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ON JUNG'S CONSTANT AND RELATED CONSTANTS IN NORMED LINEAR SPACES

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In this paper several results on certain constants related to the notion of Chebyshev radius are obtained. It is shown in the first part that the Jung constant of a finite-codimensional subspace of a space $C(T)$ is 2, where T is a compact Hausdorff space which is not extremally disconnected. Several consequences are stated, e.g. the fact that every linear projection from a space $C(T)$, T a perfect compact Hausdorff space, onto a finite-codimensional proper subspace has norm at least 2.

The second discusses mainly the "self-Jung constant" which measures "uniform normal structure." It is shown that this constant, unlike Jung's constant, is essentially determined by the finite subsets of the space.

1. Jung constant in $C(T)$ spaces.

For a bounded subset A of a normed linear space E and a subset Y of E we denote by diam A the diameter of A ($\sup_{x,y\in A}\|x-y\|$), by $r_Y(A)$ the relative Chebyshev radius of A with respect to Y ($\inf_{y\in Y}\sup_{x\in A}\|x-y\|$), and by $Z_Y(A)$ the relative Chebyshev center set of A in Y ($\{y\in Y; \sup_{x\in A}\|x-y\| = r_Y(A)\}$). The Jung constant of E is $J(E)=\sup\{2r_E(A); A\subset E, \operatorname{diam} A = 1\}$. It is easily seen that $1\leqq J(E)\leqq 2$. For n-dimensional spaces E_n, it was shown by Jung [12] that $J(l_2^n)=(2n/(n+1))^{1/2}$ and $J(E_n)=1$ if and only if $E_n = l_\infty^n$. Bohnenblust [2] showed that $J(E_n)\leqq 2n/(n+1)$, and Leichtweiss [14] characterized the extremal case (in the 2-dimensional case it is the hexagonal plane). In the infinite-dimensional case, it was shown that $J(l_2)=\sqrt{2}$ (Routledge [20]), and that $J(E)=1$ if and only if $E=C(T)$ for a Stonian T, i.e. if $E\in\mathscr{P}_1$ (Davis [5]) (cf. also [10], pages 91–92 in [11] and §6 in [4]).

Studying intersections of balls with subspaces, Franchetti [6] deduced that for every finite-codimensional subspace E of $C[a,b]$ we have $J(E)\geqq 3/2$. A stronger and more general result is true.

1.1. PROPOSITION. *If the compact Hausdorff space T is not extremally disconnected, then for every finite-codimensional subspace E of $C(T)$ we have $J(E)=2$.*

We need the following

1.2. LEMMA. *Let E be a finite-codimensional subspace of $C(T)$, T compact Hausdorff. Then for every $\varepsilon > 0$ and every infinite open $V \subset T$ there is $f \in E$ with $\| f \| = 1, f(T \setminus V) = 0$ and $f \geqq -\varepsilon$.*

Proof of the lemma. In the case where V contains no isolated points, the proof is quite short: Since V is infinite, $\{ f \in E; f(T \setminus V) = 0\}$ is infinite dimensional and there are $f_1 \in E$, $t_1 \in V$ with $\| f_1\| = 1 = f_1(t_1)$, $f_1(T \setminus V) = 0$. For $V_1 = \{t \in V; f_1(t) > 1 - \varepsilon\}$, which is infinite too, find in the same way $f_2 \in E$, $t_2 \in V_1$ with $\| f_2\| = 1 = f_2(t_2)$, $f_2(T \setminus V_1) = 0$, etc. $g = \sum_{j=1}^n f_j$ satisfies $\|g\| \geqq g(t_n) > n(1 - \varepsilon)$, while, since $f_j(t) < 0$ happens only when $f_{j-1}(t) > 1 - \varepsilon$ and $f_{j+1}(t) = 0$, $g(t) > -\varepsilon$. Normalize to get f. □

For the general case we apply

1.3. SUBLEMMA. *Given an infinite matrix $(x^j(k))_{j=1,\dots,n;\ k=1,2,\dots}$ such that $x^j(k) \to 0$ as $k \to \infty$ for $j = 1,\dots,n$ and $\varepsilon > 0$, there are k and $(\varepsilon_i)_{i=1}^\infty$ such that $|\varepsilon_i| \leqq \varepsilon$ for all i and $x^j(k) = \sum_{i \neq k} \varepsilon_i x^j(i)$ for $j = 1,\dots,n$.*

Proof of the sublemma. We may assume that the rows $x^1,\dots,x^n$ are linerly independent. Therefore there are also n independent columns, which we may assume to be the first n ones. Let $(\gamma_{r,s})_{r,s=1}^n$ be the inverse of the matrix $(x^j(k))_{j,k=1}^n$ and $c = \sum_{r,s}|\gamma_{r,s}|$. There is k such that $|x^j(k)| < \varepsilon/c$ for $j = 1,\dots,n$. Represent the kth column as a linear combination of the first n ones. □

Proof of the lemma in the general case. Take a sequence $(f_k)_{k=1}$ of disjointly supported nonnegative norm-one functions sitting in V. Apply the sublemma to $x^j(k) = \mu_j(f_k)$, where $\mu_1,\dots,\mu_n \in C(T)^*$ are such that $E = \{\mu_1,\dots,\mu_n\}_\perp$. Take $f = f_k - \sum_{i \neq k} \varepsilon_i f_i$. □

