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FREE PRODUCTS OF TOPOLOGICAL GROUPS WITH AMALGAMATION. II

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The fundamental problem is to determine if the free product with amalgamation of Hausdorff topological groups exists and is Hausdorff. This is known to be true if the subgroup being amalgamated is central or if all groups concerned are k_ω and the amalgamation subgroup is compact. In this paper a general result is proved which allows one to move outside the class of compact or central amalgamations. Using this result it follows, for example, that the amalgamated free product $F *_A G$ exists and is Hausdorff if F, G and A are k_ω-groups and A is the product of a central subgroup and a compact subgroup.

1. Introduction. The fundamental problem in this subject is to prove that the free product with amalgamation of any Hausdorff topological group exists, is Hausdorff and its underlying group structure is the amalgamated free product of the underlying groups. There have been three contributions to this problem. The first was by Ordman [9] who settled the problem for some locally invariant Hausdorff groups. The case when the amalgamated subgroup is central was settled in Khan and Morris [2]. In Katz and Morris [1] the first step was made towards handling the important class of k_ω-groups. There the case where the groups are k_ω-spaces and the amalgamated subgroup is compact is dealt with. In this paper we show that the condition that A be compact can be weakened.

We denote by $F *_C G$ the free product of the topological groups F and G with the common subgroup C amalgamated. Given k_ω-groups F and G and a common subgroup C we define the notion of the triple $(F, G;\ C)$ being beseder. If $(F, G;\ C)$ is beseder then it is readily seen that $F *_C G$ exists, is Hausdorff and has the appropriate algebraic structure. The main theorem says that $(F, G;\ C)$ is beseder if $C = A \cdot B$ where $(F, G;\ A)$ and $(F, G;\ B)$ are beseder and A is compact. So this theorem provides a procedure for progressively enlarging the family of known beseder triples.

In Katz and Morris [1] it is proved that if A is a compact subgroup of the k_ω-groups F and G then $(F, G;\ A)$ is beseder. Using results of Khan and Morris [2, 3] we prove here that if B is a closed central subgroup of the k_ω-groups F and G then the triple $(F, G;\ B)$ is beseder. Thus we can deduce from the main theorem that if the k_ω-groups F and G have a compact subgroup A and a central subgroup B then $(F, G; AB)$ is beseder.

Indeed in the above example if C is a compact subgroup of F and G such that ABC is also a subgroup of F and G then $(F, G; ABC)$ is beseder. (This result cannot be deduced in one step from the theorem as AC is not necessarily a group.) This procedure can be used repeatedly.

2. Definitions and notation. The standard references for amalgamated free products ofr groups are B. H. Neumann [6] and Magnus, Karrass and Solitar [4].

DEFINITION. Let C be a common subgroup of topological groups F and G. The topological group $F *_C G$ is said to be *the free product of the topological groups F and G with amalgamated subgroup C* if

(i) F and G are topological subgroups of $F *_C G$

(ii) $F \cup G$ generates $F *_C G$ algebraically, and

(iii) every pair ϕ_1, ϕ_2 of continuous homomorphisms of F and G, respectively, into any topological group D which agree on C extend to a continuous homomorphism of $F *_C G$ into D.

Throughout the paper we will be dealing with k_ω-spaces. Our definition of k_ω-space includes Hausdorffness.

Observe that if F and G are k_ω-groups and C is a closed common subgroup of F and G, then the k_ω-decompositions $F = \bigcup F_n$ and $G = \bigcup G_n$ can be chosen such that

(i) $F_n = F_n^{-1}$ and $G_n = G_n^{-1}$,

(ii) $F_nF_m \subseteq F_{n+m}$ and $G_nG_m \subseteq G_{n+m}$, and

(iii) $C \cap F_n \subseteq G_{n+1}$ and $C \cap G_n \subseteq F_{n+1}$.

(In verifying that requirements (ii) and (iii) are always possible recall that if $F = \bigcup F_n$ is a k_ω-space then any compact subset of F lies entirely in some F_n.)

