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ZERO SETS OF INTERPOLATING BLASCHKE PRODUCTS

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For a function h in H^∞, $Z(h)$ denotes the zero set of h in the maximal ideal space of $H^\infty + C$. It is well known that if q is an interpolating Blaschke product then $Z(q)$ is an interpolation set for H^∞. The purpose of this paper is to study the converse of the above result. Our theorem is: If a function h is in H^∞ and $Z(h)$ is an interpolation set for H^∞, then there is an interpolating Blaschke product q such that $Z(q) = Z(h)$. As applications, we will study that for a given interpolating Blaschke product q, which closed subsets of $Z(q)$ are zero sets for some functions in H^∞. We will also give a characterization of a pair of interpolating Blaschke products q_1 and q_2 such that $Z(q_1) \cup Z(q_2)$ is an interpolation set for H^∞.

Let H^∞ be the space of bounded analytic functions on the open unit disk D in the complex number plane. Identifying a function h in H^∞ with its boundary function, H^∞ becomes the (essentially) uniformly closed subalgebra of L^∞, the space of bounded measurable functions on the unit circle ∂D. A uniformly closed subalgebra B between H^∞ and L^∞ is called a Douglas algebra. We denote by $M(B)$ the maximal ideal space of B. Identifying a function h in B with its Gelfand transform, we regard h as a continuous function on $M(B)$. Sarason **[10]** proved that $H^\infty + C$ is a Douglas algebra, where C is the space of continuous functions on ∂D, and $M(H^\infty) = M(H^\infty + C) \cup D$. For a function h in H^∞, we denote by $Z(h)$ the zero set in $M(H^\infty + C)$ for h, that is,

$$Z(h) = \{x \in M(H^\infty + C);\, h(x) = 0\}.$$

For a subset E of $M(H^\infty)$, we denote by $\mathrm{cl}(E)$ the weak*-closure of E in $M(H^\infty)$. A closed subset E of $M(H^\infty)$ is called an interpolation set for H^∞ if the restriction of H^∞ on E, $H^\infty|_E$, coincides with $C(E)$, the space of continuous functions on E. For points x and y in $M(H^\infty)$, we put

$$\rho(x, y) = \sup\{|f(x)|;\, f \in H^\infty, \|f\| \le 1, f(y) = 0\}.$$

We note that if z and w are points in D, $\rho(z, w) = |z - w|/|1 - \bar{w}z|$, which is called the pseudo-hyperbolic distance on D. For a point x in $M(H^\infty)$, we put

$$P(x) = \{y \in M(H^\infty);\, \rho(x, y) < 1\},$$

which is called a Gleason part containing x. If $P(x) = \{x\}$, $P(x)$ is called trivial. For a distinct sequence $\{z_n\}_{n=1}^{\infty}$ in D satisfying $\prod_{n=1}^{\infty}(1 - |z_n|) < \infty$,

$$b(z) = \prod_{n=1}^{\infty} \left(\frac{-\bar{z}_n}{|z_n|} \frac{z - z_n}{1 - \bar{z}_n n} \right)$$

is called a Blaschke product with zeros $\{z_n\}_{n=1}^{\infty}$. A sequence $\{z_n\}_{n=1}^{\infty}$ in D is called an interpolating sequence if for every bounded sequence $\{a_n\}_{n=1}^{\infty}$ there exists a function h in H^∞ such that $h(z_n) = a_n$ for every n. By Carleson's interpolation theorem [**1**], it is characterized by $\inf_n \prod_{k:\, k \neq n} \rho(z_n, z_k) > 0$. A Blaschke product is called interpolating if its zero sequence is interpolating.

It is well known that if q is an interpolating Blaschke product, then $Z(q)$ is an interpolation set for H^∞ (see [**6**, p. 205]). Our problem in this paper is to study the converse of the above assertion.

