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INVERSE THEOREMS FOR MULTIDIMENSIONAL BERNSTEIN OPERATORS

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Let $B_n f$ be the m-dimensional Bernstein polynomials on a simplex or on a cube. The class of functions for which $\|B_n f - f\| = O(n^{-\alpha})$ is determined. That is, necessary and sufficient conditions on the smoothness of f in the simplex or the cube and especially near their boundaries are given so that $\|B_n f - f\| = O(n^{-\alpha})$. Interpolation of spaces, and in particular the characterization of the interpolation space, is one of the tools used.

For a sequence of approximation operators an inverse theorem is a result determining necessary and sufficient conditions on the rate of convergence for the function to belong to a certain class of functions generally satisfying some smoothness conditions. A more restrictive view is that which calls the necessary and the sufficient conditions above direct and inverse theorems respectively. Here the inverse results will be of the first variety.

The Bernstein polynomials on $C[0,1]$ are given by

$$(1.1)\quad B_n(f,x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right) P_{n,k}(x)$$

$$\text{where } P_{n,k}(x) \equiv \binom{n}{k} x^k (1-x)^{n-k}.$$

For $B_n(f,x)$ it was shown by Berens and Lorentz [**1**] that

$$|B_n(f,x) - f(x)| \le M\left(\frac{(x(1-x))}{n}\right)^{\alpha/2} \quad \text{for } 0 < \alpha < 2$$

occurs if and only if

$$|\Delta_h^2 f(x)| \equiv |f(x-h) - 2f(x) + f(x+h)| \le Mh^\alpha$$
$$\text{for } [x-h, x+h] \subset [0,1].$$

The Bernstein polynomial on the simplex

$$S \equiv \left\{(x_1, \ldots, x_m);\ x_i \ge 0,\ \sum_{i=1}^{m} x_i \le 1\right\},$$

is given by

$$(1.2)\qquad B_n(f,x) \equiv \sum_{\nu/n\in S} P_{n,\nu}(x) f\left(\frac{\nu}{n}\right)$$

where $x = (x_1, \dots, x_m)$, $\nu = (\nu_1, \dots, \nu_m)$ and

$$(1.3)\quad P_{n,\nu}(x) = \frac{n!}{\nu_1! \cdots \nu_m!(1-\nu_1-\cdots-\nu_m)!} x_1^{\nu_1} \cdots x_m^{\nu_m}(1-x_1-\cdots-x_m)^{n-\nu_1-\cdots-\nu_m};$$

and the Bernstein polynomial on the box B, $B = \{(x_1, \dots, x_m); 0 \le x_i \le 1\}$, is given by

$$(1.4)\qquad B^*_{n_1,\dots,n_m}(f,x) \equiv \sum_{0\le\nu_i\le n_i} \left(\prod_{i=1}^{n} P_{n_i,\nu_i}(x_i)\right) f\left(\frac{\nu_1}{n_1}, \dots, \frac{\nu_m}{n_m}\right)$$

where $P_{n_i,\nu_i}(x_i)$ is given by $P_{n,k}(x)$ in (1.1).

It will be shown that for $m > 1$ the class of functions given by Lip* α is no longer adequate to characterize the rate of convergence of $B_n(f,x) - f(x)$ or $B^*_n(f,x) - f(x)$.

It was observed by K. Ivanov [5] that for $0 < \alpha < 2$

$$(1.5)\quad \|v_n(f,\cdot) - f(\cdot)\|_{C[0,1]} = O(n^{-\alpha/2}) \Leftrightarrow E_n(f) \equiv \inf_{\deg P\le n} \|f-P\|_{C[0,1]} = O(n^{-\alpha}).$$

We conjecture this is the case for the multidimensional Bernstein polynomials too.

For inverse theorems for approximation processes on D such that span $D = R^m$ and $m > 1$ the present result seems to be the first dealing differently with points of different distance from the boundary of D. Probably this is the reason that so few inverse results are known in the multivariate case, none of which exhibit the above phenomenon. (This phenomenon was shown by many authors to be natural for one-dimensional approximation processes.)

We will show that $\|B_n f - f\| = O(n^{-\alpha/2})$ is equivalent to a certain interpolation space in stages. The direct result will be proved in §4 and the converse result in §5. We will then characterize the interpolation space and the K-functional in terms of smoothness. As the result for m dimensions is not substantially different in ideas from that for two dimensions but is somewhat loaded with indices, we will present the result for two dimensions and comment in §§8 and 9 about the m-dimensional case.

2. Preliminary results. Some preliminary results on two-dimensional Bernstein polynomials on the simplex (triangle) will be necessary in later sections. For two dimensions $B_n(f, x)$ given in (1.2) and (1.3) can also be given by

$$(2.1)\quad B_n(f,x,y) = \sum_{k=0}^{n} \sum_{l=0}^{n-k} \binom{n}{k}\binom{n-k}{l} x^k y^l (1-x-y)^{n-k-l} f\left(\frac{k}{n}, \frac{l}{n}\right)$$
$$\equiv \sum_{k=0}^{n} \sum_{l=0}^{n-k} P_{n,k,l}(x,y) f\left(\frac{k}{n}, \frac{l}{n}\right).$$

We can now prove the following lemma:

LEMMA 2.1. *For $B_n(f, x, y)$ given in* (2.1) *we have*:

$$(2.2)\quad \frac{\partial}{\partial x} B_n(f,x,y)$$
$$= \sum_{k=0}^{n} \sum_{l=0}^{n-k} f\left(\frac{k}{n}, \frac{l}{n}\right)\binom{n}{k}\binom{n-k}{l} x^{k-1} y^l$$
$$\times (1-x-y)^{n-k-l-1}[k(1-x-y) - (n-k-l)x];$$

$$(2.3)\quad \frac{\partial^2}{\partial x^2} B_n(f,x,y)$$
$$= \sum_{k=0}^{n} \sum_{l=0}^{n-k} f\left(\frac{k}{n}, \frac{l}{n}\right)\binom{n}{k}\binom{n-k}{l} x^{k-2} y^l (1-x-y)^{n-k-l-2}$$
$$\times \left[k(k-1)(1-x-y)^2 - 2k(n-k-l)x(1-x-y) + (n-k-l)(n-k-l-1)x^2\right];$$

$$(2.4)\quad \frac{\partial^2}{\partial x \partial y} B_n(f,x,y)$$
$$= \sum_{k=0}^{n} \sum_{l=0}^{n-k} f\left(\frac{k}{n}, \frac{l}{n}\right)\binom{n}{k}\binom{n-k}{l} x^{k-1} y^{l-1} (1-x-y)^{n-k-l-2}$$
$$\times \left[kl(1-x-y)^2 - (ky+lx)(n-k-l)(1-x-y) + (n-k-l)(n-k-l-1)xy\right];$$

$$(2.5)\quad \frac{\partial}{\partial x} B_n(f,x,y)$$
$$= n \sum_{k=0}^{n} \sum_{l=0}^{n-k} P_{n-1,k-1,l}(x,y)\left[f\left(\frac{k}{n}, \frac{l}{n}\right) - f\left(\frac{k-1}{n}, \frac{l}{n}\right)\right];$$