Proof of the proposition. Choose disjoint open subsets V_1, V_2 and $w \in \overline{V}_1 \cap \overline{V}_2$ (such w, V_1, V_2 exist since T is not extremally disconnected). Fix $\varepsilon > 0$. Let $A \subset E$ consist of all $f_1 - f_2$, when f_i run over all the functions f satisfying the conclusions of the lemma with respect to V_i. Then $f^* = \sup_{f \in A} f$ is 1 on V_1 and $\leqq \varepsilon$ on V_2, while $f_* = \inf_{f \in A} f$ is -1 on V_2 and $\geqq -\varepsilon$ on V_1. Thus the diameter of A is $\leqq 1 + \varepsilon$. The radius of A, however, is $\geqq 1$ since $\max_{t_1,t_2 \in V} | f^*(t_1) - f_*(t_2)| = 2$ in every neighborhood V of w. □

REMARK. Proposition 1.1 verifies also a conjecture of Franchetti ([7]): If $J(C(T)) < 2$ then T is extremally disconnected (and then $J(C(T)) = 1$ by Davis' result). This last result has been proved independently by C. Franchetti [8].

Lemma 1.2 can be applied also to improve Proposition 2 in [6], giving an alternative proof of our Proposition 1.1 in the perfect case.

1.4. PROPOSITION. *Let F be a finite-codimensional subspace of $C(T)$, T perfect compact Hausdorff space. Then for every $x \in C(T)$ and every $s > d \equiv d(x, F)$ we have*

$$Z_F(B(x, s) \cap F) = P_F x \quad \text{and} \quad r_F(B(x, s) \cap F) = s + d,$$

where $B(x, s)$ is the closed s-ball centered at x ($\{y; \|y - x\| \leqq s\}$) and $P_F x$ is the best approximation to x in F.

Proof. Given any $y_0 \in F$ with $\|x - y_0\| > d$, we want to show that there is a $y \in F$ with $\|x - y\| \leqq s$ and $\|y - y_0\| > s + d$. This will establish both claims, since if $\|x - y_1\| < d + \varepsilon$ then clearly $\|y - y_1\| \leqq \|y - x\| + \|x - y_1\| < s + d + \varepsilon$ for every such y.

Without loss of generality we may assume $y_0 = 0$, $\|x\| = x(t_0)$ for some $t_0 \in T$. If $\|x\| < s$, let

$$0 < \varepsilon < \min\left(\frac{s - \|x\|}{s + d + 1}, \frac{\|x\| - d}{2}, 1\right),$$

$V = \{t; |x(t) - x(t_0)| < \varepsilon\}$. Apply Lemma 1.2 to get $z \in F$ with $\|z\| = 1$ and $z \geqq -\varepsilon$ which vanishes off V. Let $y = (s + d + \varepsilon)z$. Clearly $\|y\| > s + d$. If $t \notin V$, then $|(x - y)(t)| = |x(t)| \leqq \|x\| < s$. If $t \in V$ then

$$-s < \|x\| - \varepsilon - (s + d + \varepsilon) \leqq (x - y)(t) \leqq \|x\| + \varepsilon(s + d + \varepsilon) < s.$$

If $\|x\| \geqq s$, let $y_1 \in F$ satisfy $d < \|x - y_1\| < s$. Let

$$0 < \varepsilon < \min\left(\frac{s - \|x - y_1\|}{s + d}, \frac{\|s - y_1\| - d}{2}, y_1(t_0)\right),$$

$V = \{t\colon |x(t) - x(t_0)| + |y_1(t) - y_1(t_0)| < \varepsilon\}$. Apply Lemma 1.2 to get $z \in F$ with $\|z\| = 1 = z(t_1)$, $z \geqq -\varepsilon$ which vanishes off V. Let $y = y_1 + (s + d)z$.

$$\|y\| \geqq y_1(t_1) + s + d > y_1(t_0) - \varepsilon + s + d > s + d.$$

If $t \notin V$, then $|(x - y)(t)| = |(x - y_1)(t)| < s$. If $t \in V$, then

$$\begin{aligned} -s &< (x - y_1)(t_0) - \varepsilon - s - d \\ &\leqq (x - y)(t) \leqq (x - y_1)(t) + (s + d)\varepsilon < s. \end{aligned}$$

□

1.5. COROLLARY. *If F is a subspace of $C(T)$, T any compact Hausdorff space with no isolated points, and* $1 \leqq \operatorname{codim} F < \infty$, *then* $J(F) = 2$.

Thus, for perfect T, the restriction in Proposition 1.1 that T be non-Stonian is necessary (for $J(F) = 2$) only in the case $F = C(T)$. Further concessions are impossible — since if $t_0 \in T$ is isolated in the Stonian space T, then $F = \{x \in C(T); x(t_0) = 0\}$ is isometric to $C(T')$, where $T' = T \setminus \{t_0\}$ is Stonian too, hence $J(F) = 1$.

Applying Franchetti's observation on the relation between projection constants of hyperplanes and radii of hypercircles [**8**], we get:

1.6. COROLLARY. *If F is a finite-codimensional proper subspace of $C(T)$, T perfect compact Hausdorff space, then every linear projection of $C(T)$ onto F has norm* $\geqq 2$.

Proof. Let $F = \{\mu_1, \ldots, \mu_n\}_\perp$, $\mu_i \in C(T)^*$, $\|\mu_i\| = 1$, $E = \{\mu_1, \ldots, \mu_{n-1}\}_\perp$ such that F is a maximal subspace of E. A linear projection of E onto F has the form $Px = x - \mu_n(x)z$, where $z \in E$ and $\mu_n(z) = 1$. But

$$\|P\| \geqq \sup_{0 \leqq \alpha < 1} \sup_{\|x\| \leqq 1} \|Px\| = \sup_{0 \leqq \alpha < 1} \sup_{\substack{y \in F \\ \|y + \alpha z\| \leqq 1}} \|y\|$$

$$\geqq \sup_{0 \leqq \alpha < 1} r_F(B(-\alpha z, 1) \cap F).$$

By Proposition 1.4, since $d(-\alpha z, F) = \alpha$, $r_F(B(-\alpha z, 1) \cap F) = 1 + \alpha$, so that $P = \sup_{0 \leqq \alpha < 1}(1 + \alpha) = 2$.