THEOREM 1. *Let h be a function in H^∞ and let $h = \mathrm{IO}$ be an inner-outer factorization of h. If $Z(h)$ is an interpolation set for H^∞, then*

(i) O *is invertible in H^∞, and*

(ii) *there is an interpolating Blaschke product b such that $Z(b) = Z(h)$ and $\mathrm{I}\bar{b} \in H^\infty$.*

We will give some applications of our theorem. The first question is; for a given interpolating Blaschke product, q, which closed subsets of $Z(q)$ are zero sets for some functions in H^∞. We will give the complete answer in Corollary 1. In [**8**], we proved that a union set of two interpolation sets of $M(L^\infty)$ for H^∞ is also an interpolation set, but there are two interpolating Blaschke products q_1 and q_2 such that $Z(q_1) \cup Z(q_2)$ is not an interpolation set. The second question is; for which pair of interpolating Blaschke products q_1 and q_2, $Z(q_1) \cup Z(q_2)$ is an interpolation set for H^∞. The answer will be given in Corollary 4.

To prove Theorem 1, we need some lemmas.

LEMMA 1 [**6**, p. 205]. *If b is an interpolating Blaschke product with zeros $\{z_n\}_{n=1}^{\infty}$, then $Z(b) = \mathrm{cl}(\{z_n\}_{n=1}^{\infty}) \setminus \{z_n\}_{n=1}^{\infty}$ and $Z(b)$ is an interpolation set for H^∞.*

The following lemma follows from Carleson's theorem [**1**].

LEMMA 2. *Let $\{z_n\}_{n=1}^{\infty}$ and $\{w_n\}_{n=1}^{\infty}$ be disjoint interpolating sequences. Then $\{z_n, w_n;\, n = 1, 2, \ldots\}$ is an interpolating sequence if and only if $\inf_{n,m} \rho(z_n, w_m) > 0$.*

The following lemma follows from [**7**, Theorem 6.2].

LEMMA 3. *Let* $\{z_n\}_{n=1}^{\infty}$ *and* $\{w_n\}_{n=1}^{\infty}$ *be sequences in* D *and* σ *be a positive constant with* $0 < \sigma < 1$. *If* $|z_n| \to 1$ $(n \to \infty)$ *and* $\rho(z_n, w_n) < \sigma$ *for every* n, *then for each point* x *in* $\mathrm{cl}(\{w_n\}_{n=1}^{\infty}) \setminus \{w_n\}_{n=1}^{\infty}$, *there is a point* y *in* $\mathrm{cl}(\{z_n\}_{n=1}^{\infty}) \setminus \{z_n\}_{n=1}^{\infty}$ *such that* $\rho(x, y) \leq \sigma$.

Proof of Theorem 1. Since $Z(h)$ is an interpolation set for H^∞, by the open mapping theorem there is a constant σ, $0 < \sigma < 1$, such that if $f \in C(Z(h))$ there is $f_1 \in H^\infty$ with $f_1 = f$ on $Z(h)$ and $\|f_1\| < \|f\|/\sigma$. Then we have

$$\rho(x, y) > \sigma \quad \text{for every } x, y \in Z(h), x \neq y. \tag{1}$$

Consequently, there are no nontrivial Gleason part P such that $Z(h) \supset P$. By the proof of [**5**, Corollary 1], O is invertible in H^∞ and I is a finitely many product of interpolating Blaschke products b_i, $i = 1, 2, \ldots, n$. We note that the above proof depends deeply on Kerr-Lawson's lemmas in [**9**].

To prove (ii), it is sufficient to show the case $I = b_1 b_2$ and $Z(b_1) \neq Z(I)$. Let $\{z_n\}_{n=1}^{\infty}$ and $\{w_n\}_{n=1}^{\infty}$ be interpolating zero sequences of b_1 and b_2. Let $\{w_{1,n}\}_{n=1}^{\infty}$ be a subsequence of $\{w_n\}_{n=1}^{\infty}$ whose pseudo-hyperbolic distances from $\{z_n\}_{n=1}^{\infty}$ are less than σ, and put $\{w_{2,n}\}_{n=1}^{\infty} = \{w_n\}_{n=1}^{\infty} \setminus \{w_{1,n}\}_{n=1}^{\infty}$. We denote by q_1 and q_2 the interpolating Blaschke products whose zero sequences are $\{w_{1,n}\}_{n=1}^{\infty}$ and $\{w_{2,n}\}_{n=1}^{\infty}$ respectively. By Lemma 2, $b_1 q_2$ is an interpolating Blaschke product. By Lemma 1 and 3, for each point x in $Z(q_1)$, there is a point y in $Z(b_1)$ such that $\rho(x, y) \leq \sigma$. Since $Z(q_1) \cup Z(b_1) \subset Z(h)$, by (1) we have $Z(q_1) \subset Z(b_1)$. Then we obtain

$$Z(h) = Z(I) = Z(b_1) \cup Z(q_1) \cup Z(q_2) = Z(b_1 q_2).$$

Thus $b = b_1 q_2$ satisfies (ii).