(2.6) $$\frac{\partial^2}{\partial x^2}B_n(f,x,y)$$
$$= n(n-1)\sum_{n=2}^{n}\sum_{l=0}^{n-k} P_{n-2,k-2,l}(x,y)$$
$$\times\left[f\left(\frac{k}{n},\frac{l}{n}\right) - 2f\left(\frac{k-1}{n},\frac{l}{n}\right) + f\left(\frac{k-2}{n},\frac{l}{n}\right)\right];$$

(2.7) $$\frac{\partial^2}{\partial x\partial y}B_n(f,x,y)$$
$$= n(n-1)\sum_{k=1}^{n}\sum_{l=1}^{n-k} P_{n-2,k-1,l-1}(x,y)$$
$$\times\left[f\left(\frac{k}{n},\frac{l}{n}\right) - f\left(\frac{k-1}{n},\frac{l}{n}\right) - f\left(\frac{k}{n},\frac{l-1}{n}\right) + f\left(\frac{k-1}{n},\frac{l-1}{n}\right)\right].$$

Proof. Equations (2.2), (2.3) and (2.4) are actually straightforward derivatives of (2.1). We derive (2.5), (2.6) and (2.7) from (2.2), (2.3) and (2.4) by comparing coefficients of $x^{k-1}y^l(1-x-y)^{n-k-l}$, $x^{k-2}y^l(1-x-y)^{n-k-l}$ and $x^{k-1}y^{l-1}(1-x-y)^{n-k-l}$ respectively. To prove (2.5), (2.6) and (2.7) we have to show

(a) $$n\binom{n-1}{k-1}\binom{n-k}{l} = k\binom{n}{k}\binom{n-k}{l}$$
$$= (n-k-l+1)\binom{n}{k-1}\binom{n-k+1}{l},$$

(b) $$n(n-1)\binom{n-2}{k-2}\binom{n-k}{l} = k(k-1)\binom{n}{k}\binom{n-k}{l}$$
$$= (k-1)(n-k-l+1)\binom{n}{k-1}\binom{n-k+1}{l}$$
$$= n(n-k-l+1)(n-k-l+2)\binom{n}{k-1}\binom{n-k+2}{l},$$

and

(c) $$n(n-1)\binom{n-2}{k-1}\binom{n-k-1}{l}$$
$$= kl\binom{n}{k}\binom{n-k}{l} = k(n-k-l+1)\binom{n}{k}\binom{n-k}{l-1}$$
$$= l(n-k-l+1)\binom{n}{n-1}\binom{n-k+1}{l}$$
$$= (n-k-l+1)(n-k-l+2)\binom{n}{k-1}\binom{n-k+1}{l-1}.$$

To show that (a), (b) and (c) are valid is a tedious but elementary computation.

REMARK. The expressions $(\partial/\partial y)B_n(f,x,y)$ and $(\partial^2/\partial y^2)\,B_n(f,x,y)$ can be obtained by interchanging l and y with k and x.

LEMMA 2.2. *For $B_n(f,x,y)$ given by (2.1) and for $\phi_i(x,y)$ given by $\phi_1(x,y)=x$, $\phi_2(x,y)=y$ and $\phi_3(x,y)=1-x-y$ we have*

$$B_n(\phi_i,x,y)=\phi_i \quad \text{and} \quad B_n(1,x,y)=1; \tag{2.8}$$

$$B_n(\phi_i^2,x,y)=\phi_i(x,y)^2+\frac{\phi_i(x,y)(1-\phi_i(x,y))}{n}; \tag{2.9}$$

and

$$B_n(\phi_i\phi_j,x,y)=\phi_i(x,y)\phi_j(x,y)(1-1/n) \quad \text{for } i\neq j. \tag{2.10}$$

Proof. We can write

$$f(x,y,z)=\sum_{k=0}^{n}\sum_{l=0}^{n-k}\binom{n}{k}\binom{n-k}{l}x^k y^l z^{n-k-l}=(x+y+z)^n.$$

From this

$$B_n(1,x,y)=F(x,y,1-x-y)=1$$

and

$$B_n(\phi_1,x,y)=\frac{x}{n}\frac{\partial}{\partial x}F(x,y,z)$$

at $z=1-x-y$ which yields (2.8). To prove (2.9), which we do only for ϕ_1, we write

$$B_n(\phi_1^2,x,y)=\frac{x^2}{n^2}\left(\frac{\partial}{\partial x}\right)^2F(x,y,z)+\frac{x}{n^2}\left(\frac{\partial}{\partial x}\right)F(x,y,z)$$

at $z=1-x-y$ and, therefore,

$$B_n(\phi_1^2,x,y)=x^2\frac{(n-1)}{n}+\frac{x}{n}=x^2+\frac{x(1-x)}{n}.$$

To prove (2.10), which we do only for $i=1$, $j=2$, we write

$$B_n(\phi_1\cdot\phi_2,x,y)=\frac{xy}{n^2}\frac{\partial^2}{\partial x\partial y}F(x,y,z)$$

at $z=1-x-y$, or

$$B_n(\phi_1\cdot\phi_2,x,y)=\frac{xy}{n}(n-1)=xy-\frac{xy}{n}.$$

3. Rate of approximation, optimal case for $x + y < 3/4$. One can use Korovkin's theorem and the fact that $1, x, y, x^2$ and y^2 is a Korovkin system to obtain $\|B_n(f, x, y) - f(x, y)\| = o(1)$ as $n \to \infty$ where $\|g\|$ will mean $\|g\|_{C(S)}$. We can also prove the following estimate.

LEMMA 3.1. *For* $f(x, y) \in C^2(S)$ *satisfying*

$$\left\|\frac{\partial^2}{\partial x^2} f(x, y)\right\| \le M, \qquad \left\|\frac{\partial^2}{\partial y^2} f(x, y)\right\| \le M$$

and

$$\left\|\frac{\partial^2}{\partial x \partial y} f(x, y)\right\| \le M,$$

we have

$$|B_n(f, x, y) - f(x, y)| \le 2M\frac{1}{n}(x(1 - x) + y(1 - y)). \tag{3.1}$$

We will need for the present paper a somewhat more delicate result and the next lemma will constitute that result for a partial domain.

LEMMA 3.2. *For* $f(x, y) \in C(S)$, f *is* C^2 *locally in the interior of* S *and*

$$\Phi_0(f) \equiv \max_{x+y \le 3/4}\left(\left|x\frac{\partial^2}{\partial x^2} f(x, y)\right|, \left|y\frac{\partial^2}{\partial y^2} f(x, y)\right|, \left|\sqrt{xy}\, \frac{\partial^2}{\partial x \partial y} f(x, y)\right|\right) \tag{3.2}$$

we have

$$\max_{x+y \le 2/3} |B_n(f, x, y) - f(x, y)| \le \frac{M(\Phi_0(f) + \|f\|)}{n} \tag{3.3}$$

where M *is independent of* n.

