf{F}' \in \Phi_H(f)$ and if $\rho(\mathbf{F}) = [H+\alpha]$ then $q(\Phi((H+\alpha)\circ\tilde{f}_\omega)) = \mathbf{F}$. In this way the index of 3.1

can be redefined as $i[H+\alpha] = i(q\Phi((H+\alpha)\circ \tilde{f}_\omega))$. It is now easy to see that the homotopy G lifts at $(H+\alpha)\circ \tilde{f}_\omega$ to a unique homotopy $\tilde{G}$: $(H+\alpha)\circ \tilde{f}_\omega \simeq (H+\alpha)\circ \tilde{g}_{\omega+G(x)}$ and the result follows from standard techniques (cf. for example [**12**, Theorem 1.4.5]). □

The proof of 3.3 is useful in that it indicates a formal way to tie together the fundamental group and the covering space approaches to Nielsen Theory.

The following Lemma is analogous to [**12**, Theorem 3.2.6].

LEMMA 3.4. *Let $H \lhd \Pi X$, $\overline{H} \lhd \Pi \overline{X}$, let f: $X \to \overline{X}$, g: $\overline{X} \to X$ be continuous functions with $f_*(H) \subset \overline{H}$ and $g_*(\overline{H}) \subset H$. Then for ω: $x \to gf(x)$ in ΠX, the function f_*:* $\operatorname{Coker}(1 - gf^{\omega}_{*/H}) \to \operatorname{Coker}(1 - fg^{f_*(\omega)}_{*/\overline{H}})$ *given by $f_*[H+\alpha] = [\overline{H} + f_*(\alpha)]$, is an index preserving bijection, hence $R_H(gf) = R_{\overline{H}}(fg)$ and $N_H(gf) = N_{\overline{H}}(fg)$.*

Proof. Since $f(gf) = (fg)f$ we see as in 2.9 that f_* above is well defined. Since $[H+\alpha] = [H + g_*f_*(\alpha)]$ in $\operatorname{Coker}(1 - gf^{\omega}_{*/H})$, the inverse of f_* is g_*. It is also easy to see that f induces bijections f_*: $\Phi(gf) \to \Phi(fg)$ and f_*: $\Phi'_H(gf) \to \Phi'_H(fg)$; so if $\Phi(gf) = \varnothing$ then f_* above is clearly index preserving. Assume $\Phi(gf) \neq \varnothing$ and let $\mathbf{F} \in \Phi'_H(gf)$, then

$$
\begin{aligned}
i(\mathbf{F}) &= i(X, gf, U) \text{ for suitable } U \\
&= i\big(Y, fg, g^{-1}(U)\big) \text{ by the commutative property of index} \\
&= i\big(f(F)\big).
\end{aligned}
$$
□

Let X and Y be compact connected ANR's, $H \lhd \Pi X$, $K \lhd \Pi Y$ and let f: $X \to X$, g: $Y \to Y$ and h: $X \to Y$ be such that $f_*(H) \lhd H$, $g_*(K) \subset K$, $f_*(H) \subset K$ further let h be a homotopy equivalence with homotopy inverse k such that $k_*(K) \subset H$ and such that the diagram

$$
\begin{array}{ccc}
X & \xrightarrow{f} & X \\
h\downarrow & & \downarrow h \\
Y & \xrightarrow{g} & Y
\end{array}
\tag{3.5}
$$

is homotopy commutative.

The next proposition is a relative version of the analogue of [**12**; Theorem 1.5.4]. Let L: $f \simeq kgh$ $X \to Y$ and G: $hkg \simeq g$: $Y \to Y$ be homotopies.

PROPOSITION 3.6. *There is an index preserving bijection*

$$h_{\square}\colon \operatorname{Coker}\left(1 - f^{\omega}_{*/H}\right) \to \operatorname{Coker}\left(1 - g^{h_*(\omega + L(x)) + G(h(x))}_{*/K}\right)$$

given by:

$$h_{\square}[H + \alpha] = [K + h_*(\alpha)];$$

hence $R_H(f) = R_K(g)$ *and* $N_H(f) = N_K(g)$.

Proof. $h_{\square}$ is the composite

$$\operatorname{Coker}\left(1 - f^{\omega}_{*/H}\right) \xrightarrow{L_\#} \operatorname{Coker}\left(1 - kgh^{\omega + L(x)}_{*/H}\right)$$

$$\xrightarrow{h_*} \operatorname{Coker}\left(1 - hkg^{h_*(\omega + L(x))}_{*/H}\right) \xrightarrow{G_\#} \operatorname{Coker}\left(1 - g^{h_*(\omega + L(x)) + G(x)}_{*/H}\right). \square$$

For the rest of this section we investigate the relationships between $R_H(f)$ and $R(f)$ on the one hand, and between $N_H(f)$ and $N(f)$ on the other. Here $f\colon X \to X$, $H \lhd \Pi X$, and f preserves H.