Let q be a non-continuous interpolating Blaschke product. By Theorem 1, if $h \in H^\infty$ satisfies $Z(h) \subset Z(q)$, then there is an interpolating Blaschke product b with $Z(b) = Z(h)$ and $h\bar{b} \in H^\infty$. It only shows that the zero sequence of b can be found in the zero sequence of h. But the following corollary shows that there is an interpolating Blaschke product b_1 such that $Z(b_1) = Z(h)$ and $q\bar{b}_1 \in H^\infty$. This fact means that the zero sequence of b_1 can be found in the zero sequence of q.

COROLLARY 1. *Let* q *be an interpolating Blaschke product and let* E *be a closed subset of* $Z(q)$. *Then the following assertions are equivalent.*

(i) E *is an open-closed subset of* $Z(q)$.

(ii) *There is an interpolating Blaschke product* b *with* $E = Z(b)$ *and* $q\bar{b} \in H^\infty$.

(iii) *There is a function* h *in* H^∞ *with* $E = Z(h)$.

Proof. Let $\{z_n\}_{n=1}^\infty$ be an interpolating zero sequence of q.

(i) $\Rightarrow$ (ii) Suppose that E is an open-closed subset of $Z(q)$. Then there are disjoint open subsets U and V of $M(H^\infty)$ such that $U \supset E$ and $V \supset Z(q) \setminus E$. We may assume that $\{z_n\}_{n=1}^\infty \subset U \cup V$. Let b be an interpolating Blaschke product with zeros $U \cap \{z_n\}_{n=1}^\infty$. Then $q\bar{b} \in H^\infty$. By Lemma 1, we get $Z(b) \subset U \cap Z(q) = E$ and $Z(q\bar{b}) \subset V$. Thus we obtain

$$E = E \cap Z(q) = E \cap \left(Z(b) \cup Z(q\bar{b})\right) = E \cap Z(b) = Z(b).$$

(ii) $\Rightarrow$ (iii) is trivial.

(iii) $\Rightarrow$ (i) By Lemma 1, $Z(h)$ is an interpolation set for H^∞. By Theorem 1, we may assume that h is an interpolating Blaschke product and $Z(h) \subsetneqq Z(q)$. We note that $Z(h) \neq Z(hq) = Z(q)$. By the proof of Theorem 1 (we put $b_1 = h$ and $b_2 = q$), there are interpolating Blaschke products q_1 and q_2 such that $q = q_1 q_2$, hq_2 is an interpolating Blaschke product and $Z(hq) = Z(hq_2)$. Since $Z(h) \cap Z(q_2) = \varnothing$ and $Z(h) \cup (q_2) = Z(hq) = Z(q)$, $Z(h)$ is an open-closed subset of $Z(q)$.

COROLLARY 2. *Let* q *be an interpolating Blaschke product. Then there exists* $h \in H^\infty$ *such that* $Z(q) \cap Z(h) \neq Z(g)$ *for every* $g \in H^\infty$.

Proof. By Corollary 1, it is sufficient to show the existence of h in H^∞ such that $Z(q) \cap Z(h)$ is not open in $Z(q)$. Let $\{z_n\}_{n=1}^\infty$ be the zero sequence of q. Let $\{E_n\}_{n=1}^\infty$ be a sequence of subsets of $\{z_n\}_{n=1}^\infty$ such that

$$E_n \text{ is an infinite subset,} \tag{2}$$

$$E_n \cap E_m = \varnothing \quad \text{if} \quad n \neq m, \quad \text{and} \tag{3}$$

$$\bigcup_{n=1}^\infty E_n = \{z_n\}_{n=1}^\infty. \tag{4}$$

Then there exists a function h in H^∞ such that

$$h = 1/n \text{ on } E_n \quad \text{for every } n = 1, 2, \ldots.$$

We obtain $Z(q) \cap Z(h) \neq \varnothing$. By (2), there exists $x_n \in Z(q)$ such that $h(x_n) = 1/n$. Thus $Z(q) \cap Z(h)$ is not an open subset of $Z(q)$.