Proof. To estimate convergence in the domain $x + y \le 2/3$ we may assume $f(x, y) = 0$ in the domian $x + y \ge 3/4$ as $f_1(x) = f(x)$ in $x + y < 3/4$ and $f_1(x) = 0$ in $x + y \ge 3/4$ satisfies there

$$\begin{aligned}|B_n(f - f_1, x, y)| &\le 12^2\|f\|B_n\big((\phi_1 + \phi_2 - x - y)^2, x, y\big)\\ &= 12^2\|f\|\frac{(x + y)(1 - x - y)}{n} = O\left(\frac{1}{n}\right).\end{aligned}$$

Recalling Taylor's formula

$$\begin{aligned}f\left(\frac{k}{n}, \frac{l}{n}\right) = f(x, y) &+ \left(\frac{k}{n} - x\right)\frac{\partial}{\partial x} f(x, y) + \left(\frac{l}{n} - y\right)\frac{\partial}{\partial y} f(x, y)\\ &+ \int_0^1 tF''(t)\, dt\end{aligned}$$

where

$$F(t) = f\left(\frac{k}{n} + t\left(x - \frac{k}{n}\right), \frac{l}{n} + t\left(y - \frac{l}{n}\right)\right)$$

and writing

$$\int_0^1 tF''(t)\,dt \equiv \psi\left(\frac{k}{n}, \frac{l}{n}, x, y\right) \equiv \psi,$$

we have for $x + y \leq 2/3$

$$|B_n(f, x, y) - f(x, y)| = \sum \psi\left(\frac{k}{n}, \frac{l}{n}, x, y\right) P_{n,k,l}(x, y) + \frac{M_1(\|f\| + \Phi_0(f))}{n}$$

as $|\partial f/\partial x|$, $|\partial f/\partial y|$ and $|f|$ are bounded on the domain $x + y \leq 2/3$ by $M_2(\|f\| + \Phi_0(f))$. We now write $\psi \equiv \psi_1 + \psi_2 + \psi_3$ where $(\partial/\partial x)^2$, $(\partial^2/\partial x \partial y)$ and $(\partial/\partial y)^2$ appear in ψ_1, ψ_2 and ψ_3 respectively. We estimate

$$\begin{aligned}
|\psi_1| &= \left|\left(x - \frac{k}{n}\right)^2 \int_0^1 t\left(\frac{\partial}{\partial x}\right)^2 f\left(\frac{k}{n} + t\left(x - \frac{k}{n}\right), \frac{l}{n} + t\left(y - \frac{l}{n}\right)\right) dt\right| \\
&\leq \left(x - \frac{k}{n}\right)^2 \int_0^1 \frac{\Phi_0(f)\,dt}{|k/n + t(x - k/n)|} \\
&= \Phi_0(f) \int_{k/n}^x \frac{(\xi - k/n)}{\xi}\,d\xi \leq \Phi_0(f) \frac{(x - k/n)^2}{x}
\end{aligned}$$

as for ξ between x and (k/n) $|(\xi - k/n)/\xi| \leq |(x - k/n)/x|$. Similarly, $|\psi_3| \leq \Phi_0(f)(y - l/n)^2/y$ and

$$\begin{aligned}
|\psi_2| &\leq \left|\left(x - \frac{k}{n}\right)\left(y - \frac{l}{n}\right) \int_0^1 t\frac{\partial^2}{\partial x \partial y} f\left(\frac{k}{n} + t\left(x + \frac{k}{n}\right), \frac{l}{n} + t\left(y - \frac{l}{n}\right)\right) dt\right| \\
&\leq \left|\Phi_0(f) \int_0^1 \frac{t(x - k/n)(y - k/n)\,dt}{|k/n + t(x - k/n)|^{1/2} |l/n + t(y - l/n)|^{1/2}}\right| \\
&\leq \Phi_0(f) \left\{\int_{k/n}^x \frac{(\xi - k/n)}{\xi}\,d\xi\right\}^{1/2} \left\{\int_{l/n}^y \frac{(\eta - l/n)}{\eta}\,d\eta\right\}^{1/2} \\
&\leq \Phi_0(f) \left|x - \frac{k}{n}\right| \left|y - \frac{l}{n}\right| / x^{1/2} y^{1/2}.
\end{aligned}$$

We now write $I_i \equiv \sum \psi_i P_{n,k,l}(x,y)$ and, using the above, we can write

$$I_1 \leq \Phi_0(f)x^{-1}B_n\big((\phi_1 - x)^2, x, y\big) \leq \Phi_0(f)/n,$$

$$I_3 \leq \Phi_0(f)y^{-1}B_n\big((\phi_2 - y)^2, x, y\big) \leq \Phi_0(f)/n$$

and

$$I_2 \leq \Phi_0(f)x^{-1/2}y^{-1/2}B_n\big(|\phi_1 - x|\,|\phi_2 - y|, x, y\big) \leq \Phi_0(f)/n.$$

4. Rate of approximation, direct theorem. It is known that if $|B_n(f,x,y) - f(x,y)| = o(1/n)$, even locally, then $f(x,y)$ satisfies in S the elliptic differential equation

$$x(1-x)\frac{\partial^2}{\partial x^2}f - xy\frac{\partial^2}{\partial x \partial y}f + y(1-y)\frac{\partial^2}{\partial x^2}f = 0$$

which for this case would be in the "trivial" class of functions for the present approximation process. Globally the result is still all solutions of the elliptic equation, but since we have the side condition $f(x,y) \in C(S)$, only constants will be admitted. Therefore, the optimal approximation rate is $O(1/n)$.

In the preceding section a condition for $B_n(f,x,y) - f(x,y)$ to behave like $O(1/n)$ in $x + y \leq 2/3$ is related to the behaviour of the derivatives of $f(x,y)$ in $x + y < 3/4$. We now generalize the result to all of S.

We now define the transformations T_i

(4.1) $$T_1(x,y) \equiv (1 - x - y, y), \quad T_2(x,y) = (x, 1 - x - y) \quad \text{and}$$
$$f_i(x,y) \equiv f(T_i(x,y))$$

and the seminorm $\Phi(f)$,

(4.2) $$\Phi(f) = \max_{i=0,1,2} \Phi_i(f) \quad \text{where } \Phi_i(f) = \Phi_0(f_i) \text{ for } i = 1,2$$

where $\Phi_0(f)$ is given in (3.2).

We observe that $\Phi_1(f)$, for example, can be written explicitly as

$$\Phi_1(f) = \max_{x \geq 1/4}\left(\left|(1-x-y)\frac{\partial^2}{\partial x^2}f(x,y)\right|, \left|y\frac{\partial^2}{\partial \xi^2}f(x,y)\right|,\right.$$
$$\left.\left|\sqrt{y(1-x-y)}\,\frac{\partial^2}{\partial \xi \partial x}f(x;y)\right|\right)$$

where $\xi = (1, -1)$.

The result on optimal rate of convergence can be written now as follows:

THEOREM 4.1. *For $f \in C(S)$ which is twice continuously differentiable in the interior of S we have*

$$|B_n(f,x,y) - f(x,y)| \le M(\Phi(f) + \|f\|)/n. \tag{4.3}$$

Proof. We can conclude the proof if we show $|B_n(f,x,y) - f(x,y)| \le M(\Phi_i(f) + \|f\|)/n$ for $x \ge 1/3$ and $y \ge 1/3$ where $i = 1$ and $i = 2$ respectively. We can write

$$\begin{aligned} B_n(f,x,y) &= \sum_{k=0}^{n} \sum_{l=0}^{n-k} f\left(\frac{k}{n}, \frac{l}{n}\right) \frac{n!}{k!l!(n-k-l)!} x^k y^l (1-x-y)^{n-k-l} \\ &= \sum_{l=0}^{n} \sum_{m=0}^{n-l} f\left(1 - \frac{l}{n} - \frac{m}{n}, \frac{l}{n}\right) \frac{n!}{(n-l-m)!l!m!} \\ &\qquad \times x^{n-l-m} y^l (1-x-y)^m. \end{aligned}$$

