PACIFIC JOURNAL OF MATHEMATICS
Vol. 120, No. 1, 1985

CONTINUITY OF HOMOMORPHISMS OF BANACH G-MODULES

B. E. JOHNSON

We consider whether, given a locally compact abelian group G and two Banach G-modules X and Y, every G-module homomorphism from X into Y is continuous. Discontinuous homomorphisms can exist only when Y has submodules on which G acts by scalar multiplication. They are also associated with discontinuous convariant forms on X so if either of these are absent them all G-module homomorphisms are continuous.

1. Introduction.

Throughout this paper G is a locally compact abelian group.

DEFINITION 1.1. A *Banach G-module* is a Banach space X with a map $(g, x) \mapsto gx$ of $G \times X$ into X such that

(i) $x \mapsto gx$ is linear on $X\,(g \in G)$.
(ii) $g(hx) = (gh)x \quad (g, h \in G, x \in X)$.
(iii) $ex = x \quad (x \in X$, e is the identity element of $G)$.
(iv) There is a $K \in \mathbf{R}$ with

$$\|gx\| \leq K\|x\| \qquad (x \in X, g \in G).$$

Note that we do not require any continuity of the map $(g, x) \mapsto gx$ in g—in fact in most of the paper we will be treating G as a discrete group.

A *G-submodule* of X is a closed linear subspace X_0 of X with $gx \in X_0$ $(g \in G, x \in X_0)$. The G-module X is *scalar* if for each $g \in G$ there is $\lambda(g) \in \mathbf{C}$ with $gx = \lambda(g)x$ $(g \in G, x \in X)$. If $X \neq \{0\}$ then $\lambda(e) = 1$, $\lambda(gh) = \lambda(g)\lambda(h)$ and $|\lambda(g)| \leq K$. Applying this last inequality to g^n $(n \in \mathbf{Z})$ we see $|\lambda(g)| = 1$ so λ is a character and mild continuity hypotheses on $g \mapsto gx$ would imply that λ is continuous.

DEFINITION 1.2. Let X, Y be Banach G-modules. Then S: $X \to Y$ is a *G-module homomorphism* if it is linear and $S(gx) = gS(x)\,(g \in G, x \in X)$.

If Y is a scalar module then $S(gx) = \lambda(g)\,S(x)$ and we say that S is λ-covariant. In the special case when $\lambda \equiv 1$ is the trivial character we say S is invariant. When $Y = \mathbf{C}$ we call S a *form*.

Invariant and covariant forms are related in many cases because if S is a λ covariant form on X and T: $X \to X$ is a linear map with

$T(gx) = \lambda(g)^{-1} gT(x)$ then ST is an invariant form because $ST(gx) = S(\lambda(g)^{-1} gT(x)) = gST(x)$. When X is a G-module of functions on which G acts by translation such a T is given by $(T\alpha)(h) = \lambda(h)^{-1}\alpha(h)$.

The main result of this paper [Theorem 4.1] is that if S is a G-module homomorphism of X into Y then the separating set of S is the direct sum of finite number of scalar G-submodules of Y. This is proved by methods similar to **[1]** involving identifying certain intersections of ranges $(\Sigma a_i g_i)Y$ where the $a_i \in \mathbf{C}$. Our method for doing this depends on doing it first of all for $Y = l^\infty(G)$ and to achieve it there we need some results on difference operators which are given in §2.

2. Two Lemmas. Throughout this section $\mathbf{R}^n$ is partially ordered by the product order (except that $x < y$ means $x_i < y_i$ for all i) and $a, b \in \mathbf{R}^n$ with $a < b$. For $x \in \mathbf{R}^n$ we put $|x| = \max|x_i|$. The standard basis vectors of $\mathbf{R}^n$ are denoted by $e_1, \ldots, e_n$. If $x, h \in \mathbf{R}^n$, $a < x - h < x < x + h < b$ and g is a complex valued function on (a, b) we define

$$\Delta_i g = \Delta_i(x, h)g = \left[g(x + h_i e_i) - 2g(x)\cosh h_i + g(x - h_i e_i)\right] h_i^{-2}.$$

Abusing notation $\Delta_i g$ is a function of x and we have $\Delta_i \Delta_j g = \Delta_j \Delta_i g$.