Let $x \in \Phi(f)$, then we have by hypothesis, an exact sequence

$$(3.7) \qquad 0 \to H(x) \xrightarrow{j^x} \Pi_1(X, x) \xrightarrow{q^x} \Pi_1(X, x)/H \to 0$$

where j^x is the inclusion and q^x the projection. Furthermore, f induces a homomorphism of 3.7; so by 1.8, there is for each $x \in \Phi(f)$ an exact sequence

$$(3.8) \quad 0 \to \operatorname{Fix}(f^x_*|H) \to \operatorname{Fix} f^x_* \xrightarrow{q_*} \operatorname{Fix} f^x_{*/H} \xrightarrow{\delta} \operatorname{Coker}(1 - f^x_*|H)$$

$$\xrightarrow{j^x_*} \operatorname{Coker}(1 - f^x_*) \xrightarrow{q^x_*} \operatorname{Coker}\left(1 - f^x_{*/H}\right) \to 0.$$

COROLLARY 3.9. *If f is eventually commutative, then by* (1.15)

$$\left[\operatorname{Fix} f^x_{*/H}\colon q\operatorname{Fix} f^x_*\right] R(f) = R(f_*|H) R_H(f). \qquad \square$$

The Jiang subgroup $J(f)$ plays a useful role. Recall that $J(f)$ consists of those elements β of $\Pi_1(X, x)$ for which there is a homotopy $G\colon f \simeq f$ with $G(x) = \beta$.

We translate a proposition of Jiang [**12**; Theorem 3.2.11] into our context.

PROPOSITION 3.10. (*Jiang*). *If $f_*(H) \subset J(f)$, then any two ordinary fixed point classes contained in an H-fixed point class have the same index.*

Proof. We refer to 3.8. Let $[\alpha]$, $[\alpha']$ in $\operatorname{Coker}(1 - f_*^x)$ be such that $q_*^x[\alpha] = q_*^x[\alpha']$. This is true if and only if $[\alpha'] = [h + \alpha]$ for some $h \in H$. Now $[h + \alpha] = [\alpha + f_*(h)]$ by 1.4. Given G: $f \simeq f$ with $G(x) = f_*(h)$; then the composite

$$\operatorname{Coker}(1 - f_*^x) \xrightarrow{G_\#} \operatorname{Coker}(1 - f_*^{f_*(h)}) \xrightarrow{r} \operatorname{Coker}(1 - f_*^x)$$

is index preserving and $rG_\#[\alpha] = [\alpha + f_*(h)] = [\alpha']$ as required. □

Note from 3.10 that for $[\lambda] \in \operatorname{Coker}(1 - f_*^x)$ the set $E(f) \cap q_*^{x^{-1}}(q_*^x[\lambda])$ is either empty, or is the whole of $(q_*^x)^{-1}((q_*^x[\lambda]))$.

We need another result of Jiang ([**12**; Lemma 2.3.7]).

PROPOSITION 3.11 (*Jiang*). $J(f) \subset Z(f_*(\Pi_1 X)\colon \Pi_1 X)$. *Here* $Z(H, G) = \{g \in G$ *with* $g + h = h + g$ *for all* $h \in H\}$ *is the centralizer of* H *in* G. □

COROLLARY 3.12. *If the quotient* $R(f_*|H(x))/[\operatorname{Fix} f_{*/H}^x\colon q\operatorname{Fix} f_*^x]$ *is independent of* x *in any essential fixed point class of* f, *and if* $f_*(H) \subset J(f)$ *then*

$$[\operatorname{Fix} f_{*/H}^x\colon q(\operatorname{Fix} f_*^x)]\, N(f) = R(f_*|H) N_H(f).$$

Proof. If $f_*(H) \subset J(f)$, then by 3.11 $f_*|H$ is eventually commutative and $R(f_*|H)$ has a group structure with δ of 3.8 a homomorphism (to prove this compare 3.8 with the corresponding sequence deduced from H_1 of 3.7); so $\operatorname{Ker} q_*^x$ has order $R(f^*|H)/[\operatorname{Fix} f_{*/H}^x;\ q\operatorname{Fix} f_*^x]$. By the independence condition, it is enough to show that for any $[\lambda] \in \mathbf{E}(f)$ there is a bijection $(q_*^x)^{-1}(q_*^x[\lambda]) \cong \operatorname{Ker} q_*^{x'}$ for some x' in an essential fixed point class of f. Now $i[\lambda] \neq 0$ means $[\lambda] = [u - f_*(u)]$ for some u: $x \to x'$, with $x' \in \Phi(f)$ and the result follows from 2.11. □

COROLLARY 3.13. *Let* $f_*(H) \subset J(f)$, *then*

(i) *If* f *is eventually commutative, the formula in* 3.12 *holds*.

