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CODIMENSION TWO ISOMETRIC IMMERSIONS BETWEEN EUCLIDEAN SPACES

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Hartman and Nirenberg showed that any C^∞ isometric immersion f: $\mathbf{E}^n \to \mathbf{E}^{n+1}$ between flat Euclidean spaces is a cylinder erected over a plane curve. We show that in the codimension two case, f: $\mathbf{E}^n \to \mathbf{E}^{n+2}$ factors as a composition of isometric immersions $f = f_1 \circ f_2$: $\mathbf{E}^n \to \mathbf{E}^{n+1} \to \mathbf{E}^{n+2}$, when $n > 1$ and f has nowhere zero normal curvature. Counterexamples are given if this assumption is relaxed.

How can paper be folded? More precisely, how can flat Euclidean 2-space $\mathbf{E}^2$ be isometrically immersed into flat Euclidean n-space $\mathbf{E}^n$ (for simplicity, assume C^∞ differentiability everywhere). For $n = 3$, A. V. Pogorelov **[4]** announced without proof that the image is a cylinder erected over a plane curve; proofs may be found in Massey **[3]** and Stoker **[5]**. In this paper, we consider $n = 4$ and show that any isometric immersion g: $\mathbf{E}^2 \to \mathbf{E}^4$ with nowhere zero normal curvature factors as a composition of isometric immersions $g = g_1 \circ g_2$: $\mathbf{E}^2 \to \mathbf{E}^2 \to \mathbf{E}^4$.

The result of Pogorelov has been generalized by Hartman and Nirenberg **[2]**. They showed that the image of any codimension-one isometric immersion between flat Euclidean spaces is a cylinder erected over a plane curve. Using a result of Hartman **[1]** we easily show that any codimension-two, isometric immersion f: $\mathbf{E}^n \to \mathbf{E}^{n+2}$, $n > 1$, with nowhere zero normal curvature factors as a composition $f = f_1 \circ f_2$: $\mathbf{E}^n \to \mathbf{E}^{n+1} \to \mathbf{E}^{n+2}$. The images of f_1 and f_2 are cylinders. The assumption of nowhere zero normal curvature is essential; counterexamples are given in §3 when the assumption is relaxed.

From another point of view, the cylinders of Pogorelov and Hartman and Nirenberg can be deformed ("unrolled") through a one-parameter family of isometric immersions to a hyperplane. This family is obtained by deforming the generating plane curve to a straight line. From our results, it follows easily that any isometric immersion f: $\mathbf{E}^n \to \mathbf{E}^{n+2}$ with nowhere zero normal curvature can be deformed through isometric immersions to a standard inclusion i: $\mathbf{E}^n \hookrightarrow \mathbf{E}^{n+2}$ (it would be interesting to know if the normal curvature assumption can be removed). In addition, we proved **[7]** that if the normal curvature is identically zero, then any isometric immersion $h: \mathbf{E}^n \to \mathbf{E}^m$ (any codimension) is deformable through isometric immersions to a standard inclusion. In codimension one, the normal curvature is always zero.

We thank J. D. Moore for bringing our attention to [1].

1. Preliminaries. We begin with some Riemannian geometry. As mentioned earlier, C^∞ differentiability is assumed everywhere.

Let $h: M \to \overline{M}$ be an isometric immersion between Riemannian manifolds. Let ∇, $\overline{\nabla}$ be the Riemannian connection on M, $\overline{M}$ respectively, and let TM and $\nu(h)$ denote the tangent and normal bundles. If $X \in \Gamma(TM)$ and $N \in \Gamma(\nu(h))$ are sections of the tangent and normal bundles, then we can decompose $\overline{\nabla}_X N$ into its tangential and normal components $\overline{\nabla}_X N = A_X N + D_X N$. The linear mapping A_p: $T_p \otimes \nu_p(h) \to T_p M$ is the second fundamental form of h at $p \in M$, and D is the normal connection. It is easy to see that $\nu(h)$ is a Riemannian vector bundle with the induced metric and Riemannian connection D. We will use $\langle\ ,\ \rangle$ to denote the metric on both $\nu(h)$ and TM. Associated to A is the second fundamental tensor B: $TM \otimes TM \to \nu(h)$ defined by $\langle B(X, Y), N\rangle = \langle A_N X, Y\rangle$. The curvature tensors associated to ∇ and D are given by

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X, Y]} Z$$

$$R^*(X, Y)N = D_X D_Y N - D_Y D_X N - D_{[X, Y]} N$$

where X, Y, Z are tangent vector fields on M, and N is a normal vector field on M.