Let $\Delta = \Delta_1 \Delta_2 \cdots \Delta_n$. Lemma 2.2 is an extension of Schwarz' Theorem to functions of several variables with the operator D^2 replaced by $D^2 - I$; Lemma 2.1 is a preparatory result.

LEMMA 2.1. *Let u, $v \in (a, b)$ with $u < v$. Suppose g is continuous $(a, b) \to \mathbf{C}$ and $g(x) = 0$ whenever $x_i = u_i$ or v_i for some i. Suppose also that $\Delta g = 0$ whenever $a < x - h < x < x + h < b$. Then g is zero throughout (a, b).*

Proof. We prove the results by induction on n. When $n = 1$ we see that if we have any three points in (a, b) in arithmetic progression and g is zero at two of them it is zero at the third. Hence g is zero at all points in (a, b) of the form $(1 - \lambda)u + \lambda v$ where $\lambda = 2^{-s}t$ $(s, t \in \mathbf{Z})$. By continuity g is zero throughout (a, b).

Suppose the result holds whenever $n = k$ and g satisfies the hypotheses for $n = k + 1$. Let $c \in (a_{k+1}, b_{k+1})$ and let $y, h \in \mathbf{R}^k$. For all y_{k+1}, h_{k+1} with $a_{k+1} < y_{k+1} - h_{k+1} < y_{k+1} < y_{k+1} + h_{k+1} < b_{k+-1}$ we have

$$\Delta(y_{k+1}, h_{k+1})G = 0$$

where

$$G(y_{k+1}) = \Delta_1(y, h)\Delta_2(y, h) \cdots \Delta_k(y, h)g(y_1, \ldots, y_k, y_{k+1}).$$

Also $G(u_{k+1}) = 0 = G(v_{k+1})$. Thus by the result for $n = 1$, $G(c) = 0$. We now apply the inductive hypothesis to the function $g(x_1, x_2, \ldots, x_n, c)$ and the result follows.

LEMMA 2.2. *Let f be continuous $(a, b) \to \mathbf{C}$ and suppose that for each $x \in (a, b)$ there are complex valued functions α_i, β_i $(i = 1, \ldots, n)$ defined in a neighbourhood of 0 in $\mathbf{R}^n$ where α_i and β_i are constant with respect to the ith variable, such that*

$$f(x+h) = \sum_{i=1}^{n} \alpha_i(x+h)\cosh h_i + \beta_i(x+h)\sinh h_i + o\left(|h|^{2n}\right)$$

as $h \to 0$. Then there are complex valued functions A_i, B_i $(i = 1, \ldots, n)$ on (a, b) where A_i and B_i are constant with respect to the ith variable such that for all $x \in (a, b)$

$$f(x) = \sum_{i=1}^{n} A_i(x)\cosh x_i + B_i(x)\sinh x_i.$$

Proof. First of all we show that if $a < x - h < x < x + h < b$ then $\Delta f = 0$. We have

$$\Delta_i(x, h)f = 4^{-1}\left(\Delta_i\left(x + \frac{1}{2}h_i e_i, \frac{1}{2}h\right)f \right.$$
$$\left. + 2\cosh\frac{1}{2}h_i \Delta_i(x, h)f + \Delta_i\left(x - \frac{1}{2}h_i e_i, \frac{1}{2}h\right)f\right).$$

Applying this to each of the factors in $\Delta = \Delta_1\Delta_2 \cdots \Delta_n$ we express $\Delta(x, h)$ as the mean of 4^n terms of the form $C\Delta(y, \frac{1}{2}h)$ where C is the product of some of the terms $\cosh\frac{1}{2}h_i$. If we denote $|\Delta(x, h)f|$ by K then for one of these y's, $y^{(1)}$ say

$$\left|\Delta\left(y^{(1)}, \frac{1}{2}h\right)\right| \geq KC\left(\frac{1}{2}h\right)^{-1}$$

where $x - h \leq y^{(1)} - \frac{1}{2}h \leq y^{(1)} + \frac{1}{2}h \leq x + h$ and $\cosh\frac{1}{2}h_1 \cdots \cosh\frac{1}{2}h_n = C(\frac{1}{2}h)$. Repeating the process we obtain a sequence $y^{(m)}$ with

$$\left|\Delta\left(y^{(m)}, 2^{-m}h\right)f\right| \geq KC_m^{-1}$$

where

$$1 \leq C_m = C\left(\frac{1}{2}h\right)C\left(\frac{1}{4}h\right) \cdots C(2^{-m}h) < C_\infty < \infty$$

where C_∞ is the infinite product $\Pi C(2^{-j}h)$. Moreover $y^{(m)} - 2^{-m}h \leq y^{(m+1)} \leq y^{(m)} + 2^{-m}h$ so the sequence $\{y^{(m)}\}$ converges to $z \in (a, b)$.