This implies $B_n(f,x,y) = B_n(f_1,u,v)$ where $(u,v) = T_1(x,y)$, and similarly, $B_n(f,x,y) = B_n(f_2,u,v)$ where $(u,v) = T_2(x,y)$. We can now apply Lemma 3.2 to the domains $x \ge 1/3$ and $y \ge 1/3$ as well. Therefore, $\Phi_0(f_1) \le M$ implies

$$|B_n(f_1,u,v) - f_1(u,v)| \le M_1(\Phi_0(f_1) + \|f_1\|)/n$$

for $u + v \le 2/3$, or $x \ge 1/3$. We have

$$\left| u \frac{\partial^2}{\partial u^2} f_1(u,v) \right| = \left| (1-x-y) \frac{\partial^2}{\partial x^2} f(x,y) \right|;$$

moreover,

$$\left| v \frac{\partial^2}{\partial v^2} f_1(u,v) \right| = \left| y \frac{\partial^2}{\partial \xi^2} f(x,y) \right|$$

as

$$\left| \frac{\partial}{\partial v} f_1(u,v) \right| = \left| \frac{\partial}{\partial \xi} f(x,y) \right|$$

and similarly

$$\left| \sqrt{vu}\, \frac{\partial^2}{\partial v \partial u} f_1(u,v) \right| = \left| \sqrt{y(1-x-y)}\, \frac{\partial^2}{\partial \xi \partial x} f(x,y) \right|.$$

DEFINITION. The subspace A of $C(S)$ is the collection of $f \in C(S)$ for which the seminorm $\Phi(f) \equiv \max(\Phi_0(f), \Phi_1(f), \Phi_2(f))$ is bounded where $\Phi(f)$ and $\Phi_i(f)$ are defined in (4.2). We assume that f is locally twice differentiable in the interior of S and that $f, (\partial/\partial x)f$ and $(\partial/\partial y)f$ are locally absolutely continuous in both variables.

DEFINITION. The interpolation space $(C, A)_\alpha$ is the collection of all $f \in C(S)$ for which $K(f, t) \le M(f)t^\alpha$ for all $t \le t_0$ where $K(f, t) \equiv \inf_{g \in A}(\|f - g\| + t\Phi(g))$.

THEOREM 4.2. *For $f \in (C(S), A)_\alpha$, $0 < \alpha < 1$, we have*

$$\|B_n(f, x, y) - f(x, y)\| \le M_1 n^{-\alpha}. \tag{4.4}$$

Proof. For $t = 1/n$ and $K(f, 1/n) \le M(F)(1/n)^\alpha$ we have $g \in A$ such that $\|f - g\| + n^{-1}\Phi(g) \le 2M(f)n^{-\alpha}$ or $\|f - g\| \le 2M(f)n^{-\alpha}$ and $\Phi(g) \le 2M(f)n^{1-\alpha}$. We write now

$$\begin{aligned}
&|B_n(f, x, y) - f(x, y)| \\
&\le |B_n(f - g, x, y) - f(x, y) + g(x, y)| + |B_n(g, x, y) - g(x, y)| \\
&\le |B_n(f - g, x, y)| + \|f - g\| + M\big(\Phi(g) + |g|\big)/n \\
&\le 2\|f - g\| + M\big(\Phi(g) + 2\|f\|\big)/n \\
&\le 2L(f)(1 + M)n^{-\alpha} + 2\|f\|/n.
\end{aligned}$$

We used $\|g\| \le 2\|f\|$ which follows the definition of the interpolation space. This concludes the proof of Theorem 4.2.

In §6 we will characterize $(C(S), A)_\alpha$ using smoothness properties of $f \in (C(S), A)_\alpha$.

5. The inverse result. We will prove in this section that the rate of approximation $O(n^{-\alpha})$ implies $f \in (C, A)_\alpha$.

THEOREM 5.1. *For $f \in C(S)$ and $\alpha < 2$, $\|f(x, y) - B_n(f, x, y)\| \le Mn^{-\alpha}$, implies $f \in (C(S), A)_\alpha$.*

Proof. Obviously $B_n(f, x, y)$ belongs to C^2 locally in the interior of S. Therefore,

$$K(f, t) \le \|f(x, y) - B_n(f, x, y)\| + t\Phi(B_n(f)).$$

If we prove the following two inequalities:

$$\Phi(B_n(f)) \le Ln\|f\| \tag{5.1}$$

and

$$\Phi(B_n(f)) \le L\Phi(f) \tag{5.2}$$

we will have $K(f,t) \le Mn^{-\alpha} + tnLK(f, n^{-1})$. The latter inequality combined with the fact that $K(f,t_0) \le \|f\|$ and the established procedure of Berens and Lorentz **[1]** yield $K(f,t) \le M_1 t^{\alpha}$. Therefore, we will finish our proof when (5.1) and (5.2) will be established in Lemmas 5.2 and 5.3 respectively.

LEMMA 5.2. *For* $f \in C(S)$ *we have* $\Phi(B_n(f)) \le Ln\|f\|$.

Proof. We first show $\Phi_0(B_n(f)) \le Ln\|f\|$. We use (2.6) and (2.7) to show

$$\left|\left(\frac{\partial}{\partial x}\right)^2 B_n(f,x,y)\right| \le 4n^2\|f\| \quad \text{and} \tag{5.3}$$

$$\left|\frac{\partial^2}{\partial x \partial y} B_n(f,x,y)\right| \le 4n^2\|f\|,$$

and the same for $(\partial^2/\partial y^2)B_n(f)$. Now we use (2.3) to obtain

$$\begin{aligned}
&\left|\left(\frac{\partial}{\partial x}\right)^2 B_n(f,x,y)\right| \\
&= \left|\frac{n^2}{x^2(1-x-y)^2}\sum_{k=0}^{n}\sum_{l=0}^{n-k} f\left(\frac{k}{n},\frac{l}{n}\right)P_{n,k,l}(x,y)\right. \\
&\qquad \times\left\{\frac{k}{n}\left(\frac{k}{n}-\frac{l}{n}\right)(1-x-y)^2 - 2\frac{k}{n}\left(1-\frac{k}{n}-\frac{l}{n}\right)x(1-x-y)\right. \\
&\qquad\qquad \left.\left. +\left(1-\frac{k}{n}-\frac{l}{n}\right)\left(1-\frac{k}{n}-\frac{l}{n}-\frac{1}{n}\right)x^2\right\}\right| \\
&= \left|\frac{n^2}{x^2(1-x-y)^2}\sum_{k=0}^{n}\sum_{l=0}^{n-l} f\left(\frac{k}{n},\frac{l}{n}\right)P_{n,k,l}(x,y)\right. \\
&\qquad \times\left\{\left(\frac{k}{n}-x\right)^2(1-x-y)^2 + \left(\frac{k}{n}+\frac{l}{n}-x-y\right)^2 x^2\right. \\
&\qquad -\frac{1}{n}\frac{k}{n}(1-x-y)^2 - \frac{1}{n}\left(1-\frac{k}{n}-\frac{l}{n}\right)x^2 \\
&\qquad\qquad \left.\left. -2\left(\frac{k}{n}-x\right)\left(x+y-\frac{k}{n}-\frac{l}{n}\right)x(1-x-y)\right\}\right|
\end{aligned}$$