(ii) *If* $f_*|H$ *is nilpotent, then* $N(f) = N_H(f)$.

Proof. If f is eventually commutative, q_*^x, can be regarded as a group homomorphism so each of the fibres of q_*^x in 3.8 has the same cardinality. As in 3.12, $\operatorname{Ker} q_*^x$ has cardinality $R(f_*|H(x))/[\operatorname{Fix} f_{*/H}^x;\ q \operatorname{Fix} f_*^x]$ and the first result follows. If $f_*|H$ is nilpotent, then by 3.8 for each $x \in \Phi(f)$, we have $R(f_*|H(x)) = 1 = [\operatorname{Fix} f_{*/H}^x;\ q \operatorname{Fix} f_*^x]$, and the second result follows from 3.12. □

If for each $x \in \Phi(f)$, 3.7 has a normal splitting preserved by f_*: $\Pi_1(X, x) \to \Pi_1(X, x)$ (cf. 1.15) we say *f splits normally over H*.

COROLLARY 3.14. *If $f_*(H) \subset J(f)$, f splits normally over H and $R(f_*|H(x))$ is independent of x in any essential fixed point class of f, then*

$$N(f) = R(f_*|H)N_H(f).$$

Proof. By 1.17, $[\operatorname{Fix} f^x_{*/H}; q^x \operatorname{Fix} f^x_*] = 1$ for all appropriate x. □

We remark again that f eventually commutative will yield $R(f_*|H(x))$ independent of $x \in \Phi(f)$. If X is the total space of a fibration q: $X \to B$ with $H = \Pi_1 F_b$ where F_b is the fibre, then by Gottlieb [7], $f_*(H) \subset J(f)$ (see also the proof of 4.7). The condition $N(f_b) \neq 0$ then implies that $N(f_b) = R(f_*|H)$ and 3.14 gives some well known result. One condition which ensures that f splits normally over H is that $\Pi_1(X)/H$ is all torsion and H is torsion free. There is no fibration assumption needed in this latter situation. Corollary 3.14 was, of course, inspired by [5] from which other conditions for f to split normally over H can be deduced.

4. Nielsen numbers of fibre maps. In 3.12 we deduced a product theorem relating the H-Nielsen number with the ordinary Nielsen number. The central results of this section use the same techniques to deduce product theorems for fibre maps.

Throughout this section let p: $E \to B$, f: $E \to E$, $\bar{f}$: $B \to B$ and K be as in the introduction; thus, the diagram

$$\text{(4.1)} \qquad \begin{array}{ccc} E & \xrightarrow{f} & E \\ p\downarrow & & \downarrow p \\ B & \xrightarrow{\bar{f}} & B \end{array}$$

is commutative. Let $x \in \Phi(f)$, $b = p(x)$ and $f_b = f|F_b$: $F_b \to F_b$; then there is an exact sequence

$$\text{(4.2)} \qquad 0 \to \Pi_1(F, x)/K \to \Pi_1(E, x) \to \Pi_1(B, p(x)) \to 0$$

a morphism of 4.2 induced by f, and from 1.8 an exact sequence

$$\text{(4.3)} \qquad 0 \to \operatorname{Fix} f^x_{b*/K} \xrightarrow{j_{b*}} \operatorname{Fix} f^x_* \xrightarrow{p_*} \operatorname{Fix} \bar{f}^b_* \xrightarrow{\delta} \operatorname{Coker}(1 - f^x_{b*/K})$$

$$\xrightarrow{j_{b*}} \operatorname{Coker}(1 - f^x_*) \xrightarrow{p_*} \operatorname{Coker}(1 - \bar{f}^b_*) \to 0.$$

By analogy with the results of section three, we will want to know when the number [Fix $\bar{f}^b_*$; p_* Fix f^x_*] is independent of x, when $N_K(f_b)$ is independent of b, and in view of You's Theorem ([**20**; Theorem 5.6]), we will want to know when the former number is one and the latter number equal to $N(f_b)$. This is because these are the necessary and sufficient conditions for Brown's formula to hold.

We shall need the following.

PROPOSITION 4.4 (*Heath–Norton* [**10**]). *If* λ: $e \to e'$ *is a path in E with* $\bar{\lambda} = p(\lambda)$, *then if* $\bar{\lambda}$ *is not a constant path, there is a regular lifting function* Γ *for p with the property that* $\Gamma(\bar{\lambda}, e) = \lambda$, *and* $\Gamma(-\bar{\lambda}, e') = -\lambda$. □

For any lifting function Γ and path $\bar{\mu}$: $b \to d$ in B the translation function $\tau_{\bar{\mu}}$: $F_b \to F_b$ is defined by $\tau_{\bar{\mu}}(x) = \Gamma(\bar{\mu}, x)(1)$. We note that 4.4 *essentially* allows us to choose base points in the fibres in advance with $\tau_{\bar{\mu}}$ base point preserving.