Now we specialize to $\overline{M} = \mathbf{E}^m$. The following equations are necessary and sufficiently conditions for the existence of $h: M \to \mathbf{E}^m$ (see [6]).

$$\left.\begin{aligned} R(X, Y)Z &= A_X B(Y, Z) - A_Y B(X, Z) \\ R^*(X, Y)N &= B(A_X N, Y) - B(X, A_Y N) \end{aligned}\right\} \quad \text{(Gauss)}$$

$$\begin{aligned} \nabla_X A_Y N - \nabla_Y A_X N - A_{[X, Y]} N & \\ = A_Y D_X N - A_X D_Y N & \end{aligned} \quad \text{(Codazzi-Mainardi).}$$

EXISTENCE THEOREM. *Let M be a simply connected Riemannian n-manifold with a Riemannian k-plane bundle ν over M equipped with a second fundamental form A, an associated second fundamental tensor B, and a compatible normal connection D (compatible with the Riemannian metric on ν). If the Gauss and Codazzi-Mainardi equations are satisfied, then M can be isometrically immersed in $\mathbf{E}^{n+k}$ with normal bundle, ν, normal connection D, and second fundamental form A.*

The rigidity theorem states that an isometric immersion is essentially determined by its Riemannian data.

RIGIDITY THEOREM. *Let $h, h'\colon M \to \mathbf{E}^{n+k}$ be isometric immersions of a connected Riemannian n-manifold (not necessarily simply connected) with normal bundles ν, ν' equipped as above with bundle metrics, connections, and second fundamental forms. Suppose that there is an isometry $\phi\colon M \to M$ that can be covered by a bundle map $\phi^*\colon \nu \to \nu'$ which preserves the bundle metrics, the connections, and the second fundamental forms. Then there is an isometry φ of $\mathbf{E}^{n+k}$ such that $\varphi \circ h = h' \circ \phi$.*

In [7], a one parameter version of the existence theorem is established. This gives the deformations discussed in the introduction.

2. The results. We will prove,

THEOREM 1. *Let $f\colon \mathbf{E}^n \to \mathbf{E}^{n+2}$, $n > 1$, be an isometric immersion with nowhere zero normal curvature. Then f factors as the composition of isometric immersions $f = f_1 \circ f_2\colon \mathbf{E}^n \to \mathbf{E}^{n+1} \to \mathbf{E}^{n+2}$.*

COROLLARY 1. *There is a deformation through isometric immersions between f and the standard inclusion $i\colon \mathbf{E}^n \to \mathbf{E}^{n+2}$.*

Proof. The deformation is obtained by first unrolling f_1 and then f_2. Q.E.D.

The proof of the theorem consists of setting up and solving the associated algebraic problem at the bundle level, and then apply the existence theorem to obtain the required isometric immersions. The assumption of nowhere zero curvature is used in its equivalent form that the second fundamental forms do not commute. These are equivalent pointwise as is seen by a straight forward calculation to establish

$$\langle R^*(X, Y)N_1, N_2\rangle = \langle [AN_1, AN_2]X, Y\rangle$$

where $[\alpha, \beta] = \alpha\beta - \beta\alpha$.

LEMMA 1. *Let $g\colon \mathbf{E}^2 \to \mathbf{E}^4$ be an isometric immersion with nowhere zero normal curvature. Then there exist unique global C^∞ unit normal vector fields N_1 and N_2 satisfying* $\det AN_1 = 0 = \det AN_2$.

Proof. The pointwise existence follows from the first Gauss equation which is equivalent to $\det AN + \det AN^\perp = 0$, where N and $N^\perp$ are orthogonal (the orthogonal operation $\perp$ requires an orientation of the normal bundle, which exists since $\mathbf{E}^2$ is contractible). The uniqueness comes from the noncommutivity of the second fundamental forms. The C^∞ differentiability is demonstrated below.

The mean curvature vector field H is well defined by $H = (\operatorname{tr} AN)N + (\operatorname{tr} AN^\perp)N^\perp$, for arbitrary orthonormal fields $N, N^\perp$. It is nowhere zero as can be easily seen by using the frame N_1, N_2 (for otherwise, the symmetry of the second fundamental form and the condition $\det AN_1 = 0 = \operatorname{tr} AN_1$ imply that AN_1 is the zero transformation, a contradiction).

Define $\sigma\colon \mathbf{E}^2 \to \mathbf{E}^1$ by $N_1 = \cos\sigma H_1 + \sin\sigma H_2$ where $H_1 = H/\|H\|$ and $H_2 = H_1^\perp$. It suffices to show that σ is C^∞. The eigenvectors of AH_ι are C^∞ (since the eigenvalues are distinct) and with respect to a basis of eigenvectors for, say, AH_2, we can write

$$AH_1 = \begin{pmatrix} p & q \\ q & r \end{pmatrix}, \qquad AH_2 = \begin{pmatrix} \tau & 0 \\ 0 & -\tau \end{pmatrix}.$$

The first Gauss equation, $\tau^2 = pr - q^2$, implies that $0 = \det AN_1 = \tau^2 \cos 2\sigma + (\tau/2)(r-p)\sin 2\sigma$. Hence $\sigma = \frac{1}{2}\operatorname{arccot}((p-r)/2\tau)$ is C^∞. Note that $\tau \neq 0$ and $\sin 2\sigma \neq 0$ since $AH_2 \neq 0$. Q.E.D.