Writing

$$f(z+k) = \sum \alpha_i(z+k)\cosh k_i + \beta_i(z+k)\sinh k_i + \tilde{f}(z+k)$$

as in the hypotheses of the theorem where $\tilde{f}(z) = 0$ and $\tilde{f}(z+k)|k|^{-2n} \to 0$ as $|k| \to 0$ and using

$$\Delta_i(y^{(m)}, 2^{-m}h)\left[\alpha_i(y^{(m)})\cosh(y_i^{(m)} - z_i) + \beta_i(y^{(m)})\sinh(y_i^{(m)} - z_i)\right] = 0$$

we see

$$KC_\infty^{-1} \leq |\Delta(y^{(m)}, 2^{-m}h)f| = |\Delta(y^{(m)}, 2^{-m}h)\tilde{f}|.$$

The points at which $\tilde{f}$ is evaluated in calculating the right hand side of this lie in $[z - 2 \cdot 2^{-m}h, z + 2 \cdot 2^{-m}h]$ so that

$$KC_\infty^{-1} \leq 4^n C(2^{-m}h)|\tilde{f}(z^{(m)})|\left[(h_1 h_2 \cdots h_n)^2 4^{-mn}\right]^{-1}$$

where $z^{(m)}$ is the evaluation point at which $\tilde{f}$ takes its greatest modulus (so we have $z^{(m)} \neq z$). As $|z^{(m)} - z| \leq 4 \cdot 2^{-m}|h|$ this gives

$$KC_\infty^{-1} \leq 4^n C(2^{-m}h)|\tilde{f}(z^{(m)})|\,|z^{(m)} - z|^{-2n}(4|h|)^{2n}(h_1 h_2 \cdots h_n)^{-2}.$$

Letting $m \to \infty$, $C(2^{-m}h) \to 1$ and we see from the hypotheses on $\tilde{f}$ that $K = 0$.

Let $a < u < v < b$. There are complex valued functions A_i, B_i $(i = 1, \ldots, n)$ on (a, b) where A_i and B_i are constant with respect to the ith variable such that

$$g(x) = f(x) - \sum A_i(x)\cosh x_i + B_i(x)\sinh x_i$$

takes the value 0 whenever $x_i = u_i$ or v_i for some i—more precisely put

$$g(x) = \sum f(w) \prod \frac{\sinh(x_i - w_i')}{\sinh(w_i' - w_i)}$$

where the sum is over all n tuples $(w_1, \ldots, w_n)$ of symbols with $w_i = u_i$, x_i or v_i for all i, the product is over all i for which $w_i \neq x_i$ and $w_i' = u_i$ if $w_i = v_i$ and $w_i' = v_i$ if $w_i = u_i$. Using the addition formula for the sinh function shows that g is of the form required. We see $\Delta g = \Delta f$ because $\Delta_i\, A_i(x)\cosh x_i + B_i(x)\sinh x_i = 0$. An application of Lemma 2.1 completes the proof.

3. Spectral subspaces as intersections of ranges. Throughout this section G is a discrete abelian group.

DEFINITION 3.1. For an open set E in $\hat{G}$ we define

$$I_0(E) = \{a;\, a \in l^1(G), \operatorname{supp} \hat{a} \subseteq E\}$$

where supp $\hat{a}$ is the closed support of $\hat{a}$.

If Y is a Banach G-module, $a \in l^1(G)$ then we define

$$ay = \sum_{g \in G} a(g) gy$$

and Y is a left module over the Banach algebra $l^1(G)$.