LEMMA 4.5 (*see* [**8**]). *If* $\bar{\mu} \sim \bar{\sigma}$: $b \to d$ *in B, then* $\tau_{\bar{\mu}} \simeq \tau_{\bar{\sigma}}$: $F_b \to F_d$. *If* $b, d \in \Phi(\bar{f})$ *and* $\bar{\mu}$: $b \to d$ *in B, then the diagram*

$$\begin{array}{ccc} F_b & \xrightarrow{f_b} & F_b \\ \tau_{\bar{\mu}} \downarrow & & \downarrow \tau_{\bar{f}(\bar{\mu})} \\ F_d & \xrightarrow{f_d} & F_d \end{array}$$

is homotopy commutative. □

COROLLARY 4.6. *For* $H = K$ *or H trivial, the following conditions are independent of* $b \in \Phi(\bar{f})$:

(i) f_b: $F_b \to F_b$ *is nilpotent* mod H.

(ii) f_b: $F_b \to F_b$ *is eventually commutative* mod H.

Moreover, f nilpotent respectively eventually commutative implies f_b *is nilpotent* mod H, *respectively eventually commutative* mod H. *The latter statements are implied by* $f_*|\text{Ker } q_*$ *is nilpotent respectively eventually commutative.*

Proof. The independence conditions follow from iterations of the diagram in 4.5. □

The fibration p is said to be *orientable* if, for any two paths λ, μ: $b \to d$ in B, $\tau_\lambda \simeq \tau_\mu$: $F_b \to F_d$. This condition is easily seen to be equivalent to: for any loop λ: $b \to b$ in B, $\tau_\lambda \simeq 1$: $F_b \to F_b$ (see [**12**; Definition 2.1]).

Part (i) of the next corollary is due to You [**20**; p. 235].

COROLLARY 4.7. *For* $H = K$ *or* H *trivial* $N_H(f_b)$ *is independent of* b *in any essential fixed point class of* $\bar{f}$ *if one of the following holds*:

(i) *If* p *is orientable*

(ii) *If* f_b *is nilpotent* mod H *and* $N(f_b) \neq 0$ *for any* b *in any essential fixed point class of* $\bar{f}$. *In this case* $N_H(f_b) = 1$.

(iii) *If* $R(\bar{f}) = 1$, *in particular if* $\bar{f}$ *is nilpotent.*

(iv) *If* $N(\bar{f}) = 1$.

Proof. If p is orientable, we can replace $\tau_{\bar{f}(\bar{\mu})}$ by $\tau_{\bar{\mu}}$ in diagram 4.5, then apply 3.6. Gottlieb in [**6**] shows that the image of $\delta\colon \Pi_2 B \to \Pi_1 F$ in the exact sequence of p is contained in $J(1_x)$. But $\delta(\Pi_2 B) = K$ and $J(1_x) \subset J(f)$ (cf. [**1**]), so (ii) follows from 3.12. For (iii), either $N(\bar{f}) = 0$ in which case the proposition is trivial or (iii) reduces to (iv). For (iv), we need only consider $b, d \in \Phi(\bar{f})$ for which there is a $\lambda\colon b \to d$ in B with $\lambda \sim \bar{f}(\lambda)$. The result in this case follows from 3.6 and 4.5. □

From 1.3 we see that if $\Pi_1 B$ is finite and $\operatorname{Fix} \bar{f}^b = 0$ for some $b \in \Phi(\bar{f})$, then (iii) holds. This is also true if $\bar{f}$ is eventually commutative and $H_1(\bar{f})$ is nilpotent. The number $N(\bar{f}_b) \neq 0$ for all b in any essential fixed point class if the natural projection $\mathbf{E}(f) \to \mathbf{E}(\bar{f})$ is surjective (this follows from 4.8 below); this will happen, for example, if $J(f) = \Pi_1 E$ and $N(f) \neq 0$.