Thus, every projection of E onto F, and therefore also every projection of $C(T)$ onto f, has norm $\geqq 2$. □

2. Jung constants and normal structure coefficients. By a classical result of Garkavi and Klee (cf., e.g. [**13**]) $r_A(A) = r(A)$ for all convex closed and bounded $A \subset E$ is equivalent to E having dimension $\leqq 2$ or being an inner product space. Therefore, besides the Jung constant $J(E)$, one may study also the "self-Jung constant" $J_s(E) = \sup\{2r_A(A); A \subset E$ convex, $\operatorname{diam} A = 1\}$. Clearly $J_s(E) \geqq J(E)$. E is said to have "normal structure" if for every such A we have $r_A(A) < \operatorname{diam} A$. Thus $J_s(E)$ measures to what extent E has "uniform normal structure". Bynum [**3**] introduced the "normal structure coefficient"; $N(E) = 2/J_s(E)$, and two other coefficients, $BS(E)$ and $WCS(E)$, analogously defined by the "asymptotic diameter" and the "asymptotic radius" of bounded, or weakly convergent, sequences in E, respectively, i.e.

$$\inf\left\{\frac{\lim_k \sup_{m,n>k}\|x_n - x_m\|}{\inf \lim_k \sup_{n\geq k}\|y - x_n\|;\ y \in \overline{\operatorname{conv}}(x_k)_{k=1}^{\infty}}\right\},$$

where the infinum is taken over all bounded nonconvergent sequences $(x_n) \subset E$ in the $BS(E)$ case, and over all weakly convergent, non-norm-convergent sequences in the $WCS(E)$ case. Clearly $1 \leq N(E) \leq BS(E) \leq WCS(E)$ and $WCS(E) \leq 2$ unless E has the Schur property (i.e. unless in E norm and weak sequential convergence coincide).

It is easy to see, and hinted in **[3]**, that $BS(E) = \sup\{N(F);\ F \subset E$ separable$\}$ and $WCS(E) = \sup\{WCS(F);\ F \subset E$ separable$\}$.

In **[15]**, Lim shows that $J_s(E) = \sup\{2r_A(A);\ A \subset E$ convex and separable, $\operatorname{diam} A = 1\}$, hence $N(E) = BS(E)$ for every normed E. This can be further improved, using the following observations:

2.1. PROPOSITION. (a) *If E is a dual Banach space, then*

$$J(E) = \sup\{2r_E(K);\ K \subset E \textit{ finite}, \operatorname{diam} K = 1\}.$$

(b) *If E is a reflexive Banach space, then*

$$J_s(E) = \sup\{2r_{\operatorname{conv} K}(K);\ K \subset E \textit{ finite}, \operatorname{diam} K = 1\}.$$

Proof. (a) Let $A \subset E$ be any with $\operatorname{diam} A = 1$, $r < r_E(A)$ any. Then $\bigcap_{x\in A} B(x, r) = \emptyset$ and by w*-compactness of the balls there is a finite $K = \{x_1, \ldots, x_n\} \subset A$ with $\bigcap_{x\in K} B(x, r) = \emptyset$, i.e. $r < r_E(K)$.

(b) Let $A \subset E$ be convex closed with $\operatorname{diam} A = 1$, $r < r_A(A)$ any. Then $\bigcap_{x\in A} B(x, r) \cap A = \emptyset$ and by w-compactness of the balls and of A there is a finite $K \subset A$ with $\bigcap_{x\in K} B(x, r) \cap \operatorname{conv} K \subset \bigcap_{x\in K} B(x, r) \cap A = \emptyset$, i.e. $r < r_{\operatorname{conv} K}(K)$. □

2.2. PROPOSITION. (*Maluta*, **[16]**.) *If E is a non reflexive Banach space, then $J_s(E) = 2$.*

Proof. By a theorem of D. P. Milman and V. D. Milman **[18]** there is, in every nonreflexive Banach space E and for every $\varepsilon > 0$, a sequence $(x_n)_{n=1}^{\infty}$ in E such that for every $m \geq 1$ and every $y' \in \operatorname{conv}(x_1, \ldots, x_m)$, $y'' \in \operatorname{conv}(x_{m+1}, x_{m+2}, \ldots)$ we have $1 - \varepsilon < \|y' - y''\| < 1 + \varepsilon$. Taking $A = \operatorname{conv}(x_n)_{n=1}^{\infty}$, one has $\operatorname{diam} A \leq 1 + \varepsilon$ while $r_A(A) \geq 1 - \varepsilon$. □

2.3. COROLLARY. (a) (Lim) $J_s(E) = \sup\{2r_{\operatorname{conv} A}A;\ A \subset E$ *separable*, $\operatorname{diam} A = 1\} = \max\{J_s(F);\ F$ *a separable subspace of* $E\}$.

(b) *If* $J_s(E) < 2$, *then* $J_s(E) = \sup\{2r_{\mathrm{conv}\,K}K;\ K \subset E$ *finite*, $\mathrm{diam}\,K = 1\} = \sup\{J_s(F);\ F$ *a finite dimensional subspace of* $E\}$.