DEFINITION 3.2. For an open set $E \subseteq \hat{G}$ put

$$Y(E) = \{y;\ y \in Y,\ ay = 0 \text{ for all } a \in I_0(E)\}$$

and put

$$Y^{\perp} = \{a;\ a \in l^1(G),\ ay = 0 \text{ for all } y \in Y\}.$$

$Y^{\perp}$ is a closed ideal in $l^1(G)$. Its hull is the Arveson spectrum of Y

$$\operatorname{spec} Y = \{\chi;\ \chi \in \hat{G},\ \hat{a}(\chi) = 0 \text{ for all } a \in Y^{\perp}\}.$$

Let $g_1, \ldots, g_n \in G$, $\varepsilon > 0$ and $\psi \in \hat{G}$. Put

$$E(g_1, \ldots, g_n, \psi, \varepsilon) = E = \{\chi;\ \chi \in \hat{G}, |\chi(g_j) - \psi(g_j)| < \varepsilon, i = 1, \ldots, n\}$$

and

$$\cap(E) = \cap_Y(E) = \bigcap_{\chi \in E} \left[\sum_{j=1}^{n} \left(g_j^{-1} - \chi(g_j)^{-1} e\right)^{n+1} \left(g_j - \chi(g_j) e\right)^{n+1} \right] Y.$$

THEOREM 3.3. $Y(E) = \cap(E)$.

Proof. (i) Let $y \in Y(E)$ and $\chi \in E$. As $A(\hat{G})$ is a regular algebra and

$$Z = \{\chi';\ \chi' \in \hat{G}, \chi'(g_j) = \chi(g_j), j = 1, \ldots, n\}$$

is a compact subset of E, there is $a \in l^1(G)$ with $\hat{a}(\chi') \neq 0$ for all $\chi' \in Z$ and $\hat{a}(\gamma) = 0$ for all γ in a neighborhood of $\hat{G} \setminus E$, that is $a \in I_0(E)$. Put

$$\Sigma = \Sigma_{\chi} = \sum_{j=1}^{n} \left(g_j^{-1} - \chi(g_j)^{-1} e\right)^{n+1} \left(g_j - \chi(g_j) e\right)^{n+1} \in l^1(G).$$

Then $\hat{\Sigma}(\gamma) \geq 0$ with equality only for $\gamma \in Z$ so $(\Sigma + a^*a)\hat{}$ is nowhere zero on $\hat{G}$ which implies $b = \Sigma + a^*a$ is invertible in $l^1(G)$. We have $y = b^{-1}by = b^{-1}\Sigma y = \Sigma b^{-1} y \in \Sigma Y$. Thus $y \in \cap(E)$.

(ii) To prove the opposite inclusion, first consider the case $G = \mathbf{Z}^n$, $Y = l^{\infty}(G)$ where G acts on $l^{\infty}(G)$ by translation, that is

$$(gf)(h) = f(g^{-1}h) \qquad (g, h \in G, f \in l^{\infty}(G)),$$

and $g_1, \ldots, g_n$ are the usual generators of $\mathbf{Z}^n$. We consider $\mathbf{T} = \mathbf{Z}\hat{}$ as $\mathbf{R} \bmod 2\pi\mathbf{Z}$ and functions on $\mathbf{T}$ as 2π periodic functions on $\mathbf{R}$. Let

$y \in \bigcap(E)$, $\xi \in E$ so that

$$\sum_{\xi} = \sum_{j=1}^{n} \left[g_j^{-1} - (\exp - i\xi_j)e\right]^{n+1}\left[g_j - (\exp i\xi_j)e\right]^{n+1}.$$

There is $z \in Y$ with $y = \Sigma z$. The Fourier transforms $\hat{y}, \hat{z}$ of y, z are Schwarz distributions on $\mathbf{T}^n$. For $m = (m_1, m_2, \ldots, m_n) \in \mathbf{Z}^n$ put

$$\Delta(m) = \left[(1 + m_1^2)(1 + m_2^2) \cdots (1 + m_n^2)\right]^{-1}.$$

Then Δ, $\Delta \cdot y$ and $\Delta \cdot z$ (the pointwise product) are in $l^1(G)$ so their Fourier transforms are in $C(\mathbf{T}^n)$. Put $f = (\Delta \cdot y)^\wedge = \Delta^\wedge * \hat{y}$, $g = (\Delta \cdot z)^\wedge = \Delta^\wedge * \hat{z}$. As Δ^{-1} is the inverse Fourier transform of the distribution