The following corollary shows that the assertion of Corollary 2 is also true if $Z(h)$ is replaced by $M(B)$ for some Douglas algebra B.

COROLLARY 3. *Let q be an interpolating Blaschke product. Then there is a Douglas algebra B such that $Z(q) \cap M(B) \neq Z(g)$ for every $g \in H^\infty$.*

Proof. For a subset J of L^∞, we denote by $[J]$ the uniformly closed subalgebra generated by J. By [8, Proposition 6.3], there exists a maximal Douglas algebra B contained in $[H^\infty, \bar{q}]$ properly. Then we have $\bar{q} \notin B$. So we get $Z(q) \cap M(B) \neq \varnothing$. We shall show that B satisfies our assertion. To show this, suppose not. By Corollary 1, there exists an interpolating Blaschke product b such that

$$q\bar{b} \in H^\infty \quad \text{and} \quad Z(b) = Z(q) \cap M(B). \tag{5}$$

Then we have $\bar{b} \in [H^\infty, \bar{q}]$. By [3, Theorem 1], there is an interpolating Blaschke product ψ such that

$$b\bar{\psi} \in H^\infty \quad \text{and} \quad H^\infty + C \subsetneqq \left[H^\infty, \bar{\psi}\right] \subsetneqq \left[H^\infty, \bar{b}\right] \subset \left[H^\infty, \bar{q}\right]. \tag{6}$$

This implies that there exists x_0 in $M(H^\infty + C)$ such that

$$|\psi(x_0)| = 1 \quad \text{and} \quad b(x_0) = 0. \tag{7}$$

By (5), $q(x_0) = 0$ and $x_0 \notin M([H^\infty, \bar{q}])$. By (5) and (7), we have $x_0 \in M(B)$ and $x_0 \in M([B, \bar{\psi}])$, consequently $[H^\infty, \bar{q}] \neq [B, \bar{\psi}]$. By (6), we get $\bar{\psi} \in [H^\infty, \bar{q}]$. Since B is maximal in $[H^\infty, \bar{q}]$, we get $\bar{\psi} \in B$. But by (5) and (6), we have

$$\varnothing \neq Z(\psi) \subset Z(b) \subset M(B),$$

so we obtain $\bar{\psi} \notin B$. This is a contradiction.

COROLLARY 4. *Let q_1 and q_2 be interpolating Blaschke products. Then the following conditions are equivalent.*

(i) *$Z(q_1) \cup Z(q_2)$ is an interpolation set for H^∞.*

(ii) *$Z(q_1) \cap Z(q_2)$ is an open-closed subset of $Z(q_1)$.*

(iii) *There exists an interpolating Blaschke product q_3 such that $Z(q_3) = Z(q_1) \cap Z(q_2)$.*

Proof. (i) $\Rightarrow$ (ii) We put $q = q_1 q_2$. By (i), $Z(q) = Z(q_1) \cup Z(q_2)$ is an interpolation set for H^∞. By Theorem 1, we may assume that q is an interpolating Blaschke product. By Corollary 1, $Z(q_2)$ is an open-closed subset of $Z(q_1) \cup Z(q_2)$. Then $Z(q_1) \cap Z(q_2)$ is an open-closed subset of $Z(q_1)$.

(ii) $\Rightarrow$ (iii) follows from Corollary 1.

(iii) $\Rightarrow$ (i) By Corollary 1, (iii) implies that $Z(q_1) \cap Z(q_2)$ and $Z(q_1) \setminus Z(q_2)$ are open-closed subsets of $Z(q_1)$, and $Z(q_2) \setminus Z(q_1)$ is an open-closed subset of $Z(q_2)$. Again by Corollary 1, there are interpolating

Blaschke products b_1, b_2 and b_3 such that $Z(b_1) = Z(q_1) \cap Z(q_2)$, $Z(b_2) = Z(q_1) \setminus Z(q_2)$ and $Z(b_3) = Z(q_2) \setminus Z(q_1)$. By Lemmas 1, 2 and 3, we may assume that $b_1 b_2 b_3$ is an interpolating Blaschke product. Consequently, $Z(q_1) \cup Z(q_2) = Z(b_1 b_2 b_3)$ is an interpolation set for H^∞.

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