(c) *If* E *has "uniform normal structure", so does every reflexive* G *which is finitely representable in* E (*i.e. such that for every finite dimensional subspace* F *of* G *and every* $\varepsilon > 0$ *there is an isomorphism* T *of* F *onto a subspace of* E *with* $\|T\|\,\|T^{-1}\| < 1 + \varepsilon$).

Proof. Immediate from Propositions 2.1(b) and (2.2) and from the fact that every non reflexive Banach space contains a separable non reflexive subspace. □

REMARK. It is not clear, however, from the above whether "uniform normal structure" is a superproperty, i.e. whether "reflexive" can be dropped in (c) or, equivalently, whether "uniform normal structure" implies superreflexivity.

We observe here that the (absolute) Jung constant $J(E)$ cannot be estimated from either side by the Jung constants of its subspaces in a similar way. Any space E is a subspace of some $\mathscr{P}_1$-space $F = l_\infty(\Gamma)$ for some Γ (e.g. the dual ball) and $J(F) = 1$ while $J(E)$ can be any. Thus we may have $J(E) > J(F)$ when $E \subset F$. We cannot also get lower bounds for $J(E)$ by considering finite or separable subsets, as shown by:

2.4. EXAMPLES. (a) $J(c_0) = 2$ by Proposition 1.1 (e.g. take $A = \{(-1)^n e_n;\ n = 1,2,\dots\}$, then $\mathrm{diam}\,A = 1 = r_{c_0}(A)$). However, for every finite $A = \{x_1,\dots,x_n\} \subset c_0$, $\bar{x} = \frac{1}{2}(\max_{1\le i\le n} x_i - \min_{1\le i\le n} x_i) \in c_0$ satisfies $r(\bar{x}, A) = \frac{1}{2}\,\mathrm{diam}\,A$.

(b) Let Γ be an uncountable set, $E = \{x \notin m(\Gamma);\ \mathrm{spt}\ x \text{ countable}\}$ (where $\mathrm{spt}\,x = \{\gamma;\ x(\gamma) \neq 0\}$). E is a closed subspace of $m(\Gamma)$, hence Banach. Every separable subset of E is contained in a subspace of $m(\Gamma_0)$, where $\Gamma_0 \subset \Gamma$ is countable (the union of the supports of a dense sequence). $m(\Gamma_0)$ is a subspace of E isometric to $m = l_\infty$ which has Jung constant $J(M) = 1$. On the other hand, let $\Gamma = \Gamma_1 \cup \Gamma_2$, where Γ_1, Γ_2 are uncountable and disjoint, $A_i = \{x \in E;\ 0 \leqq x \leqq x_{\Gamma_i}\}$ $(i = 1,2)$, $A = A_1 \cup (-A_2)$. It is easily seen that $\mathrm{diam}\,A = 1$ but $r(A) = 1$. Thus $J(E) = 2$.

On the other hand, we have:

2.5. PROPOSITION. *Let* $(E_\alpha)_{\alpha\in D}$ *be a net of linear subspaces of the Banach space* E, *directed by inclusion, such that* $\overline{\bigcup_{\alpha\in D} E_\alpha} = E$. *Then*: (a) *If* E *is reflexive, then* $J_s(E) = \sup_\alpha J_s(E_\alpha) = \lim_{\alpha\in D} J_s(E_\alpha)$.

(b) *If* E *is a dual space and each* E_α *admits a norm*-1 *linear projection* P_α, *then* $J(E) = \sup_\alpha J(E_\alpha) = \lim_{\alpha\in D} J(E_\alpha)$.

Proof. If P is a norm-1 projection of E onto F, then for every $A \subset F$, $x \in E$ we have $r(Px, A) \leq r(x, A)$, hence $r_F(A) = r_E(A)$, thus $J(F) \leq J(E)$. Therefore for every $\alpha \leq \beta$ we have $J_s(E_\alpha) \leq J_s(E_\beta) \leq J_s(E)$ or $J(E_\alpha) \leq J(E_\beta) \leq J(E)$, respectively. In either case it is enough to consider $A = \operatorname{conv}(x_1,\ldots,x_n) \in E$ with

$$\frac{r_A(A)}{\operatorname{diam} A} > \frac{1}{2}J_s(E) - \varepsilon \quad \text{or} \quad \frac{r_E(A)}{\operatorname{diam} A} > \frac{1}{2}J(E) - \varepsilon,$$

respectively. But taking $x_1',\ldots,x_n' \in E_\alpha$ with $\|x_i - x_i'\| < \varepsilon$ for $i = 1,\ldots,n$ we get $A' = \operatorname{conv}\{x_1',\ldots,x_n'\} \subset E_\alpha$ (for some α) satisfying, respectively,

$$\frac{r_{A'}(A')}{\operatorname{diam} A'} > \frac{r_A(A) - \varepsilon}{\operatorname{diam} A + \varepsilon} \quad \text{or} \quad \frac{r_E(A')}{\operatorname{diam} A'} > \frac{r_E(A) - \varepsilon}{\operatorname{diam} A + \varepsilon}. \qquad \square$$

2.6. COROLLARY. *For every $1 \leq p < \infty$ and every infinite dimensional $L_p(\mu)$ space we have*
(a) $J_s(L_p(\mu)) = J_s(l_p) = \sup_n J_s(l_p^n) = \lim_n J_s(l_p^n)$ *and*
(b) $J(L_p(\mu)) = J(l_p) = \sup_n J(l_p^n) = \lim_n J(l_p^n)$.