$$\gamma = (I - \tilde{D}_1^2)(I - \tilde{D}_2^2) \cdots (I - \tilde{D}_n^2)$$

on $\mathbf{T}^n$ where $D_j = \partial/\partial\eta_j$ and $\tilde{D}_j\phi = (D_j\phi)(0)$ for $\phi \in \mathscr{D}$ so that $D_j\phi = \tilde{D}_j * \phi$. $\tilde{D}_j^2$ is $\tilde{D}_j * \tilde{D}_j$. Thus $\hat{y} = \Gamma * f$, $\hat{z} = \Gamma * g$, and

$$f = \Delta^\wedge * \left(\Sigma^\wedge \cdot \hat{z}\right) = \Delta^\wedge * \left[\Sigma^\wedge \cdot (\Gamma * g)\right]$$

where $\cdot$ denotes the pointwise product of a function and a distribution [**4**; p. 117] and

$$\Sigma^\wedge(\eta) = \sum_{j=1}^{n} \left[2 \sin \frac{1}{2}(\eta_j - \xi_j)\right]^{2n+2}.$$

For each j we have

$$\Sigma^\wedge\left[(I - \tilde{D}_j^2) * g\right] = (I - \tilde{D}_j^2) * (\Sigma^\wedge \cdot g) + 2\tilde{D}_j * \left[(D_j\Sigma^\wedge) \cdot g\right] - (D_j^2\Sigma^\wedge) \cdot g$$

so that, because $D_jD_k\Sigma^\wedge = 0$ for $j \neq k$ we have

$$\Sigma^\wedge \cdot (\Gamma * g) = \Gamma * (\Sigma^\wedge \cdot g) - \sum_j 2\Gamma * (I - \tilde{D}_j^2)^{-1} * \tilde{D}_j * (D_j\Sigma^\wedge \cdot g) + \sum_j \Gamma * (I - \tilde{D}_j^2)^{-1}(D_j^2\Sigma^\wedge \cdot g)$$

so that

$$(\dagger) \quad f = \Delta^\wedge * \left[\Sigma^\wedge \cdot (\Gamma * g)\right] = \Sigma^\wedge \cdot g - \sum_j 2(I - \tilde{D}_j^2)^{-1} * \tilde{D}_j * (\tilde{D}_j\Sigma^\wedge \cdot g) + (I - \tilde{D}_j^2)^{-1} * (\tilde{D}_j^2\Sigma^\wedge \cdot g).$$

However $(I - \tilde{D}_j^2)^{-1}$ is the functional

$$(I - \tilde{D}_j^2)^{-1}(\phi) = \frac{1}{2\pi}\int_0^{2\pi} c(\eta_j)\phi(0, 0, \ldots, \eta_j, 0, \ldots, 0)\, d\eta_j$$

where

$$c(\eta_j) = \pi(\sinh\pi)^{-1}\cosh(\eta_j - \pi) \quad \text{for } 0 \le \eta_j < 2\pi$$

and $(I - \tilde{D}_j^2)^{-1} * \tilde{D}_j$ the functional

$$\left[\left(I - \tilde{D}_j^2\right) * D_j\right](\phi) = \frac{1}{2\pi}\int_0^{2\pi} s(\eta_j)\phi(0,\ldots,0,\eta_j,0,\ldots,0)\,d\eta_j$$

where $s(\eta_j) = \pi(\sinh\pi)^{-1}\sinh(\eta_j - \pi)$ for $0 \le \eta_j < 2\pi$ and so these distributions are measures. Thus all the terms in (†) are continuous functions and, considered as an equation between functions it holds almost everywhere and hence everywhere. We consider c and s as extended to 2π periodic functions on $\mathbf{R}$.

As $\eta \to \xi$ we have $\Sigma^\wedge = O(|\eta - \xi|^{2n+2})$, $D_j\Sigma^\wedge = O(|\eta - \xi|^{2n+1})$ and $D_j^2\Sigma^\wedge = O(|\eta - \xi|^{2n})$. However, if $\tilde{g} \in C(\mathbf{T}^n)$ with $\tilde{g}(\eta) = O(|\eta - \xi|^{2n})$ as $\eta \to \xi$ then