(continues)

$$\leq \|f\| \left\{ \frac{n^2}{x^2} B_n\left((\phi_1 - x)^2, x, y\right) \right.$$

$$+ \frac{n^2}{(1-x-y)^2} B_n\left((\phi_3 - 1 + x + y)^2, x, y\right)$$

$$+ \frac{n}{x^2} B_n(\phi_1, x, y) + \frac{n}{(1-x-y)^2} B_n(\phi_3, x, y)$$

$$\left. + \frac{2n^2}{x(1-x-y)} B_n\left(|\phi_1 - x|\,|\phi_3 - 1 + x + y|, x, y\right) \right\}$$

$$\leq \left\{ \frac{n(1-x)}{x} + \frac{n(x+y)}{1-x-y} + \frac{n}{x} + \frac{n}{1-x-y} \right.$$

$$\left. + \frac{2n^2}{x(1-x-y)} \left(\frac{x(1-x)}{n}\right)^{1/2} \left(\frac{(1-x-y)(x+y)}{n}\right)^{1/2} \right\} \|f\|$$

$$\leq 3\left[\frac{n}{x} + \frac{n}{1-x-y}\right] \|f\|.$$

Actually we proved the part of the estimate of $(\partial^2/\partial x^2) B_n(f, x, y)$ in $x + y < 3/4$ where we use for $x \leq 1/n$ (or $1 - x - y \leq 1/n$)

$$\left| \left(\frac{\partial}{\partial x}\right)^2 B_n(f, x, y) \right| \leq 4n^2 \|f\|$$

and for $x \geq 1/n$ and $x + y \leq 3/4$ (or $1 - x - y \geq 1/n$ and $x \geq 1/4$) the estimate

$$\left| \left(\frac{\partial}{\partial x}\right)^2 B_n(f, x, y) \right| \leq 3\left[\frac{n}{x} + \frac{n}{1-x-y}\right] \|f\|$$

with $L \leq 15$. Of course, the estimate $|(\partial/\partial y)^2 B_n(f, x, y)|$ is similar. To estimate $(\partial/\partial x \partial y) B_n(f, x, y)$ in addition to using (5.3) we use (2.4) and, after some computation and using the Cauchy-Schwarz inequality, we write

$$\begin{aligned}
&\left|\frac{\partial^2}{\partial x \partial y} B_n(f, x, y)\right| \\
&\quad = \frac{n^2}{xy(1-x-y)} \left| \sum_{k=0}^{n} \sum_{l=0}^{n-k} f\left(\frac{k}{n}, \frac{l}{n}\right) P_{n,k,l}(x, y) \right. \\
&\qquad \times \left\{ \left(\frac{k}{n} - x\right)\left(\frac{l}{n} - y\right)(1-x-y)^2 \right. \\
&\qquad + \left(x + y - \frac{k}{n} - \frac{l}{n}\right)^2 xy - \left(1 - \frac{k}{n} - \frac{l}{n}\right)\frac{1}{n} xy \\
&\qquad - \left(\frac{k}{n} - x\right)\left(x + y - \frac{k}{n} - \frac{l}{n}\right) y(1-x-y) \\
&\qquad \left. \left. - \left(\frac{l}{n} - y\right)\left(x + y - \frac{k}{n} - \frac{l}{n}\right) x(1-x-y) \right\} \right| \\
&\quad \le \|f\| \left\{ \frac{n^2}{xy} \left(\frac{x(1-x)}{n}\right)^{1/2} \left(\frac{y(1-y)}{n}\right)^{1/2} \right. \\
&\qquad + \frac{n^2}{(1-x-y)^2} \frac{(x+y)(1-x-y)}{n} \\
&\qquad + \frac{n}{(1-x-y)} + \frac{n^2}{x(1-x-y)} \\
&\qquad \left. \times \left(\frac{x(1-x)}{n}\right)^{1/2} \left(\frac{(x+y)(1-x-y)}{n}\right)^{1/2} \right\}.
\end{aligned}$$

Therefore for $x + y \le 3/4$ we have

$$\left|\frac{\partial^2}{\partial x \partial y} B_n(f, x, y)\right| \le \frac{L\|f\|n}{\sqrt{xy}}.$$

We now have essentially proved the result for $\Phi_0(B_n(f))$. To prove the result for $\Phi_1(B_n(f))$ we use the transformation $u = 1 - x - y$ and $v = y$ and the identity $B_n(f, 1 - u - v, v) = B_n(f_1, u, v)$ where $f_1(u, v) = f(1 - u - v, v)$ and following (4.1) and (4.2), $\Phi_1(f) = \Phi_0(f_1)$ and $\|f\| = \|f_1\|$. Similarly, we prove the result for $\Phi_2(B_n(f))$.

LEMMA 5.3. *For $f \in C(S)$, $f \in C^2$ locally in the interior of S and $f \in A$ we have $\Phi(B_n(f)) \le L\Phi(f)$.*

Proof. We first examine the expression Φ_0^* for which we take the maximum only on the region $x + y \le 2/3$ (rather than on $x + y \le 3/4$ as done for Φ_0). This would not matter, as a short computation shows

$$\Phi(g) = \max_{i=0,1,2} (\Phi_i(g)) \le C \max_{i=0,1,2} \left(\Phi_i^*(g)\right)$$

where Φ_i^* is maximum on the regions $x + y \le 2/3$, $x \ge 1/3$ and $y \ge 1/3$ for $i = 0$, $i = 1$ and $i = 2$ respectively.

Let us denote $\Delta_{he}f(\cdot) = f(\cdot + he) - f(\cdot)$. For $e_1 = (1,0)$, $k \ge 1$, $l \ge 0$ and $k + l \le 3n/4$ we have

$$n^2\left|\Delta^2_{e_1/n}f\left(\frac{k}{n}, \frac{l}{n}\right)\right| \max_{k/n\le x\le(k+2)/n} \frac{n}{k}\left|x\left(\frac{\partial}{\partial x}\right)^2 f\left(x, \frac{l}{n}\right)\right| \le \frac{n\Phi_0(f)}{k}$$

Similarly, for $l \ge 1$ and $k \ge 0$ we have

$$n^2\left|\Delta^2_{e_2/n}f\left(\frac{k}{n}, \frac{l}{n}\right)\right| \le \frac{n\Phi_0(f)}{k}$$

where $e_2 = (0,1)$ and for $k \ge 1$ and $l \ge 1$

$$n^2\left|\Delta_{e_1/n}\Delta_{e_2/n}f\left(\frac{k}{n}, \frac{l}{n}\right)\right| \le \frac{n\Phi_0(f)}{\sqrt{kl}}.$$

For $k = 0$ we can write

$$n^2\left|\Delta^2_{e_1/n}f\left(0, \frac{l}{n}\right)\right| \le 2n^2\left\{\int_0^{2/n} x\left|\left(\frac{\partial}{\partial x}\right)^2 f\left(x, \frac{l}{n}\right)\right| dx\right\} \le 4n\Phi_0(f).$$

For $k = l = 0$ we have

$$\begin{aligned} n^2\left|\Delta_{e_1/n}\Delta_{e_2/n}f(0,0)\right| &\le n^2\int_0^{1/n}\int_0^{1/n}\left|\frac{\partial^2}{\partial x\partial y}f(x,y)\right| dx\,dy \\ &\le \Phi_0(f)n^2\int_0^{1/n}\int_0^{1/n}(xy)^{-1/2}\,dx\,dy \\ &= \Phi_0(f)n. \end{aligned}$$