Proof. For every measurable partition $D = \{D_0, D_1,\ldots,D_n\}$ of the measure space, with $0 < \mu(D_i) < \infty$ for $i = 1,\ldots,n$, the characteristic fucntions $\{\chi_{D_1},\ldots,\chi_{D_n}\}$ span in $L_p(\mu)$a subspace F_D isometric to l_p^n, and admitting the norm-1 projection $P_D f = \sum_{i=1}^n (\int_{D_i} f\,d\mu)\chi_{D_i}/\mu(D_i)$. The F_D clearly form a net directed by inclusion whose union is dense in $L_p(\mu)$, so that we can apply Proposition 2.5. $\square$

In order to give lower bounds for J and J_s in n-dimensional spaces, consider "(n, m, r)-symmetric block designs", i.e. 0-1 symmetric $n \times n$ matrices $A = (a_{ij})_{i,j=1}^n$ such that

$$\sum_{j=1}^n a_{ij}a_{kj} = \begin{cases} m & \text{if } i = k, \\ r & \text{if } i \neq k, \end{cases}$$

where $n > m > 0$ and r is, necessarily, $m(m-1)/(n-1)$.

2.7. LEMMA. *If E is an n-dimensional space with a symmetric basis $(e_k)_{k=1}^n$ (i.e. such that $\|\sum_{k=1}^n |\alpha_k| e_k\| = \|\sum_{k=1}^n \alpha_{\pi(k)} e_k\|$ for all scalars $\alpha_1,\ldots,\alpha_n$ and all permutations π of $\{1,\ldots,n\}$), and if there is an (n, m, r)-symmetric block design $(a_{ij})_{i,j=1}^n$, then*

$$J(E) \geq 2 \min_{0 \leq \alpha \leq 1} \left\| (1-\alpha)\sum_{i=1}^m e_i + \alpha \sum_{i=m+1}^n e_i \right\| \left\| \sum_{i=1}^{2(m-r)} e_i \right\|^{-1}$$

and

$$J_s(E) \geq 2\left\|\left(1-\frac{m}{n}\right)\sum_{i=1}^{m} e_i + \frac{m}{n}\sum_{i=m+1}^{n} e_i\right\| \left\|\sum_{i=1}^{2(m-r)} e_i\right\|^{-1}.$$

If there is an $(n, m, m/2)$*-symmetric block design (hence, necessarily,* $m = (n+1)/2$*), then also*

$$J_s(E) \geq \left\|\sum_{i=1}^{n} e_i\right\| \left\|\sum_{i=1}^{m} e_i\right\|^{-1}.$$

Proof. Consider the points $x_i = \sum_{j=1}^{n} a_{ij} e_j$ and the sets $A = \mathrm{conv}(x_1,\ldots,x_n)$ or $A_0 = \mathrm{conv}(0, x_1,\ldots,x_n)$, respectively. By symmetry, center points are multiples of $\sum_{i=1}^{n} e_i$. Also,

$$\min_{0\leq\alpha\leq m/n} \max\left(\left\|(1-\alpha)\sum_{i=1}^{m} e_i + \alpha\sum_{i=m+1}^{n} e_i\right\|, \alpha\left\|\sum_{i=1}^{n} e_i\right\|\right) = \frac{1}{2}\left\|\sum_{i=1}^{n} e_i\right\|. \quad \square$$

2.8. Corollary. *If there is an* (n, m, r)*-symmetric block design then, for every* $1 \leq p \leq \infty$*, we have*

$$J(l_p^n) \geq \left(\frac{2^{p-1}(n-1)}{\|(m, n-m)\|_{q-1}}\right)^{1/p} \quad \left(\text{where } \frac{1}{p} + \frac{1}{q} = 1\right)$$

(since the minimizing α *for* $p > 1$ *is* $m^{1/p-1}/(m^{1/p-1} + (n-m)^{1/p-1})$ *and for* $p = 1$ *it is* 1 *if* $m \geq 2n$ *and* 0 *if* $m \leq 2n$*), and*

$$J_s(l_p^n) \geq \frac{\left(2^{p-1}(n-1)\|(m, n-m)\|_{p-1}^{p-1}\right)^{1/p}}{n}.$$

If there is an $(n, m, m/2)$*-symmetric block design, then*

$$J_s(l_p^n) \geq (2n/(n+1))^{1/p}.$$

2.9. Lemma. *There are* (n, m, r)*-symmetric block designs in each of the following cases:*

(a) *n is any,* $m = 1, r = 0,$ *or* $m = n-1, r = n-2$.

(b) $n = 2^{2t}, m = 2^{t-1}(2^t - 1), r = 2^{t-1}(2^{t-1} - 1)$ *or* $m = 2^{t-1}(2^t + 1)$, $r = 2^{t-1}(2^{t-1} + 1)$.

(c) $n = 2^t - 1, m = 2^{t-1}, r = 2^{t-2}$, *or* $m = 2^{t-1} - 1, r = 2^{t-2} - 1$.