Let K_C be the kernel of the canonical homomorphism of the free product $F * G \to F *_C G$. Then K_C = the normal subgroup generated by $\{f(c)g(c)^{-1}\colon c \in C\}$, where f and g are the embedding maps of C in F and G respectively. We let $X = \bigcup_{n=1}^{\infty} X_n$ where

$$X_n = \Big\{uvu^{-1}\colon u \in (F_n \cup G_n)^n, v = f(c)g(c)^{-1} \text{ or } v = g(c)f(c)^{-1} \text{ and } f(c) \in F_n \text{ and } g(c) \in G_n\Big\}$$

and $Y_n = (X_n)^n$; that is, the set of all words which are a product of at most n elements of X_n.

Each X_n and Y_n is compact and $K_C = \bigcup_{n=1}^{\infty} Y_n$.

3. The Main Theorem.

DEFINITION. Let C be a common closed subgroup of k_ω-groups F and G. The triple $(F, G;\ C)$ is said to be *beseder*[1] if for each positive integer n there exists an integer m such that

$$K_C \cap (F_n \cup G_n)^n \subseteq Y_m$$

where $(F_n \cup G_n)^n \subseteq F * G$, and K_C and Y_m are as previously defined.

REMARK. If the triple $(F, G;\ C)$ is beseder then it is easily checked that the amalgamated free product $F *_C G$ exists and is Hausdorff.

EXAMPLE. It is shown in Katz and Morris [**1**] that if F and G are k_ω-groups and C is a common compact subgroup then the triple $(F, G;\ C)$ is beseder.

THEOREM 1. *Let Fand G be k_ω-groups with a common closed subgroup $C = A \cdot B$, where A and B are subgroups of C. If A is compact and B is such that the triple $(F, G;\ B)$ is beseder then $(F, G;\ C)$ is beseder.*

Proof. Let f, f_1 and f_2 be the embeddings of C, A and B, respectively, into F. Let g, g_1 and g_2 be the embeddings of C, A and B, respectively into G. Let K_C, K_A and K_B be the kernels of the canonical maps $F * G \to F *_C G$, $F * G \to F *_A G$, and $F * G \to F *_B G$, respectively.

Since $f(A)$ is compact and so is in F_n, for some n, we can assume that $f(A) \subseteq F_1$ and, similarly, $g(A) \subseteq G_1$.

Let $w \in (F_n \cup G_n)^n \cap K_C$. Then, by Proposition 2 of [**1**], w has a representation

$$\begin{aligned} w &= \left(t_{1,1} \cdots t_{1,q_1} f(c_1) g(c_1)^{-1} t_{1,q_1}^{-1} \cdots t_{1,1}^{-1}\right) \\ &\quad \cdots \left(t_{s,1} \cdots t_{s,q_s} f(c_s) g(c_s)^{-1} t_{s,q_s}^{-1} \cdots t_{s,1}^{-1}\right) \\ &= B_0 \cdot B_1 \cdots B_s \end{aligned}$$

where

$$B_0 = t_{1,1} \cdots t_{1,q_1} f(c_1)$$

$$B_i = g(c_i)^{-1} t_{i,q_i}^{-1} \cdots t_{i,1}^{-1} t_{i+1,1} \cdots t_{i+1,q_{i+1}} f(c_{i+1}), \qquad 1 \le i < s,$$

and

$$B_s = g(c_s)^{-1} t_{s,q_s}^{-1} \cdots t_{s,1}^{-1}.$$

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[1] A Hebrew word meaning "okay".

It is shown in [1] that the representation (*) can be chosen such that the reduced form of w is the juxtaposition of the reduced forms of the blocks B_i.

Let $c_i = b_i a_i$ where $a_i \in A$ and $b_i \in B$. Put

$$w' = \left(t_{1,1} \cdots t_{1,q_1} f(b_1) g(b_1)^{-1} t_{1,q_1}^{-1} \cdots t_{1,1}^{-1}\right) \cdots \left(t_{s,1} \cdots t_{s,q_s} f(b_s) g(b_s)^{-1} t_{s,q_s}^{-1} \cdots t_{s,1}^{-1}\right).$$

Clearly w and w' have the same image in $F *_A G$ which is topologically isomorphic to $(F * G)/K_A$. Therefore $w = w' \cdot d$, where $d \in K_A$.