For $k = 0$ and $l \ne 0$ (or similarly for $l = 0$ and $k \ne 0$) we have

$$\begin{aligned} n^2\left|\Delta_{e_1/n}\Delta_{e_2/n}f\left(0, \frac{l}{n}\right)\right| &\le n^2\int_0^{1/n}\int_{l/n}^{(l+1)/n}\left|\frac{\partial^2}{\partial x\partial y}f(x,y)\right| dx \\ &\le n\Phi_0(f)\left(\frac{n}{l}\right)^{1/2}\int_0^{1/n} x^{-1/2}\,dx \le 2nl^{-1/2}\Phi_0(f). \end{aligned}$$

Using (2.6) for $x + y \le 2/3$ the fact that, for $k \ge 1$, $k/(k+1) \le 2$, and Lemma 3.2 of [2], and (3.6), we have

$$\begin{aligned}\left|\frac{\partial^2}{\partial x^2} B_n(f, x, y)\right| &\le 4\Phi_0(f) \sum_{k=2}^{n} \sum_{l=0}^{n-k} P_{n-2,k-2,l}(x, y) \frac{n}{k-1} \\ &\le 4\frac{n}{n-2}\Phi_0(f) \sum_{k=0}^{n-2} \sum_{l=0}^{n-k-2} P_{n-2,k,l}(x, y) \frac{n-2}{k+1} \\ &\le 8\Phi_0(f) \sum_{k=0}^{n-2} P_{n-2,k}(x) \frac{n-2}{k+1} \le 8\Phi_0(f)\frac{1}{x}\end{aligned}$$

(for $n < 2(n-2)$ or $n > 4$). Using (2.7) for $x + y \le 2/3$, and the estimates above we have

$$\begin{aligned}\left|\frac{\partial^2}{\partial x \partial y} B_n(f, x, y)\right| &\le 8\frac{n}{n-2}\Phi_0(f) \sum_{k=1}^{n} \sum_{l=1}^{n-k} P_{n-2,k-1,l-1}(x, y) \frac{n-2}{\sqrt{l}\sqrt{k}} \\ &\le 16\Phi_0(f)\left(\sum_{k=0}^{n-2} \sum_{l=0}^{n-k} P_{n-2,k,l}(x, y) \frac{n-2}{k+1}\right)^{1/2} \\ &\quad\times \left(\sum_{k=0}^{n-2} \sum_{l=0}^{n-k} P_{n-2,k,l}(x, y) \frac{n-2}{l+1}\right)^{1/2} \\ &\le 16\Phi_0(f)\frac{1}{\sqrt{x}\sqrt{y}}.\end{aligned}$$

This completes the proof that $\Phi_0^*(B_n(f)) \le 16\Phi_0(f)$. To prove that the same result is true for the relation between $\Phi_1^*(B_n(f))$ and $\Phi_1(f)$ we recall again the transformation $u = 1 - x - y$ and $v = y$ for $B_n(f, x, y)$ and that

$$B_n(f, 1 - u - v, v) = B_n(f_1, u, v)$$

where $f_1(u, v) \equiv f(1 - u - v, v)$.

6. The equivalence relation. We will use the symmetric difference $\Delta_{he} f(v) = f(v + \frac{1}{2}he) - f(v - \frac{1}{2}he)$ and $\Delta^2_{he} f(v) = \Delta_{he}(\Delta_{he} f(v))$ for a vector v, $h \in R_+$, and a fixed vector e. Actually, earlier we used forward differences because of convenience as they naturally appeared in the derivatives of Bernstein polynomials. However, we will use only the final estimates achieved earlier which do not involve difference (forward or otherwise) and therefore, using the present form should cause neither difficulty nor confusion.

We let $e_1 = (1,0)$, $e_2 = (0,1)$ and $e_3 = (1,-1)$ and may now state our equivalence (inverse) results.

THEOREM 6.1. *The function f with domain S belongs to $(C, A)_\alpha$, where A was given in* §4, *if and only if for $(x, y) \in S$ we have*:

(a) *In case $x + y \le 3/4$: $|x^\alpha \Delta^2_{he_1} f| \le Mh^{2\alpha}$ for $x \ge h$ and $y \ge 0$; $|y^\alpha \Delta^2_{he_2} f| \le Mh^{2\alpha}$ for $y \ge h$ and $x \ge 0$; and $|(xy)^{\alpha/2}\Delta_{he_1}\Delta_{ke_2} f| \le M(hk)^\alpha$ for $x \ge h/2$ and $y \ge k/2$.*

(b) *In case $x \ge 1/4$: $|(1 - x - y)^\alpha \Delta^2_{he_1} f| \le Mh^{2\alpha}$ for $1 - x - y \ge h$ and $y \ge 0$, $|y^\alpha \Delta^2_{he_3} f| \le Mh^{2\alpha}$ for $y \ge h$ and $1 - x - y \ge 0$; and*

$$\left|((1 - x - y)y)^{\alpha/2}\Delta_{he_1}\Delta_{ke_3} f\right| \le M(hk)^\alpha$$

for $1 - x - y \ge h/2$ and $y \ge k/2$. (That is, (a) *is valid for $f_1(x, y) = f(T_1(x, y))$.)*

(c) *In case $y \ge 1/4$ the roles of x and y in* (b) *are interchanged.* ((a) *is valid for $f_2(x, y) = f(T_2(x, y))$.)*

Note that with the above restrictions if h and k are small enough, say $h, k \le 1/16$, all points mentioned will be in S.

We can also have the following somewhat different description of $(C, A)_\alpha$.

THEOREM 6.2. *The function $f(x, y)$ belongs to $(C, A)_\alpha$ if and only if the following conditions are satisfied.*

(a) *For $x + y \le 3/4$: $|\Delta_{h\sqrt{x}e_1} f| \le Mh^{2\alpha}$ for $x \ge h^2$ and $y \ge 0$; $|\Delta^2_{h\sqrt{y}e_2} f| \le Mh^{2\alpha}$ for $y \ge h^2$ and $x \ge 0$; and $|\Delta_{h\sqrt{x}e_1}\Delta_{k\sqrt{y}e_2} f| \le M(hk)^\alpha$ for $x \ge \frac{1}{4}h^2$ and $y \ge \frac{1}{4}k^2$.*

(b) *Condition* (a) *is valid for $f_1(x, y)$ where $f_1(x, y) \equiv f(1 - x - y, y)$.*

(c) *Condition* (a) *is valid for $f_2(x, y)$ where $f_2(x, y) \equiv f(x, 1 - x - y)$.*

Proof of Theorem 6.2 *assuming Theorem* 6.1. We have to show that (a), (b) and (c) of Theorem 6.1 and 6.2 are equivalent but for a fixed (x, y) (for (a) say) $h = h_1$ of Theorem 6.2 correspond to $h = h_1\sqrt{x}$ in Theorem 6.1 and they are the same conditions etc.

This phenomenon is particular to $C(S)$, in L_p such forms would not be equivalent (see Totik **[9]** and **[10]**). The second form was introduced here too as this and not the form in Theorem 6.1 is the likely candidate for generalizing to L_p.