Proof. For (a) take the unit matrix, $a_{ij} = \delta_{ij}$ or its complement $a_{ij} = 1 - \delta_{ij}$. For (b) define, inductively, $A_0 = (1)$, $B_0 = (0)$,

$$A_{t+1} = \begin{pmatrix} B_t & A_t & A_t & A_t \\ A_t & B_t & A_t & A_t \\ A_t & A_t & B_t & A_t \\ A_t & A_t & A_t & B_t \end{pmatrix}, \qquad (B_{t+1})_{ij} = 1 - (A_{t+1})_{ij}.$$

For (c), let $W_t = (w^t_{ij})^{2^t}_{i,j=1}$ be the Walsh matrix, defined inductively by $W_0 = (1)$,

$$W_{t+1} = \begin{pmatrix} W_t & W_t \\ W_t & -W_t \end{pmatrix},$$

and consider $(\frac{1}{2}(1 - w^t_{ij}))^{2^t}_{i,j=2}$. □

2.10. Corollary. (a) $J(l_p^{2^t}) \geq ((2^t - 1)/2^{t-1})^{1/p}$.
(b) $J_s(l_p^n) \geq 2^{p-1/p}[(n-1) + (n-1)^p]^{1/p}/n$.
(c) $J_s(l_p^{2^t-1}) \geq ((2^t - 1)/2^{t-1})^{1/p}$.
(d) $J(l_p) \geq 2^{1/p}$.
(e) $J_s(l_p) \geq \max(2^{1/p}, 2^{p-1/p})$.
((e) follows also from Corollary 2.6 and Bynum's estimate $WCS(L_p) \leq \min(2^{p-1/p}, 2^{1/p})$.)

2.11. Corollary. (a) $J_s(E) \geq 2^{1/p_E}$, *where* $p_E = \inf\{p; l_p$ *is finitely represented in* $E\}$ = *the maximal "type" of E in the sense of Maurey and Pisier* [**17**]. *Thus, if E has uniform normal structure, it is "B-convex"* ([**13**]). *In fact, stronger conditions are imposed on E* (cf. [**1**]).
(b) *For every infinite-dimensional* E, $J_s(E) \geq \sqrt{2}$ (*Maluta*, [**16**]) (*since* $p_E \leq 2$ *by Dvoretzky*'s *theorem*).

Now we observe some upper bounds.

2.12. Proposition. *If* $\dim E \leq n$, *then* $J_s(E) \leq 2n/(n+1)$.

Proof. Given a convex $A \subset E$ with $\operatorname{diam} A = 1$, take any $r < r_A(A)$. Then $\bigcap_{x \in A} B(x, r) \cap \bar{A} = \varnothing$ hence by Helly's theorem, there are $x_0, \ldots, x_n \in A$ with $\bigcap_{i=0}^n B(x_i, r) \cap A = \varnothing$. But, taking

$$\bar{x} = \frac{1}{n+1} \sum_{i=0}^n x_i \in A,$$

we have

$$\|\bar{x} - x_j\| = \frac{1}{n+1}\left\|\sum_{i=0}^{n}(x_i - x_j)\right\| = \frac{1}{n+1}\left\|\sum_{i\neq j}(x_i - x_j)\right\|$$
$$\leq \frac{1}{n+1}\max_{i\neq j}\|x_i - x_j\| \leq \frac{n}{n+1},$$

hence $r < n/(n+1)$. Since $r < r_A(A)$ was arbitrary, $r_A(A) \leq n/(n+1)$.□

If $(x_0, x_1, \ldots, x_n) \in E$, the "$n$-volume" of $\operatorname{conv}(x_0, \ldots, x_n)$ is

$$V(x_0, \ldots, x_n) = \sup\left\{\det\begin{pmatrix} 1, f_1, \ldots, f_n \\ x_0, x_1, \ldots, x_n \end{pmatrix}; f_i \in B(E^*), i = 1, \ldots, n\right\}.$$

Following Sullivan [21], we define the modulus of n-convexity of E,

$$\delta_E^{(n)}(\varepsilon) = \inf\left\{1 - \frac{1}{n+1}\left\|\sum_{i=0}^{n} x_i\right\|; x_i \in B(E),\right.$$
$$\left. i = 0, \ldots, n, V(x_0, \ldots, x_n) \geq \varepsilon\right\}$$

(so that

$$\delta_E^{(1)}(\varepsilon) = \delta_E(\varepsilon) = \inf\left\{1 - \left\|\frac{x_0 + x_1}{2}\right\|; x_0, x_1 \in B(E), \|x_0 - x_1\| \geq \varepsilon\right\}$$

is the ordinary modulus of convexity). Sullivan showed that if E is "n-uniformly convex", i.e. if $\delta_E^{(n)}(\varepsilon) > 0$ for all $\varepsilon > 0$, then E is superreflexive and has normal structure. Bynum [3] observed that $J_s(E) \leq 2(1 - \delta_E(1))$. One can push this argument one step further:

2.13. **Proposition.**

$$J_s(E) \leq 2\min_{\varepsilon}\max\left(1 - \delta_E^{(2)}(\varepsilon), \frac{2}{3}\varepsilon + \frac{1}{2}\right).$$

Proof. Let $A \subset E$ be convex with $\operatorname{diam} A = 1$. Suppose $r_A(A) > r > 1 - \delta_E^{(2)}(\varepsilon)$. Take $\eta > 0$ and x_0, $x_1 \in A$ with $\|x_1 - x_0\| > 1 - \eta$ and $x_2 \in A$ with

$$\left\|x_2 - \frac{x_0 + x_1}{2}\right\| > r,$$

$x_3 \in A$ with

$$\left\|x_3 - \frac{x_0 + x_1 + x_2}{3}\right\| > r.$$

Translating, we may assume $x_3 = 0$. Take $f_1 \in B(E^*)$ with

$$f_1(x_1 - x_0) > 1 - \eta \quad \text{and} \quad f_2 \in B(E^*)$$

with

$$f_2\left(x_2 - \frac{x_0 + x_1}{2}\right) > r.$$

Then

$$\begin{aligned} V(x_0, x_1, x_2) &= \begin{vmatrix} 1 & 1 & 1 \\ f_1(x_0) & f_1(x_1) & f_1(x_2) \\ f_2(x_0) & f_2(x_1) & f_2(x_2) \end{vmatrix} \\ &= \begin{vmatrix} f_1(x_1 - x_0) & f_1(x_2 - (x_0 + x_1)/2) \\ f_2(x_1 - x_0) & f_2(x_2 - (x_0 + x_1)/2) \end{vmatrix} \\ &> (1 - \eta)r - f_1\left(x_2 - \frac{x_0 + x_1}{2}\right) f_2(x_1 - x_0). \end{aligned}$$