We saw in the proof of 4.7 that if $\bar{\mu} \sim \bar{f}(\bar{\mu})$ in B, then we can replace $\tau_{\bar{f}(\bar{\mu})}$ in 4.5 by $\tau_{\bar{\mu}}$. In this case $\tau_{\bar{f}(-\bar{\mu})}$ is a homotopy inverse of τ_{μ}. Let $x \in \Phi(f_b)$, $x' \in \Phi(f_b)$; then without changing the class of $\bar{\mu}$ in ΠB, we can lift $\bar{\mu}$ to a path $\mu\colon x \to x'$ in E. If $\bar{\mu}$ is not the constant path, there is, by 4.4, a lifting function Γ with $\Gamma(\bar{\mu}, x) = \mu$ and $\Gamma(-\bar{\mu}, x) = -\mu$. Let $h = \tau_{\bar{f}(-\bar{\mu})} \circ f_d \circ \tau_{\bar{\mu}}$, $h' = \tau_{\bar{\mu}} \circ \tau_{\bar{f}(-\bar{\mu})} \circ f_d$, then, using the above Γ, [**20**; Lemma 2.2] specializes to give explicit homotopies $L\colon f_b \simeq h$ and $G\colon h' \simeq f_d$ with $\tau_{\bar{\mu}}(L(x)) + G(x') \sim -\mu + f_*(\mu)$ in E.

LEMMA 4.8. *If* $[K + \mu] \in \operatorname{Coker}(1 - f^x_{b*/K})$ *and* $[\mu] = j_{b*}[K + \mu] \in \operatorname{Coker}(1 - f^x_*)$, *then if either* $i[K + \mu] \neq 0$ *or* $i[\mu] \neq 0$, *there is a commutative diagram*

$$\begin{array}{ccc} \operatorname{Coker}\left(1 - f^x_{b'/K}\right) & \overset{j_{b*}}{\to} & \operatorname{Coker}(1 - f^x_*) \\ \Xi \downarrow & & \Psi \downarrow \\ \operatorname{Coker}\left(1 - f^{x'}_{b'*/K}\right) & \overset{j_{b'*}}{\to} & \operatorname{Coker}\left(1 - f^{x'}_*\right) \end{array}$$

for some $x' \in \Phi(f)$ with $\Psi([\mu]) = [O_{x'}]$ the class of the constant path, and Ξ and Ψ index preserving bijections.

Proof. If $i[K + \mu] \neq 0$, then $[K + \mu] = [K + u - f_{b*}(u)]$ for some u: $x \to x'$ in ΠF_b with $x' \in \Phi(f_b)$. Then $\Xi = u_*^{f_b}$ and $\Psi = u_*^f$ are the required bijections. If $i[\mu] \neq 0$, then $[\mu] = [u - f_*(u)]$ for some u: $x \to x'$ in $\Pi_1 E$ with $x' \in \Phi(f)$. If u (and therefore x') is contained in F_b, then proceed as above; otherwise let $\bar{u} = p(u)$ as a path in B, then by 4.4, there is a regular lifting function Γ for p with $\Gamma(\bar{u}, x) = u$. Using this Γ, the left hand square of 4.9

$$\begin{array}{lccccc} & \operatorname{Coker}\left(1 - f^{x}_{b^*/K}\right) & \xrightarrow{\tau_{u\square}} & \operatorname{Coker}\left(1 - f^{\tau_u(L(x)) + G(x')}_{b'^*/K}\right) & \xrightarrow{r} & \operatorname{Coker}\left(1 - f^{x'}_{b'^*/K}\right) \\ (4.9) & j_{b^*}\downarrow & & j_{b'^*}\downarrow & & j_{b'^*}\downarrow \\ & \operatorname{Coker}(1 - f^{x}_{*}) & \xrightarrow{u_*} & \operatorname{Coker}\left(1 - f_*^{-u + f_*(u)}\right) & \xrightarrow{r} & \operatorname{Coker}\left(1 - f^{x'}_{*}\right) \end{array}$$

is commutative where $(\tau_{\bar{u}})_\square$ is defined as in 3.6 using the explicit homotopies mentioned above. To see this, we merely observe that properties of our lifting functions ensure that for any loop d at x in F_b, and any loop u: $x \to x$ in E, if $\bar{u} = q(u)$, then $\tau_{\bar{u}}(d) = -u + d + u$ (compare 4.4 and [**20**; Lemma 2.1]). Clearly, $\Xi = r(\tau_{\bar{u}})_\square$ and $\Psi = u_*^f$ do the job. □

The reader will note that $(\tau_{\bar{u}})_\square$ is a simplified form of the transformation $T_{\bar{u}}$ of [**20**]; we have developed sufficient machinery to indicate below the proofs of the following two propositions (4.10 is part of [**20**; Corollary 3.4]) in our notation.

PROPOSITION 4.10. *If in the situation of* 4.3, $[\sigma] = j_{b*}[K + \sigma] = j_{b*}[K + \theta] = [\theta]$, *then* $i([K + \sigma]) = i([K + \theta])$, *for* $[K + \sigma]$, $[K + \theta] \in \operatorname{Coker}(1 - f^x_{b^*/K})$.