Proof of Theorem 6.1. We first show that if $f \in (C, A)_\alpha$, the 3 conditions (a), (b) and (c) are satisfied. We observe that it is enough to show those conditions for $x + y \le 2/3$, $x \ge 1/3$ and $y \ge 1/3$ instead of showing them for $x + y \le 3/4$, $x \ge 1/4$ and $y \ge 1/4$ for (a), (b) and (c) respectively. We will actually show just condition (a) but the transformations mentioned in (4.1) T_i imply $\Phi_i(f) = \Phi_0(f_i)$ and $B_n(f, x, y) = B_n(f_i, u, v)$, where $T_i(x, y) = (u, v)$, will imply that it is sufficient. For $f \in (C, A)_\alpha$ we have, for any t, $f = f_{t,1} + f_{t,2}$ where $|x(\partial/\partial x)^2 f_{t,2}(x, y)| \le Mt^{\alpha-1}$ and $|f_{t,1}(x, y)| \le Mt^\alpha$ where M is independent of t. Choosing $t = h^2/x$, we have $|\Delta^2_{he_1} f_{t,1}(x, y)| \le 4Mh^{2\alpha}/x^\alpha$. We can now write

$$I \equiv \left|\Delta^2_{he_1} f_{t,2}(x, y)\right| \le 2\max\left|\int_x^{x\pm h} (u - x \mp h)\frac{\partial^2}{\partial u^2} f_{t,2}(u, y)\, du\right|$$

$$\le 2Mt^{\alpha-1}\max\left|\int_x^{x\pm h} \frac{(u - x \mp h)}{u}\, du\right|.$$

Since $|\int_x^{x+h}(u - x - h)/u\, du| \le h^2/2x$,

$$\left|\int_{x-h}^{x} \frac{(u - x + h)}{u}\, du\right| < \frac{h^2}{2(x - h)} \le \frac{h^2}{4x} \quad \text{for } x > 2h$$

and

$$\left|\int_{x-h}^{x} \frac{(u - x + h)}{u}\, du\right| \le \int_0^x du \le 2h \quad \text{for } x \le 2h,$$

and using the choice $t = h^2/x$, we have in all three cases $I \le M_1(h^2/x)^\alpha$. To estimate $|(xy)^{\alpha/2}\Delta_{he_1}\Delta_{ke_2} f(x, y)|$ we choose $f_{t,1}$ and $f_{t,2}$ to match $t = hk/(xy)^{1/2}$ for which

$$|f_{t,1}(x, y)| \le Mt^\alpha \quad \text{and} \quad \left|\sqrt{xy}\,\frac{\partial^2}{\partial x \partial y} f_{t,2}(x, y)\right| \le Mt^{\alpha-1}.$$

Therefore, we have

$$\left|\Delta_{he_1}\Delta_{ke_2} f_{t,1}(x, y)\right| \le 4M(hk)^\alpha/(xy)^{\alpha/2}$$

and

$$I_2 \equiv \left|\Delta_{he_1}\Delta_{ke_2} f_{t,2}(x, y)\right| = \left|\int_{x-h/2}^{x+h/2}\int_{y-k/2}^{y+k/2} \frac{\partial^2}{\partial u \partial v} f_{t,2}(u, v)\, du\, du\right|$$

$$\le M\left|\int_{x-h/2}^{x+h/2}\int_{y-k/2}^{y+k/2} \frac{du\, dv}{\sqrt{uv}}\right|.$$

The estimate of I_2 can now be written as follows: for $x \geq h$ and $y \geq k$

$$I_2 \leq 2Mt^{\alpha-1}hk/\sqrt{xy} = 2M(hk)^{\alpha}/(xy)^{\alpha/2};$$

for $x \geq h$ and $y < k$

$$\begin{aligned} I_2 &< \sqrt{2}\,Mt^{\alpha-1}\frac{h}{\sqrt{x}}\int_0^{3k/2}\frac{dv}{\sqrt{v}} \leq 4Mt^{\alpha-1}\frac{h}{\sqrt{x}}\sqrt{k} \\ &\leq 4M\frac{(hk)^{\alpha}}{(xy)^{\alpha/2}}\frac{\sqrt{k}\sqrt{k}}{y} \leq 4M\frac{(hk)^{\alpha}}{(xy)^{\alpha/2}}; \end{aligned}$$

for $x < h$ and $y \geq k$

$$I_2 \leq 4M(hk)^{\alpha}/(xy)^{\alpha/2},$$

and for $x < h$ and $y < k$

$$I_2 \leq Mt^{\alpha-1}\int_0^{3h/2}\frac{du}{\sqrt{u}}\int_0^{3k/2}\frac{dv}{\sqrt{v}} < 8Mt^{\alpha-1}(hk)^{1/2} \leq 8M\frac{(hk)^{\alpha}}{(xy)^{\alpha/2}}.$$

All other estimates follow similarly, and therefore $f \in (C, A)_{\alpha}$ implies (a), (b), and (c).

We now prove that conditions (a), (b) and (c) imply that for every t there exists a function g_t such that $\|f - g_t\| \leq Mt^{\alpha}$ and $\Phi(g_t) \leq Mt^{\alpha-1}$. We first observe that it is enough to find such functions g_t that will satisfy $\|f - g_t\| \leq Mt^{\alpha}$ and $\Phi_i(g_t) \leq Mt^{\alpha-1}$ for $i = 0, 1, 2$, that is, find g_t that will fit Φ_0, then a function g_t that will fit Φ_1 etc. This is possible since we can have $\psi_i \geq 0$ for $i = 0, 1, 2$ satisfying $\sum \psi_i(x, y) = 1$, $\psi_i \in C^{\alpha}$, $\psi_0 = 1$ in $x + y < 1/3$ and $\psi_0 = 0$ in $x + y \geq 2/3$, $\psi_1 = 1$ for $x \geq 2/3$ and $\psi_1 = 0$ for $x \leq 1/3$, and $\psi_2 = 1$ in $y \geq 2/3$ and $\psi_2 = 0$ for $y \leq 1/3$. We now only have to construct g_t to fit one of the functionals, Φ_0 say, as all of them can be achieved from the same construction if we use the affine transformation discussed earlier first, then construct the function and then use the inverse transformation which is actually the same transformation. The construction of the function g_t follows our method in **[2]** and **[4]** but here we have the added difficulty of dealing with the two-dimensional problem (which makes it more interesting). The multidimensional problem is treated in a very similar way. As a preliminary to our construction we define

$$\begin{aligned} (6.1)\quad & F_{t_1,t_2}(x, y) \\ &= \left(\frac{2}{t_1}\right)^2\left(\frac{2}{t_2}\right)^2\int_0^{t_2/2}\int_0^{t_2/2}\int_0^{t_1/2}\int_0^{t_1/2}[2f(x + u_1 + u_2,\, y + v_1 + v_2) \\ &\qquad -f(x + 2u_1 + 2u_2,\, y + 2v_1 + 2v_2)]\,du_1\,du_2\,dv_1\,dv_2. \end{aligned}$$

Elementary manipulations yield

$$(6.2)\quad \begin{aligned}&\Delta^2_{\tau_1 e_1+\tau_2 e_2} f(x+\tau_1,\, y+\tau_2)\\ &\quad\equiv f(x,y)-2f(x+\tau_1,\, y+\tau_2)+f(x+2\tau_1,\, y+2\tau_2)\\ &\quad= \Delta^2_{\tau_1 e_1} f(x+\tau_1,\, y)+\Delta^2_{\tau_2 e_2} f(x+2\tau_1,\, y+\tau_2)\\ &\qquad+2\Delta_{\tau_1 e_1}\Delta_{\tau_2 e_2} f\left(x+\tfrac{3}{2}\tau_1,\, y+\tfrac{1}{2}\tau_2\right).\end{aligned}$$