The reduced form of w is just the juxtaposition of the reduced forms of the blocks $B_0, \ldots, B_s$. Consider one block, B_i, say. As in the proof of Proposition 4 of [1], each element in the reduced form of w and hence of B_i lies in $F_{n^2} \cup G_{n^2}$. Thus $B_i \in (F_{n^2} \cup G_{n^2})^n$. Now

$$\begin{aligned} B_i &= g(b_i a_i)^{-1} t_{i,q_i}^{-1} \cdots t_{i+1,q_{i+1}} f(b_{i+1} a_{i+1}), \qquad i \neq 0, s, \\ &= g(a_i)^{-1} g(b_i)^{-1} t_{i,q_i}^{-1} \cdots t_{i+1,q_{i+1}} f(b_{i+1}) f(a_{i+1}) \\ &= g(a_i)^{-1} B_i' f(a_{i+1}) \end{aligned}$$

where B_i' is defined formally as the ith block of w', in a similar fashion to the definition of B_i. Also $B_0 = B_0' f(a_i)$ and $B_s = g(a_s)^{-1} B_s'$.

As $g(a_i)^{-1}$ and $f(a_{i+1})$ are in $F_1 \cup G_1$ we see that

$$B_i' \in (F_{n^2+1} \cup G_{n^2+1})^n.$$

Observe that $w' = B_0' \cdots B_s'$ and from [1] $s \leq n$ so that

$$w' \in (F_{n^2+1} \cup G_{n^2+1})^{n^2}.$$

As $w \in (F_n \cup G_n)^n$ and $w = w' \cdot d$ we have that

$$d \in (F_{n^2+1} \cup G_{n^2+1})^{n^2+n} \subseteq (F_{n^2+n} \cup G_{n^2n+n})^{n^2+n}.$$

By [1] the triple $(F, G; A)$ is beseder, and by assumption the triple $(F, G; B)$ is beseder. So

$$w' \in (F_{n^2+1} \cup G_{n^2+1})^{n^2+1} \cap K_B \quad \text{and} \quad d \in (F_{n^2+n} \cup G_{n^2+n})^{n^2+n} \cap K_A$$

implies that

$w' \in Y_{k_1}^B$ and $d \in Y_{k_2}^A$, for some positive integers k_1 and k_2, where $Y_{k_1}^B$ (and $Y_{k_2}^A$) is defined in a similar fashion to Y_k.

But $Y_{k_1}^A \subseteq Y_{k_1}^C$ and $Y_{k_2}^B \subseteq Y_{k_2}^C$. Therefore

$$w = w' \cdot d \subseteq Y_{k_1}^C \cdot Y_{k_2}^C \subseteq Y_{k_1+k_2}^C.$$

So $(F, G; C)$ is a beseder triple, as required.

REMARK. Theorem 1 allows us to prove the Hausdorffness of some free products with non-compact amalgamation. Obviously if A and B are compact, then C is compact and Theorem 1 does not give anything new. Thus we need another class of beseder triples $(F, G;\ B)$ where B is not compact. One such class is produced in §4.

4. Central amalgamations. In [2] Khan and Morris proved that if F and G are any Hausdorff groups and A is a common closed central subgroup, then $F *_A G$ is Hausdorff. We use the proof of this theorem to show that if F and G are k_ω-groups, then the triple $(F, G;\ B)$ is beseder.

THEOREM 2. *Let F and G be k_ω-groups with a common closed central subgroup B. Then the triple $(F, G;\ B)$ is beseder.*

Proof. It suffices to show that $K \cap (F_n \cup G_n)^n \subseteq Y_m$, for some m, where K is the kernel of the canonical map $F * G \to F *_B G$ and Y_m is what we previously called Y_m^B.

By Morris, Ordman and Thompson [5] for each fixed n there exists a positive integer $r \geq n$ such that

$$(F_n \cup G_n)^n \subseteq F_r \cdot G_r \cdot \mathrm{gp}_r[F_r, G_r]$$

where $[F_r, G_r] = \{f^{-1}g^{-1}fg\colon f \in F_r,\ g \in G_r\}$ and $\mathrm{gp}_r[F_r, G_r]$ denotes the set of elements which are products of at most r elements of $[F_r, G_r]$ and their inverses.