$$\begin{aligned}
\left(I - \tilde{D}_j^2\right)^{-1} * \tilde{g}(\eta) &= \frac{1}{2\pi}\int_0^{2\pi} c(t)\tilde{g}(\eta - te_j)\,dt \\
&= \frac{1}{2\pi}\int_0^{2\pi} c(\eta_j - t)\tilde{g}\left(\eta - (\eta_j - t)e_j\right)dt \\
&= (2\sinh\pi)^{-1}\left[\cosh\eta_j\int_0^{\eta_j}\cosh(t+\pi)\tilde{g}\left(\eta - (\eta_j - t)e_j\right)dt\right. \\
&\qquad + \sinh\eta_j\int_0^{\eta_j}\sinh(t+\pi)\tilde{g}\left(\eta - (\eta_j - t)e_j\right)dt \\
&\qquad + \cosh\eta_j\int_{\eta_j}^{2\pi}\cosh(\pi - t)\tilde{g}\left(\eta - (\eta_j - t)e_j\right)dt \\
&\qquad \left. + \sinh\eta_j\int_{\eta_j}^{2\pi}\sinh(\pi - t)\tilde{g}\left(\eta - (\eta_j - t)e_j\right)dt\right].
\end{aligned}$$

Since $\int_0^{\xi_j}\cosh(t+\pi)\tilde{g}(\eta - \eta_je_j + te_j)\,dt$ is independent of η_j and

$$\int_{\xi_j}^{\eta_j}\cosh(t+\pi)\tilde{g}(\eta - \eta_je_j + te_j)\,dt = o\left(|\eta - \xi|^{2n}\right) \quad \text{as } \eta \to \xi$$

we see that the first integral in this expression is of the form $\tilde{A}(\eta)\cosh\eta_j + o(|\eta - \xi|^{2n})$ and hence of the form

$$A(\eta)\cosh(\eta_j - \xi_j) + B(\eta)\sinh(\eta_j - \xi_j) + o\left(|\eta - \xi|^{2n}\right)$$

where A and B are independent of the jth variable. The other three are similar and so $(I - \tilde{D}_j^2)^{-1} * \tilde{g}$ is of this form. By a similar argument

$(I - \tilde{D}_j^2)^{-1} * \tilde{D}_j * \tilde{g}$ is of this form so that the decomposition (†) shows that f satisfies the hypotheses of Lemma 2.2 and so f is of the form $\Sigma A_i(x)\cosh x_i + B_i(x)\sinh x_i$ on E so that $\hat{y} = \Gamma * f = 0$ on E because the support of Γ is $\{0\}$ and $(I - D_j^2)(A_j(x)\cosh x_j + B_j(x)\sinh x_j) = 0$. Thus, if $a \in I_0(E) \cap \mathscr{D}^{\wedge}$ we have $(ay)^{\wedge} = \hat{a} \cdot \hat{y} = 0$ because a is a function in $\mathscr{D}$ with support in E and so $ay = 0$.

Taking an infinitely differentiable approximate identity in $L^1(\hat{G})$ with support $\to E$ we see that $I_0(E) \cap \mathscr{D}^{\wedge}$ is l^1 dense in $I_0(E)$ and so, since the product ay is continuous in a we see $ay = 0$ for all $a \in I_0(E)$ and hence $y \in Y(E)$.

(iii) Consider now the case in which $G = \mathbf{Z}^n$, $g_1, \dots, g_n$ are its generators and Y is an arbitrary Banach G-module. Let $y_0 \in \bigcap(E)$, $a \in I_0(E)$, $f \in Y^*$ and consider the map $Y \to l^\infty(G)$ given by

$$\Phi(y)(g) = f(g^{-1}y).$$

We have

$$[h(\Phi(y))](g) = [\Phi(y)](h^{-1}g) = f(g^{-1}hy) = \Phi(hy)(g)$$

so Φ is a G-module and hence an $l^1(G)$ module map. Thus,

$$\Phi(y_0) \in \bigcap\nolimits_{l^\infty(G)}(E) = [l^\infty(G)](E)$$

so $\Phi(ay_0) = a\Phi(y_0) = 0$. However,

$$\Phi(ay_0)(e) = \sum_{g \in G} a(g)\Phi(gy_0)(e) = \sum a(g) f(gy_0) = f(ay_0)$$

so $f(ay_0) = 0$ for all $f \in Y^*$ showing that $ay_0 = 0$ and hence $y_0 \in Y(E)$.