Proof. Since $[\sigma] = [\theta]$ in $\operatorname{Coker}(1 - f^x_*)$, there is a $u \in \Pi_1(E, x)$ with $\theta = -u + \sigma + f_*(u)$. If $u \subset F_b$ as a path, then $[K + \sigma] = [K + \theta]$. Otherwise, let Γ be a regular lifting function for p with $\Gamma(\bar{u}, x) = u$ (where $\bar{u} = p(u)$). Then as in the proof of the second part of 4.8, we have

$$r(\tau_{\bar{u}})_\square([K + \sigma]) = \left[K + \tau_{\bar{u}}(\sigma) + \tau_{\bar{u}}(L(x)) + G(x)\right].$$

Now, $\tau_{\bar{u}}(\sigma) + \tau_{\bar{u}}(L(x)) + G(x) \sim (-u + \sigma + u) + (-u + f_*(u))$ in E, so we have

$$r(\tau_{\bar{u}})_\square([K + \sigma]) = [K + \theta]$$

since $\Pi_1(F, x)/K \to \Pi_1(E, x)$ is injective. Because $r(\tau_{\bar{u}})_\square$ is index preserving, we are done. $\square$

PROPOSITION 4.11 (*You*). *If in* 4.3 $[K + \mu] \in \operatorname{Coker}(1 - f^x_{b*/K})$, $[\mu] = j_{b*}[K + \mu] \in \operatorname{Coker}(1 - f^x_*)$ *and* $[\bar{\mu}] = q_*[\mu] \in \operatorname{Coker}(1 - \bar{f}^b_*)$, *then* $i[\mu] \neq 0$, *if and only if both* $i[K + \mu] \neq 0$ *and* $i[\bar{\mu}] \neq 0$.

Proof. Transformations of the type shown in 4.8 used in combination with fibre homotopies can be used to reduce 4.11 to the case where $[\mu] = 0$ and B is a finite polyhedron with $\bar{f}$ having only isolated fixed points. The result then follows from a known result (cf. [3]) involving the index in the total space being the product of indices in the fibre and base. This is exactly the technique of You. $\square$

THEOREM 4.12 (*You*). *If* $N_K(f_b)$ *is independent of* b *in any essential fixed point class of* $\bar{f}$, *and if* $[\operatorname{Fix} \bar{f}^b_*;\, p_* \operatorname{Fix} f^x_*]$ *is independent of* x *in any essential fixed point class of* f, *then*

$$\left[\operatorname{Fix} \bar{f}^b_* : p_* \operatorname{Fix} f^x_*\right] N(f) = N_K(f_b) N(\bar{f}).$$

Proof. If $N(f) = 0$, then by 4.10 and 4.11 $N_K(f_b) = 0$ for any b in an essential fixed point class of $\bar{f}$, or $N(\bar{f}) = 0$. So we assume $N(f)$, and hence, $N_K(f_b)$ and $N(\bar{f})$ are non-zero. Let $x \in \Phi(f)$ be our base point. Using the technique of 2.11 and the fact (2.5) that $u^f_*|\mathbf{E}(f)$ is the identity, we see that for each $[\lambda] \in \mathbf{E}(f)$

$$\text{(A)} \qquad \#\left((p^x_*)^{-1} p^x_*[\lambda] \cap \mathbf{E}(f)\right) = \#\left(\operatorname{Ker} p^{x'}_* \cap \mathbf{E}(f)\right)$$

for some x' in the class represented by $[\lambda]$. Now 4.3, 4.10, and 4.11 show that for each $[\mu] \in \operatorname{Ker} p^x_* \cap \mathbf{E}(f)$, we have $j^{-1}_{b*}[\mu] \subset \mathbf{E}_K(f_b)$ and so

$$\mathbf{E}_K(f_b) = \bigcup_{[\mu] \in \operatorname{Ker} p^x_* \cap \mathbf{E}(f)} j^{-1}_{b*}[\mu].$$

The technique of 3.3 and the exactness of 4.3 shows for some x'' in an essential fixed point class of f that $\# j_{b*}[\mu] = [\operatorname{Fix} \bar{f}^{b''}_*;\, p_* \operatorname{Fix} f^{x''}_*]$. The independence hypothesis now yields

$$\text{(B)} \qquad N_K(f_b) = \#(\operatorname{Ker} p^x_* \cap \mathbf{E}(f))\left[\operatorname{Fix} \bar{f}^b_* : p_* \operatorname{Fix} f^x_*\right].$$

The formulae (A) and (B) and our hypotheses together show that for any $[\lambda] \in \mathbf{E}(f)$, the order $\#((p^x_*)^{-1} p^x_*[\lambda] \cap \mathbf{E}(f))$ is independent of x in any essential fixed point class. To conclude the proof, we observe from 4.10 that $p_*(\mathbf{E}(f)) = \mathbf{E}(\bar{f})$ and invoke 1.10. $\square$

The hypotheses of 4.12 are fulfilled if p is orientable (4.7) and f is eventually commutative; with these hypotheses 4.12 is then exactly Corollary 5.9 of [**20**].