But

$$\begin{aligned} f_1\left(x_2 - \frac{x_0 + x_1}{2}\right) &= f_1(x_1 - x_0) + f_1\left(\frac{2x_2 + x_0}{2} - \frac{3x_1}{2}\right) \\ &= f_1(x_1 - x_0) + \frac{3}{2} f_1\left(\frac{2x_2 + x_0}{3} - x_1\right) \\ &> (1 - \eta) - \frac{3}{2} = -\frac{1}{2} - \eta \end{aligned}$$

and also

$$f_1\left(x_2 - \frac{x_0 + x_1}{2}\right) = -f_1(x_1 - x_0) + f_1\left(\frac{2x_2 + x_1}{2} - \frac{3x_0}{2}\right) < \frac{1}{2} + \eta.$$

Similarly,

$$f_2(x_1 - x_0) = f_2\left(x_2 - \frac{x_0 + x_1}{2}\right) + f_2\left(\frac{3x_1}{2} - \frac{2x_2 + x_0}{2}\right) > r - \frac{3}{2}$$

and

$$f_2(x_1 - x_0) = -f_2\left(x_2 - \frac{x_0 + x_1}{2}\right) + f_2\left(\frac{x_1 + 2x_2}{2} - \frac{3x_0}{2}\right) < \frac{3}{2} - r.$$

Thus

$$\varepsilon \ge V(x_0, x_1, x_2) > (1 - \eta)r - \left(\tfrac{1}{2} + \eta\right)\left(\tfrac{3}{2} - r\right).$$

Since $\eta > 0$ was arbitrary, $3r/2 - 3/4 \le \varepsilon$ or $r \le 2\varepsilon/3 + 1/2$. □

If we use this estimate for l_2 we get $J_s(l_2) \leq 1.61$ (while $2(1-\delta_{l_2}(1)) = \sqrt{3}$ and $J_s(l_2) = \sqrt{2}$). In any E, if $\delta_E^{(2)}(3/4) > 0$, then $J_s(E) < 2$.

2.14. PROPOSITION. *For every n and every $\varepsilon > 0$, we have $J_s(E) \leq 2\max(1-(1-\varepsilon)/n!\varepsilon, 1-\delta_E^{(n)}(\varepsilon))$, so that if $\delta_E^{(n)}(1) > 0$ then E has uniform normal structure.*

Proof. Let $A \subset E$ be convex with $\operatorname{diam} A = 1$. Take any $r < r_A(A)$ and any $\eta > 0$. Find x_0, $x_1 \in A$ with $\|x_0 - x_1\| > 1 - \eta$ and $x_k \in A$, $k = 2,3,\ldots,n+1$ with $\|x_k - k^{-1}\sum_{i=0}^{k-1} x_i\| > r$ (such x_k exist since $k^{-1}\sum_{i=0}^{k-1} x_i \in A$ and $r_A(A) > r$). Translate to get $x_{n+1} = 0$, so that $x_i \in B(E)$, $i = 0,\ldots,n$. Find $f_1 \in B(E^*)$ with $f_1(x_1 - x_0) > 1 - \eta$ and $f_k \in B(E^*)$, $k = 2,\ldots,n$, with $f_k(x_k - k^{-1}\sum_{i=0}^{k-1} x_i) > r$. Consider

$$V(x_0,\ldots,x_n) \geq \det\left(\frac{1, f_1,\ldots,f_n}{x_0, x_1,\ldots,x_n}\right)$$

$$= \det\left(\frac{f_1, f_2, f_3,\ldots,f_n}{x_1 - x_0,\, x_2 - \tfrac{1}{2}(x_0 + x_1),\, x_3 - \tfrac{1}{3}\sum_{i=0}^{2} x_i,\ldots,x_n - \tfrac{1}{n}\sum_{i=0}^{n-1} x_i}\right).$$

All the entries in the last determinant have absolute value ≤ 1, but the subdiagonal ones, $f_m(x_k - k^{-1}\sum_{i=0}^{k-1} x_i)$ for $m > k$, are small for r close to 1: since $m^{-1}\sum_{i=0}^{m-1} f_m(x_m - x_i) > r$ and $|f_m(x_m - x_i)| \leq 1$, we have $1 - m(1-r) < f_m(x_m - x_i) \leq 1$ for $i < m$, hence

$$|f_m(x_k - x_i)| = |f_m(x_m - x_i) - f_m(x_m - x_k)| < m(1-r)$$

and

$$\left|f_m\left(x_k - \frac{1}{k}\sum_{i=0}^{k-1} x_i\right)\right| = \left|\frac{1}{k}\sum_{i=0}^{k-1} f_m(x_k - x_i)\right| < m(1-r),$$

too. Thus

$$V(x_0,\ldots,x_n) > (1-\eta)r^{n-1} - (n!-1)n(1-r) > \varepsilon$$

provided $r > 1 - (1-\varepsilon-\eta)/n!n$. Therefore for such r we must have

$$r < \left\|1/(n+1)\sum_{i=0}^{n} x_i\right\| < 1 - \delta_E^{(n)}(\varepsilon).$$

Since $\eta > 0$ and $r < r_A(A)$ were arbitrary, we get

$$r_A(A) \leq \max\left(1 - \delta_E^{(n)}(\varepsilon), 1 - (1-\varepsilon)/n!n\right). \qquad \square$$

REMARK. The rough estimate we used above can be improved, but since the computation of $\delta_E^{(n)}$ seems to be quite complicated, it is not clear whether finer estimates will yield more results.