If the normal curvature is allowed to be zero, then the normal fields N_1 and N_2 may not even be continuous. An example is given in §3.

If N is any unit normal vector field for $g\colon \mathbf{E}^2 \to \mathbf{E}^4$, then the associated normal connection 1-form is denoted by Θ_N. The associated tangent vector field Z_N is defined by $\Theta_N(\cdot) = \langle \cdot , Z_N \rangle$. Observe that $Z_N = -Z_{N^\perp}$. We assume throughout that the tangent and normal bundles over $\mathbf{E}^2$ have a preferred orientation so that the orthogonal operation $\perp$ is well-defined.

LEMMA 2. *Let $g\colon \mathbf{E}^2 \to \mathbf{E}^4$ be an isometric immersion with nowhere zero normal curvature. Then $Z_{N_1}^\perp$ lies in the kernel of either AN_1 or AN_2.*

Proof. We will first show that $Z_{N_1}^\perp \in \ker(AN_2)$ at $p \in \mathbf{R}^2$, under the assumption that $Z_{N_1}^\perp \notin \ker(AN_1)$ at p. Let $X_i, X_i^\perp$ be the eigenvectors of AN_i, with corresponding eigenvalues λ_i and 0, $i = 1,2$. Let α (resp. β) be an integral curve of $X_1^\perp$ (resp. $Z_{N_1}^\perp$), through $p \in \mathbf{E}^2$.

We intend to construct two tangent fields X, Y and one normal field N along α and β (and then extend these fields to a neighborhood of p). They will be used in a calculation of the Codazzi-Mainardi equation. Let $X = X_1^\perp$ and define N along β by $N|_\beta = N_1|_\beta$. Set $Y|_\beta = Z_{N_1}^\perp|_\beta$. Then $(A_X N)|_\beta = 0$ and so

(1) $$(\nabla_Y A_X N)_p = 0.$$

Now choose N and Y along α so that $\|N\| = 1$, $Y_p = (Z_{N_1}^\perp)_p$, and $A_Y N$ is parallel along X. The existence of N and Y is demonstrated as follows. Let V be the parallel vector field along α with $V_p = (X_1)_p$. Set $N = \cos\rho N_1 + \sin\rho N_2$ and $V = b_1 X_1 + b_2 X_2$ with $\rho(p) = 0 = b_2(p)$. We want $V = A_Y N$, or equivalently,

$$b_1 X_1 + b_2 X_2 = \lambda_1 \cos\rho\langle Y, X_1\rangle X_1 + \lambda_2 \sin\rho\langle Y, X_2\rangle X_2.$$

Choose an arbitrary $\tilde{Y}$ along α so that

$$\langle \tilde{Y}, X_2\rangle \neq 0,$$

$$X_p \cdot \left[\frac{b_2}{\lambda_2\langle \tilde{Y}, X_2\rangle}\right] + \langle X, Z_{N_1}\rangle_p \neq 0$$

(this is easily seen to be possible since $\langle X, Z_{N_1}\rangle_p \neq 0$ by assumption).

Let $\rho = \arcsin(b_2/\lambda_2\langle \tilde{Y}, X_2\rangle)$ and obtain $Y|_\alpha$ by adding to $\tilde{Y}$ a multiple of $X_2^\perp$ so that $\langle Y, X_1\rangle = b_1/\lambda_1 \cos\rho$. This completely defines N and Y along α, and we obtain

$$(2) \qquad (\nabla_X A_Y N)_p = 0.$$

Finally extend N and Y arbitrarily to a neighborhood of p. Now,

$$D_X N = (X\cdot\cos\rho)N_1 + \cos\rho D_X N_1 + (X\cdot\sin\rho)N_2 + (\sin\rho)D_X N_2.$$

Since $\rho(p) = 0$,

$$(3) \qquad (D_X N)_p = (X\cdot\sin\rho)_p N_{2p} + (D_X N_1)_p$$

$$= \left[X\cdot\frac{b_2}{\lambda_2\langle \tilde{Y}, X\rangle} + \langle X, Z_{N_1}\rangle\right]_p N_{2p} \neq 0$$

From equations (1), (2), and (3), the Codazzi-Mainardi equation becomes

$$-\left(A_{[X,Y]}N\right)_p = \left(A_{\gamma Z_N^\perp} N^\perp\right)_p$$

where γ is a nonzero function which depends on the lengths of and the angle between X and Y. Both sides of this equation vanish because they are parallel to X_1 and X_2 respectively. Hence $(Z_N^\perp)_p$ lies in the kernel of AN_2. But $N|_\beta = N_1|_\beta$, i.e. N is parallel in the normal bundle along β, and so Z_N is perpendicular to $Z_{N_1}^\perp$ along β. Thus $Z_{N_1}^\perp$ is in the kernel of AN_2 at $p \in \mathbf{E}^2$.