COROLLARY 4.13 (*You*). *If p is orientable, then $N(f) = N(f_b)N(\bar{f})$ if and only if*

(i) $N_K(f_b) = N(f_b)$ *and*

(ii) *For each* $x \in \mathbf{F} \in \mathbf{E}(f)$, $[\mathrm{Fix}\, \bar{f}_*^{p(x)}; p_* \mathrm{Fix}\, f_*^x] = 1$. □

We shall investigate 4.13 (i) and (ii) separately.

The next definition and example generalize 2.1 and 2.9 of [**5**]. The fibration $p\colon E \to B$ admits a *fibre splitting mod K with respect to f* if for each $x \in \Phi(f)$, the sequence 4.2 has a normal splitting preserved by f_*^x.

EXAMPLES 4.14. In each of the following situations $p\colon E \to B$ admits a fibre splitting mod K with respect to f

(a) $\Pi_1 B = 0$

(b) $j_{b*} = 0\colon \Pi_1(F, x) \to \Pi_1(E, x)$ for some x

(c) for each $x \in E$, the sequence 4.2 has a normal splitting with $\Pi_1 B$ all torsion and $\Pi_1(F, x)/K$ torsion free.

COROLLARY 4.15. *If $N_K(f_b)$ is independent of b in any essential fixed point class of $\bar{f}$, then*

$$N(f) = N_K(f_b)N(\bar{f})$$

provided one of the following hold:

(i) *The fibration $p\colon E \to B$ admits a fibre splitting* mod K *with respect to f*.

(iii) Fix $\bar{f}_*^b = 0$ *for any b in any essential fixed point class of $\bar{f}$.*

Proof. Condition (ii) clearly implies $[\mathrm{Fix}\, \bar{f}_*^b; p_* \mathrm{Fix}\, f_*^x] = 1$ for all appropriate x as does (i) by 1.17. □

COROLLARY 4.16. *Let $p\colon E \to B$ be orientable; then $N(f) = N(\bar{f})$ provided one of the following hold*:

(i) $R_K(f_b) = 1$ *for all* $b \in \Phi(\bar{f})$, *and* $N_K(f_b) \neq 0$ *for some* $b \in \Phi(\bar{f})$ *or*

(ii) $N_K(f_b) = 1$ *for some* $b \in \Phi(\bar{f})$.

Proof. For both cases $N(\bar{f}) = 0$ if and only if $N(f) = 0$ so as in 4.12, we assume $N(f) \neq 0$. Now conditions (i) and (ii) imply condition (ii) for all $b \in \Phi(\bar{f})$. In this situation, by 4.10, $\operatorname{Ker} j_{b*} = 0$ and so for any $x \in \Phi(f)$, $[\operatorname{Fix} \bar{f}_*^b : p_* \operatorname{Fix} f_*^x] = 1$ by 4.3. □

Among the conditions that imply 4.16 (i) we have (a) f_b nilpotent mod K for some $b \in \Phi(\bar{f})$, (b) the restriction of f_*^x to $\operatorname{Ker} p_*$ is nilpotent, (c) $\operatorname{Fix} f_{*/K}^x = 0$ for all $x \in \Phi(f)$ and $\Pi_1(F, x)/K$ finite, (d) f is eventually commutative and $j_*\Pi_1(F, x)/K \subset (\Pi_1 E)'$ the commutator subgroup of $\Pi_1 E$. For $K = 0$, 4.16 (ii) is [**5**; Theorem 6.2].

We note there is no orientability condition in the next result.

Corollary 4.17. *If* $\bar{f}_*: \Pi_1 B \to \Pi_1 B$ *is nilpotent, then*

$$N(f) = N_K(f_b) N(\bar{f}).$$

Proof. If $\bar{f}_*$ is nilpotent, then $R(\bar{f}) = 1$ by 1.5 so $N(\bar{f})$ is zero or one by 2.5. If $N(\bar{f}) = 0$, then so is $N(f)$ by 4.11. If $N(\bar{f}) = 1$, then $N_K(f_b)$ is independent of b in *the* essential fixed point class of $\bar{f}$ by 4.7. The proof is completed by using 1.5 and 4.15 (ii). □

The conclusion of 4.17 is valid under the assumptions that $\operatorname{Fix} \bar{f}_*^b = 0$ for any b in an essential fixed point class of $\bar{f}$ and $\Pi_1 B$ is finite. Again we do not need orientability.

Proposition 4.18. *Let* $x \in \Phi(f)$, $b = p(x)$, *then* $N_K(f_b) = N(f_b)$ *provided one of the following holds*:

(i) p_*^x: $\operatorname{Coker}(1 - f_{b*}^x) \to \operatorname{Coker}(1 - f_{b*/K}^x)$ *in injective*

(ii) $\bar{f}_*: \Pi_2(B, b) \to \Pi_2(B, b)$ *is nilpotent* mod $p_*\Pi_2(E, x)$.