Lim [15] gave the following upper bound for $J_s(l_p), p > 2$:

$$J_s(l_p) \le 2\left(1 + \frac{1 + t^{p-1}}{(1+t)^{p-1}}\right)^{-1/p},$$

where $0 \le t \le 1$ solves $(p-2)t^{p-1} + (p-1)t^{p-2} = 1$.

Maluta [16] defined another related constant for a normed E:

$$D(E) = \sup\left\{\lim_k \sup_{n \ge k} d\left(x_{n+1}, \operatorname{conv}(x_i)_{i=1}^n\right); (x_n) \subset E, \operatorname{diam}(x_n)_{n=1}^\infty = 1\right\},$$

and showed that:

(i) $D(E) = \sup\{D(F); F \subset E \text{ separable}\}$;

(ii) $D(E) = 0$ if and only if E is finite-dimensional.

(iii) If $D(E) < 1$ then the Banach space E is reflexive and has normal structure (but $E = (\Sigma \oplus l_n)_2$ is reflexive and has normal structure although $D(E) = 1$).

(iv) $2D(E) \le J_s(E)$ and, if E is reflexive, $D(E) \le 1/WCS(E)$.

Maluta asked if $D(E) = 1/WCS(E)$ for every reflexive E. She showed that this is the case for l_p, i.e. $D(l_p) = 2^{-1/p}$ (Bynum showed $WCS(l_p) = 2^{1/p}$); $D((\Sigma \oplus l_\infty^n)_2) = 2^{-1/2}$ (Bynum showed $WCS((\Sigma \oplus l_\infty^n)_2) = \sqrt{2} > 1 = N((\Sigma \oplus l_\infty^n)_2)$). For the space $l_{p,1}$, i.e. l_p with the norm $\|x\|_{p,1} = \|x^+\|_p + \|x^-\|_p$, which is of special interest since in it $\delta(1) = 0$, one still has $D(l_{p,1}) = 1/WCS(l_{p,1}) = 2^{-1/p}$. We can give an affirmative answer to Maluta's question in the case that E satisfies the (weak) Opial condition: $w_n \xrightarrow{w} 0 \Rightarrow \liminf \|x_n - x\| \ge \liminf \|x_n\| \; \forall x \ne 0$ [19]. The l_p spaces ($1 < p < \infty$) satisfy this condition, but the $L_p[0,1]$ spaces do not, unless $p = 2$.

2.15. PROPOSITION. *If E satisfies Opial's condition, then $D(E) \ge 1/WCS(E)$.*

Proof. For any $0 \le r < 1/WCS(E)$, we can find $(x_n) \subset E$ with $x_n \xrightarrow{w} 0$, $\operatorname{diam}(x_n) = 1$ and $\limsup \|x_n - x\| > r$ for every $x \in \overline{\operatorname{conv}}(x_n)$. In particular, $\limsup \|x_n\| > r + \varepsilon$ for some $\varepsilon > 0$, so that we can take a subsequence (x_n') with $\|x_n'\| > r + \varepsilon$, $\forall n$. By Opial's condition we have $\liminf \|x_n' - x\| \ge r + \varepsilon$, $\forall x$. Let $n_1 = 1$. If $n_1, \ldots, n_k$ have been chosen, take a finite $\varepsilon/2$-net, $(y_1, \ldots, y_{m_k})$, for $\operatorname{conv}(x_n', \ldots, x_{n_k}')$, and find n_{k+1} so that $\|x_n' - y_j\| > r + \varepsilon/2$ for every $n \ge n_{k+1}$, $j \le m_k$. Then $d(x_{n_{k+1}}', \operatorname{conv}(x_{n_i}')_{i=1}^k) > r$, so that $D(E) \ge r$. □

The parameters $J(E)$, $2D(E)$ and $R(E)$ ([**1**]), although all of them between 1 and $J_s(E)$, are incomparable even for reflexive infinite dimensional spaces:

2.15. EXAMPLES.
(a) $J(l_2) = 2D(l_2) = \sqrt{2} > 1 = R(l_2)$.
(b) $E = (\Sigma \oplus l_\infty^n)_2$. Here $J(E) = 1$; $2D(E) = \sqrt{2}$ and $R(E) = 2$.
(c) $E = (\Sigma \oplus l_1^n)_2$. Here $2D(E) = \sqrt{2}$ again, but $J(E) = R(E) = 2$.

In concluding, we remark that none of the convexity properties $J(E) < 2$, $J_s(E) < 2$, $WCS(E) > 1$ or $D(E) < 1$ is isomorphy invariant. In fact, the "best" spaces have "worst" equivalent renormings. For J this follows from Proposition 1.1 ($m = C(\beta N)$ has a maximal subspace 2-isomorphic to it of the type $C(T)$, T non-Stonian). For J_s, WCS or D, it was observed by Maluta that $D(l_2, \|\ \|_J) = 1$, where $\|x\|_J = \max(\|x\|_2, \sqrt{2}\,\|x\|_\infty)$.

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