By Proposition 1 of [3] we observe that an element

$$\omega = f \cdot g \cdot [f_1, g_1]^{\varepsilon_1}[f_2, g_2]^{\varepsilon_2} \cdots [f_s, g_s]^{\varepsilon_s},$$

where each $\varepsilon_i = 1$ or -1, is in K if and only if for some $b \in B$, $f = f(b)$, $g = g(b)^{-1}$ and

$$\omega' = [f_1, g_1]^{\varepsilon_1}[f_2, g_2]^{\varepsilon_2} \cdots [f_s, g_s]^{\varepsilon_s} \in \mathrm{ngp}\{[F, g(B)] \cup [f(B), G]\},$$

the normal subgroup generated by $[F, g(B)] \cup [f(B), G]$. For convenience write $[f_1, g_1]^{\varepsilon_1} \cdots [f_s, g_s]^{\varepsilon_s}$ as

$$x_1 x_2 \cdots x_{k_1} y_1 x_{k_1+1} x_{k_1+2} \cdots x_{k_2} y_2 x_{k_2+1} \cdots x_{k_l},$$

where each x_i and y_i is $[f_j, g_j]^{\varepsilon_j}$, for some j, and we write x_i if $f_j \notin f(B)$ and $g_j \notin g(B)$ and y_i, otherwise. (So any k_i may be zero).

Now

$$\omega' = \left(x_1 \cdots x_{k_1} y_1 x_{k_1}^{-1} \cdots x_1^{-1}\right)\left(x_1 \cdots x_{k_1} x_{k_1+1} \cdots x_{k_2} y_2 x_{k_2}^{-1} \cdots x_1^{-1}\right)$$
$$\cdots\left(x_1 \cdots x_{k_1} x_{k_1+1} \cdots x_{k_2} x_{k_2+1} \cdots x_{k_{l-1}} y_{l-1} x_{k_{l-1}}^{-1} \cdots x_1^{-1}\right)$$
$$\cdot\left(x_1 \cdots x_{k_1} x_{k_1+1} \cdots x_{k_2} x_{k_2+1} \cdots x_{k_{l-1}} x_{k_{l-1}+1} \cdots x_{k_l}\right).$$

The image of each y_i is e in $F *_B G$, since B is central in F and G, so the image in $F *_B G$ of each of all but the last bracket is e. As the image of the whole word is e, the image of the last bracket is also e. But Proposition 1 of [2] shows this is possible if and only if the last bracket equals e.

Observe that each y_i is of the form $f(b_i)^{-1}c^{-1}f(b_i)c$ or $d^{-1}g(b_i)^{-1}dg(b)$ (or the inverse of one of these), where $b_i \in B$, $c \in G$ and $d \in F$. Now

$$\begin{aligned} y_i &= f(b_i)^{-1}c^{-1}f(b_i)c = \left(f(b_i)^{-1}g(b_i)\right)\left(c^{-1}g(b_i)^{-1}f(b_i)c\right) \\ &= \left(f(b_i^{-1})g(b_i^{-1})^{-1}\right)\left(c^{-1}g(b_i^{-1})f(b_i^{-1})^{-1}c\right) = z_i \cdot z_i', \quad \text{say.} \end{aligned}$$

So

$$\begin{aligned} \omega = {} & \left(f(b)g(b)^{-1}\right)\cdot\left(x_1 \cdots x_{k_1} z_1 x_{k_1}^{-1} \cdots x_1^{-1}\right)\cdot\left(x_1 \cdots x_{k_1} z_1' x_{k_1}^{-1} \cdots x_1^{-1}\right) \\ & \cdot\left(x_1 \cdots x_{k_1} x_{k_1+1} \cdots x_{k_2} z_2 x_{k_2}^{-1} \cdots x_1^{-1}\right) \\ & \cdot\left(x_1 \cdots x_{k_1} x_{k_1+1} \cdots x_{k_2} z_2' x_{k_2}^{-1} \cdots x_1^{-1}\right) \cdots \\ & \cdot\left(x_1 \cdots x_{k_1} x_{k_1+1} \cdots x_{k_2} \cdots x_{k_{l-1}} z_{l-1} x_{k_{l-1}}^{-1} \cdots x_1^{-1}\right) \\ & \cdot\left(x_1 \cdots x_{k_1} \cdots x_{k_{l-1}} z_{l-1}' x_{k_{l-1}}^{-1} \cdots x_1^{-1}\right). \end{aligned}$$