It remains to show that $Z_{N_1}^\perp$ cannot be zero along a curve so that, on opposite sides of this zero curve, $Z_{N_1}^\perp$ lies in $\ker(AN_1)$ and $\ker(AN_2)$ respectively. We have already shown that $Z_{N_1}^\perp = k_1 X_1^\perp + k_2 X_2^\perp$ where

k_1, k_2 are C^∞ functions satisfying $k_1 \cdot k_2 = 0$. If k_1 and k_2 vanish on opposite sides of a curve, then their derivatives vanish on the curve. But this means that the normal curvature has a zero, contradiction. Q.E.D.

LEMMA 3. *Let g: $\mathbf{E}^2 \to \mathbf{E}^4$ be an isometric immersion with nowhere zero normal curvature. Then g factors as the composition of isometric immersions $g = g_1 \circ g_2$: $\mathbf{E}^2 \to \mathbf{E}^3 \to \mathbf{E}^4$.*

Proof. Let N_1 and N_2 be normals in the previous lemmas with $Z_{N_1}^{\perp}$ in $\ker AN_2$. Let M be the subbundle of the normal bundle $\nu(g)$ generated by N_1 with the induced Riemannian data from $\nu(g)$. It is trivial to see that the Gauss and Codazzi-Mainardi equations are satisfied by M. From the existence theorem, M is the normal bundle to an isometric immersion g_2: $\mathbf{E}^2 \to \mathbf{E}^3$.

The mapping g_1 essentially identifies $\mathbf{E}^3$ with the normal (sub)bundle M. To be more precise, first recall that a focal point is, by definition, a singularity of the identification of $\mathbf{E}^3$ with M. The nonfocal points form an open dense subset $G \subset \mathbf{E}^3$ on which this identification is, locally, an isometric diffeomorphism. Regarding M as a subset of $\mathbf{E}^4$, we obtain an isometric immersion from G to $\mathbf{E}^4$ which extends, by continuity, to all of $\mathbf{E}^3$. It is denoted by g_1. Q.E.D.

So far, our arguments have been local in nature and simple connectivity has been the only topological assumption needed to apply the existence theorem. We now state a general result.

THEOREM 2. *Let U be an open 2-dimensional flat manifold and consider an isometric immersion g: $U \to \mathbf{E}^4$ with nowhere zero normal curvature. Then there exists an open 3-dimensional flat manifold N and two isometric immersions g_1: $U \to N$ and g_2: $N \to \mathbf{E}^4$ so that $g = g_2 \circ g_1$.*

The manifold N may be chosen as a tubular neighborhood of zero section of an appropriate line subbundle of $\nu(g)$. If U (and hence N) is simply connected, then both may be regarded as open subsets of Euclidean space. Simple connectivity is not needed in Theorem 2 because the desired mappings already exist—our main effort has been to find a subbundle of $\nu(g)$ which is flat, viewed as a submanifold of $\mathbf{E}^4$.

Proof of Theorem 1. Hartman **[1]** showed that any isometric immersion f: $\mathbf{E}^n \to \mathbf{E}^{n+2}$, $n \geq 2$, (no assumption on the normal curvature) factors as a Riemannian product $f = g \times \text{id}: \mathbf{E}^2 \to \mathbf{E}^{n-2} \to \mathbf{E}^4 \times \mathbf{E}^{n-2}$. If f has nonzero normal curvature, then so does g. By Lemma 3, g factors as a composition, and hence so does f. Q.E.D.

3. Counterexamples. If the normal curvature is allowed to vanish, then $f: \mathbf{E}^n \mapsto \mathbf{E}^{n+2}$ may not factor as a composition. The idea behind the counterexample is to roll $\mathbf{E}^2$, regarded as complementary half-planes P_1 and P_2, into two distinct hyperplanes in $\mathbf{E}^4$. To insure that the two rollings fit together smoothly, we may assume that near the boundary of each half-plane, the rolling "flattens out" (like $x \to e^{-1/x^2}$ near 0). In this way, we obtain unique normals N_1 and N_2 on each open half-plane P_i. But these do not extend continuously across their common boundary for they are uniquely determined on each half-plane by rollings into different hyperplanes.

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