(iv) Finally, consider the general case. Denote the injection map $\mathbf{Z}^n \to G$ given by $g_i' \to g_i$ by ι where the g_i' are the generators of $\mathbf{Z}^n$. ι^* is a map $\hat{G} \to \mathbf{T}^n$ and putting $ky = \iota(k)y$ $(k \in \mathbf{Z}^n, y \in Y)$, Y becomes a $\mathbf{Z}^n$-module. Let $E' = \iota^* E$, $\psi' = \iota^*\psi$. Let $\iota^{*-1}E' = E$ and if $\varepsilon_1 < 1$ then $I_0(E)$ is the ideal in $l^1(G)$ generated by $\iota\ I_0(E')$ and $\bigcap(E') = \bigcap(E)$. Hence if $y \in \bigcap(E)$ then $0 = ay = \iota(a)y$ for all $a \in I_0(E')$ and, because$\{b;\ b \in l^1(G),\ by = 0\}$ is an ideal in $l^1(G)$ containing $\iota\,(I_0(E'))$ it contains $I_0(E)$ which implies $y \in Y(E)$.

4. Automatic continuity results.

THEOREM 4.1. *Let G be an abelian group and let X, Y be Banach G-modules. Let S: $X \to Y$ be a G-module homomorphism and let $\mathfrak{S}$ be the separating space of S. Then $\mathfrak{S}$ is the direct sum of a finite number of scalar submodules of Y. The separating space is defined in* [**6**; *p.* 7].

Proof. We apply [**6**, Theorem 2.3] with $\Omega = \hat{G}$, Γ as the set of all $E(g_1,\ldots,g_n; \psi, \varepsilon)$ $(n \in \mathbf{Z}^+, g_1,\ldots,g_n \in G, \psi \in \hat{G}, 0 < \varepsilon < 1)$ and $X(E)$ and $Y(E)$ as in Definition 3.2. By the regularity of $l^1(G)$, if $F_1,\ldots,F_n \in \Gamma$ with $\overline{F}_i \cap \overline{F}_j = \varnothing$ for $i \neq j$ there is $b \in l^1(G)$ with $\hat{b} = 0$ on $F_1 \cup F_2 \cup \cdots \cup F_{n-1}$ and $\hat{b} = 1$ on F_n. Let $x \in X$. Then $x = bx + (e - b)x$. If for some j with $1 \leq j \leq n - 1$ we have $a \in I_0(F_j)$ then $ab = 0$ so $bx \in X(F_j)$ $(j = 1,\ldots,n - 1)$. Similarly $(e - b)x \in X(F_n)$ so that [**6**: Conditions 2.2] apply.

For any $a \in l^1(G)$ with finite support we have $S(ax) = aS(x)$ and so $S(aX) = aS(X) \subseteq aY$. Hence for each $E \in \Gamma$, $S(\bigcap_X(E)) \subseteq \bigcap_Y(E)$. By Theorem 3.3, this implies $S(X(E)) \subseteq Y(E)$ so that the hypotheses of [**6**: Theorem 2.3] are satisfied. Hence the set Λ of discontinuity points of S is finite. Thus for each $\lambda \in \hat{G} \setminus \Lambda$ there is $E \in \Gamma$ with $\mathfrak{S} \subseteq Y(E)$. Let $a \in I_0(\hat{G} \setminus \Lambda)$. By the regularity of $l^1(G)$ we have $a = \sum_{i=1}^n a \cdot \rho_i$ where $\rho_i \in I_0(E_i)$ and $\mathfrak{S} \subseteq Y(E_i)$. Thus, if $s \in \mathfrak{S}$ then $as = \sum a\rho_i s = 0$. Hence $\mathfrak{S}^\perp \supseteq I_0(\hat{G} \setminus \Lambda)^-$ which, by [**3**; p. 170], implies $\mathfrak{S}^\perp \supseteq \{a; a \in l^1(G), \hat{a}(\lambda) = 0, \lambda \in \Lambda\} = Z(\Lambda)$ and so $\operatorname{spec} \mathfrak{S} \subseteq \Lambda$. As $l^1(G)/Z(\Lambda) = \mathbf{C}^n$ and $\mathfrak{S}$ is an $l^1(G)/Z(\Lambda)$ module it is a $\mathbf{C}^n$ module and hence a direct sum of n $\mathbf{C}$ modules. These summands are scalar G-modules.

COROLLARY 4.2. *If in* 4.1, *S is discontinuous then there is an element χ of $\hat{G}$ for which*

(i) *X has a discontinuous χ-covariant linear form.*

(ii) *Y has a non-trivial scalar submodule corresponding to the character χ.*

Conversely if such a χ exists then there are discontinuous G module homomorphisms $X \to Y$.