Recalling that $f(b_i) \in F_n$ implies $g(b_i) \in G_{n+1}$ we see that each bracketed term is an element of X_{5r}. (See §2.) [Recall that each x_i is an $[f_j, g_j]$ where f_j and $g_j \in F_r \cup G_r$.] As there are at most $2r + 1$ bracketed terms, ω clearly lies in Y_{5r}. Thus

$$K \cap F_r \cdot G_r \cdot \operatorname{gp}_r[F_r, G_r] \subseteq Y_{5r}.$$

So

$$K \cap (F_n \cup G_n)^n \subseteq Y_{5r}.$$

This completes the proof.

5. Conclusion and problems. We now apply Theorems 1 and 2 to some specific examples. In so doing we reveal the Hausdorffness of some amalgamated free products.

COROLLARY. *Let F and G be k_ω-groups with a common subgroup C such that $C = AB$, where A and B are subgroups of C. If A is compact and B is closed and central in F and G, then $F *_C G$ is a k_ω-group.*

REMARK. The above Corollary includes the important special case that C is the direct product of A and B.

In this paper we have applied Theorem 1 in the case that B is a central subgroup of F and G, because we proved that this yields a beseder triple $(F, G;\ B)$. So we would like to know precisely what conditions, other than centrality or compactness of B, give rise to a beseder triple.

Problem 1. (a) Under what conditions on F, G and B is $(F, G;\ B)$ a beseder triple?

(b) In particular, if $F *_B G$ is a k_ω-group (and has the appropriate algebraic structure) is $(F, G;\ B)$ necessarily a beseder triple?

In Theorem 1, in order to show that $(F, G;\ C)$ is a beseder triple we required not only that $(F, G;\ A)$ and $(F, G;\ B)$ are beseder triples but also that A be compact.

Problem 2. (a) Let A and B be common closed subgroups of k_ω-groups F and G such that $C = AB$ is a closed subgroup of F and G. If $(F, G;\ A)$ and $(F, G;\ B)$ are beseder triples, is $(F, G;\ C)$ necessarily a beseder triple?

(b) If the answer to (a) is in the negative, is it true under some condition on A weaker than compactness?

Another way to move outside the compactness restriction is to put restrictions on F and G. In particular, it is interesting to consider the case when F and G are free topological groups. It is readily proved that $F(X) *_A F(Y)$ is Hausdorff if $F(X)$ and $F(Y)$ are free topological groups on completely regular Hausdorff spaces X and Y and A is the free topological group on $Z = X \cap Y$. This is clear since $F(X) *_A F(Y)$ is the free topological group $F(X \cup_A Y)$; where $X \cup_A Y$ is an adjunction space. [In general the subgroups of $F(X)$ and $F(Y)$ generated by Z do not have the same topology. They do, for example, when both are the free topological group on Z. Two cases when this occurs are when Z is compact or when X and Y are k_ω-spaces and $X \cap Y$ is closed in X and Y.] However we do not know if $F(X) *_A F(Y)$ is Hausdorff when we do not alssume that $Z = X \cap Y$ but only that the group generated by Z is a closed subgroup of $F(X)$ and $F(Y)$.

Problem 3. (a) If $F(X)$ and $F(Y)$ are Hausdorff free topological groups with a common closed free topological subgroup $F(Z)$, is $F(X) *_{F(Z)} F(Y)$ necessarily Hausdorff?

(b) Is (a) true under the additional assumption that X and Y are k_ω-spaces?

(c) Is (b) true under the additional assumption that Z is compact?

Nickolas [7] observed that the map $(x, y) \to xyx$ from $[0,1] \times [0,1] \to F([0,1])$ extends to an embedding α of $F([0,1] \times [0,1])$ into $F([0,1])$. Nickolas [**8**] also proved that there is an embedding β of $F((0,1))$ into $F([0,1])$.

Problem 4. (a) Is $F([0,1]) *_{F([0,1]\times[0,1])} F([0,1])$ Hausdorff?

(b) Is $F([0,1]) *_{F((0,1))} F([0,1])$ Hausdorff where the embedding is given by β?

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