(iii) $f_{b*}|K$ *is nilpotent*

(iv) f_b *is eventually commutative and* $K \subset (\Pi_1, F)'$

(v) $N(f_b) = 1$.

Proof. Statements (ii) through (iv) imply (i) and the results follow from section three, as does (v). □

The condition that K is a subgroup of $(\Pi_1 F)'$ can arise as follows. Let M be a group for which $M' \subset Z(M)$, the centre of M, then for any $K \subset M'$, the realization of the crossed module $M \to M/K$ gives a fibration $F \to Y \to B$ with $\Pi_1 F = M$, $\Pi_1 Y = M/K$, $\Pi_2 B = K$ and all other groups zero (for further details see [**14**]). I am grateful to P. J. Higgins and R. Brown (Bangor) for help with this example.

Putting some of these results together, we have

COROLLARY 4.19. *If p is orientable, admits a fibre splitting* mod K *with respect to f, and if for all b in an essential fixed point class of $\bar{f}$, $f_{b*}|K$ or $\bar{f}_*$: $\Pi_2 B \to \Pi_2 B$ is nilpotent, then*

$$N(f) = N(f_b)N(\bar{f}). \qquad \square$$

A map is said to be *homotopy nilpotent* if f^n is null homotopic for some positive integer n. R. F. Brown has shown me the following example of a homotopy nilpotent map: Let B be the wedge of a space with itself thus $B = A \vee A$ for some A. The map g: $B \to B$ which takes a pair (a_1, a_2) to the pair $(a_2, *)$ where $*$ is a base point, is homotopy nilpotent.

Note that our final result does not require orientability.

COROLLARY 4.20. *If $\bar{f}$ is homotopy nilpotent then*

$$N(f) = N(f_b)N(\bar{f}). \qquad \square$$

REFERENCES

[1] R. F. Brown, *The Nielsen number of a fibre map*, Ann. of Math., **85** (1967), 483–493.

[2] ______, *The Lefchetz Fixed Point Theorem*, Scott-Foresman (1971).

[3] ______, *On the Product Theorem for the Fixed Point Index*, Fixed Point Theory (Proceedings, Sherbrooke, Quebec, (1980)). Lecture Notes in Mathematics, Vol. 886, Springer, 1981, 44–48.

[4] R. F. Brown and E. Fadell, *Corrections to "The Nielsen number of a fibre map"*, Ann. of Math., **95** (1972), 365–367.

[5] E. Fadell, *Natural fibre splittings and Nielsen numbers*, Houston J. Math., **2**(1) (1976), 71–84.

[6] E. C. Giessmann, *Product formulae for the Nielsen numbers of bundle maps*, Funkcional. Anal. i Prilozen, **12**(4) (1978), 81–82.

[7] D. H. Gottlieb, *On fibre spaces and the evaluation map*, Annals of Math., **87** (1968), 42–55.

[8] P. R. Heath, *Groupoid operations and fibre homotopy equivalence*, Math. Z., **130** (1973), 207–233.

[9] P. R. Heath and K. H. Kamps, *On exact orbit sequences*, Illinois J. Math., **26** (1) (1982), 149–154.

[10] P. R. Heath and G. H. Norton, *Equiconnectivity and cofibrations II*, (in preparation).

[11] B. J. Jiang, *Estimation of the Nielsen numbers*, Chinese Math., **5** (1966), 330–339.

[12] ______, *Lecture on Nielsen fixed point theory*, Contemporary Math., **14** (1983).

[13] T. H. Kiang, *The Theory of Fixed Point Classes*, (in Chinese), Scientific Press, Peking, 1979.

[14] S. K. Kim, D. Mcgavran and J. Pak, *On the obstruction $P(\mathbf{F}, f)$ to the product theorems of fibre-preserving maps*, Indiana U. Math. J., **25** (8) (1976), 725–732.

[15] J. L. Loday, *Spaces with finitely many non-trivial homotopy groups*, J. Pure and Appl. Algebra, **24** (1982), 179–202.

[16] D. McCord, *An estimate of the Nielsen number and an example concerning the Lefchetz fixed point theorem*, Pacific J. Math., **66** (1) (1976), 195–203.

[17] D. G. Northcott, *Homological Algebra*, Cambridge University Press, (1960).

[18] J. Pak, *On the Reidemeister numbers and Nielsen numbers of the fibre preserving maps*, Preprint.

[19] E. Spanier, *Algebraic Topology*, McGraw-Hill Book Co., New York, 1966.

[20] C. Y. You, *Fixed Point Classes of a Fibre Map*, Pacific J. Math., **100** (1), (1982), 217–241.

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