Proof. As S is not continuous, $\mathfrak{S}$ is not $\{0\}$ so there is $s \in \mathfrak{S}$ with $s \neq 0$ and $\chi \in \hat{G}$ with $gs = \chi(s)g$ $(g \in G)$. Let $f \in Y^*$ with $f(s) \neq 0$. For $y \in Y$ let α_y: $G \to \mathbf{C}$ be the function $g \mapsto \chi(g)f(g^{-1}y)$. Then $y \mapsto \alpha_y$ is a bounded linear map $Y \to l^\infty(G)$. Let M be a translation invariant mean on $l^\infty(G)$ and put $F(y) = M(\alpha_y)$. Then $F \in Y^*$, α_s is the constant element $g \mapsto f(s)$ so $F(s) = f(s) \neq 0$ and $FS(gx) = FgS(x) = \chi(g)FS(x)$ because $\alpha_{gy} = \chi(g)\tau_t\alpha_y$ where τ_g is translation by g. Thus FS is a χ-covariant linear form.

For the converse if Φ is a discontinuous χ-covariant form on X and $y \neq 0$ lies in a scalar submodule then $S(x) = \phi(x)y$ is a discontinuous G-module homomorphism.

REMARK 4.3. If G is a locally compact abelian group, X, Y are Banach G-modules, S is a G-module homomorphism $X \to Y$ and the product gy is continuous in g in some way which ensures that scalar submodules of Y correspond to *continuous* characters then we see that 4.2 applies with $\hat{G}$ as the topological dual of G.

EXAMPLES 4.4. If Y is a Banach G-module containing no scalar submodule then every G-module homomorphism into Y is continuous. If $X = L^p(G)$ $(1 < p < \infty)$ where G is an extension of a locally compact abelian group by a discrete group with uncountably infinite torsion free rank or $p = 2$ and G is compact and weakly polythetic, there are no discontinuous translation invariant forms on X [**2** and **7**] and hence, by the remarks after Definition 1.2, no discontinouus χ-covariant forms for any $\chi \in \hat{G}$. Thus, if Y is a continuous Banach G-module then every G-module homomorphism $X \to Y$ is continuous.

The results in this paper can be extended to the case of G-modules which satisfy Definition 1.1 with (iv) replaced by

(iv)′ For each $g \in G$ there is $K \in \mathbf{R}$ and an integer k with

$$\|g^n x\| \le K n^k \|x\| \qquad (x \in X, n \in \mathbf{Z}).$$

The main changes needed are to replace $l^1(\mathbf{Z}^n)$ by the space of functions of rapid decrease [**5**; p. 83] and $l^\infty(\mathbf{Z})$ by the space of functions of slow increase. We now define $\bigcap(E)$ by

$$\bigcap_{k=1}^{\infty} \bigcap_{\chi \in E} \sum_{j=1}^{n} \left(g_j^{-1} - \chi(g_j)^{-1} e\right)^k \left(g_j - \chi(g_j) e\right)^k Y$$

and need higher order versions of 2.1 and 2.2.

REFERENCES

[1] B. E. Johnson, *Continuity of linear operators commuting with continuous linear operators*, Trans. Amer. Math. Soc., **128** (1967), 88–102.

[2] ______, *A proof of the translation invariant form conjecture for $L^2(G)$*, Bull. Sci. Math., (2) **107** (1983), 301–310.

[3] W. Rudin, *Fourier Analysis on Groups*, Interscience tracts in pure and applied mathematics, Wiley, New York, 1962.

[4] L. Schwartz, *Theorie des distributions*, Tome 1, Act. Sci. Ind. 1245, Hermann, Paris, 1957.

[5] ______, *Theorie des distributions*, Tome II, Act. Sci. Ind. 1122, Hermann, Paris, 1951.

[6] A. M. Sinclair, *Automatic Continuity of Linear Operators*, Cambridge University Press, Cambridge, 1976.

[7] G. S. Woodward, *Translation-invariant linear forms on* $C_0(G)$, $C(G)$, $L^p(G)$ *for noncompact groups*, J. Funct. Analysis, **16** (1974), 205–220.

Received June 28, 1983.

THE UNIVERSITY
NEWCASTLE UPON TYNE
ENGLAND