This will imply

$$\begin{aligned}&\left|f(x,y)-F_{t_1,t_2}(x,y)\right|\\ &\quad\le\left(\frac{2}{t_1}\right)^2\int_0^{t_1/2}\int_0^{t_1/2}\left(\frac{u_1+u_2}{\sqrt{x+u_1+u_2}}\right)^{2\alpha} du_1\,du_2\\ &\qquad+\left(\frac{2}{t_2}\right)^2\int_0^{t_2/2}\int_0^{t_2/2}\left(\frac{v_1+v_2}{\sqrt{y+v_1+v_2}}\right)^{2\alpha} dv_1\,dv_2\\ &\qquad+2\left(\frac{2}{t_1}\right)^2\left(\frac{2}{t_2}\right)^2\int_0^{t_1/2}\int_0^{t_1/2}\int_0^{t_2/2}\int_0^{t_2/2}\left(\frac{v_1+v_2}{\sqrt{y+\frac{1}{3}(v_1+v_2)}}\right)^{\alpha}\\ &\qquad\qquad\times\left(\frac{u_1+u_2}{\sqrt{x+\frac{3}{2}(u_1+u_2)}}\right)^{\alpha} du_1\,du_2\,dv_1\,dv_2.\end{aligned}$$

Therefore

$$(6.3)\quad \begin{aligned}&\left|f(x,y)-F_{t_1,t_2}(x,y)\right|\\ &\quad\le M\left\{\min\left(t_1^{2\alpha}/x^{\alpha},\, t_1^{\alpha}\right)+\min\left(t_2^{2\alpha}/y^{\alpha},\, t_2^{\alpha}\right)\right.\\ &\qquad\left.+2\min\left((t_1t_2)^{\alpha}/(xy)^{\alpha/2},\, t_1^{\alpha}t_2^{\alpha/2}/x^{\alpha/2},\, t_2^{\alpha}t_1^{\alpha/2}/y^{\alpha/2},\, (t_1t_2)^{\alpha/2}\right)\right\}.\end{aligned}$$

Following the standard techniques of Stekelov-type averages, we have

$$\begin{aligned}&\left(\frac{\partial}{\partial x}\right)^2 F_{t_1,t_2}(x,y)\\ &\quad=\left(\frac{4}{t_1t_2}\right)^2\int_0^{t_2/2}\int_0^{t_2/2}\left\{2\Delta^2_{t_1t_2/2} f(x+t_1/2,\, y+v_1+v_2)\right.\\ &\qquad\qquad\left.-\Delta^2_{t_1e_1} f(x+t_1,\, y+2v_1+2v_2)\right\} dv_1\,dv_2\end{aligned}$$

and, therefore, using the conditions in the theorem,

$$(6.4)\qquad \left|\left(\frac{\partial}{\partial x}\right)^2 F_{t_1,t_2}(x,y)\right|\le Mt_1^{-2}\min\left(\frac{t_1^{2\alpha}}{x^{\alpha}},\, t_1^{\alpha}\right).$$

Similarly,

$$\left|\left(\frac{\partial}{\partial y}\right)^2 F_{t_1,t_2}(x,y)\right| \le M t_2^{-2} \min\left(\frac{t_2^{2\alpha}}{y^\alpha}, t_2^\alpha\right)$$

and

$$(6.5)\quad \left|\frac{\partial^2}{\partial x \partial y} F_{t_1,t_2}(x,y)\right|$$
$$\le M(t_1t_2)^{-1}\min\left\{\left(\frac{t_1t_2}{\sqrt{xy}}\right)^\alpha, \left(\frac{t_1}{\sqrt{x}}\right)^\alpha t_2^{\alpha/2}, \left(\frac{t_2}{\sqrt{y}}\right)^\alpha t_1^{\alpha/2}, (t_1t_2)^{\alpha/2}\right\}.$$

To construct the functions in question we remember that the function that would fit Φ_0 does not have even to be defined for $x + y \ge 2/3$, etc. We restrict t so that for $x + y \le 2/3$, $x + 2t + y + 2t \le 3/4$ or in other words $t < 1/50$; this is not a serious restriction as the interesting part is when t tends to zero, and otherwise it just modifies the constants. We choose $\psi(x)$ to satisfy $\psi(x) = 1$ for $x < 1/4$, $\psi(x) = 0$ for $x \ge 3/4$, $\psi(x)$ decreasing and $\psi(x) \in C^\alpha$. We define also $\psi_l(x) \equiv \psi(4^l x)$. We are now able to define in $x + y \le 2/3$, $f_{t,2}$ for Φ_0 which we denote by g_t and $f_{t,1}$ will therefore be just $f - f_{t,2}$. For $2^{-l-1} < t \le 2^{-l}$ (and of course $t < 1/50$), we write

$$(6.6)\quad g_{t^2}(x,y) = \sum_{k=0}^{l-1}\sum_{m=0}^{l-1} F_{t2^{-k},t2^{-m}}(x,y)\psi_k(x)\psi_m(y)$$
$$\times(1-\psi_{k+1}(x))(1-\psi_{m+1}(y))$$
$$+\sum_{k=0}^{l-1} F_{t2^{-k},t2^{-l}}(x,y)\psi_l(y)\psi_k(x)(1-\psi_{k+1}(x))$$
$$+\sum_{m=0}^{l-1} F_{t2^{-l},t2^{-m}}(x,y)\psi_l(x)\psi_m(y)(1-\psi_{m+1}(y))$$
$$+F_{t2^{-l},t2^{-l}}(x,y)\psi_l(x)\psi_l(y)$$
$$\equiv \sum_{k=0}^{l}\sum_{m=0}^{l} F_{t2^{-k},t2^{-m}}(x,y)\Psi_{k,m}(x,y).$$

Of course, the preparation up to now was in order that for $x + y \le 2/3$ we have $|f(x,y) - g_{t^2}(x,y)| \le Mt^{2\alpha}$. We observe first that if in the definition of $g_{t^2}(x,y)$ in (6.6) $f(x,y)$ would replace $F_{t2^{-k},t2^{-m}}$ for all m and k (including $m = l$ and $k = l$), $f(x,y)$ would also replace $g_{t^2}(x,y)$

on the other side, or in other words in the region prescribed the coefficients are a partition of unity. We observe that for every x and y at most four terms are different from zero. Moreover, for $4^{-r-1} \leq x \leq 4^{-r}$, $\psi_k(x)(1 - \psi_{k+1}(x)) \neq 0$ only at most for $k \leq r + 1$ and $r - 1 < k$, or only $k = r + 1$ and $k = r$ are possible (not always both). For $x \leq 4^{-l-1}$, $\psi_l(x)$ is the only non-zero coefficient of $F_{t2^{-k},t2^{-m}}$. Therefore, using (6.3) for $t_1 = 2^{-k}t$ and $t_2 = 2^{-m}t$ when $x \sim 4^{-k}$ and $y \sim 4^{-m}$ respectively, and for $t_1 = 2^{-l}t$ and $t_2 = 2^{-l}t$ when $x \leq 4^{-l-1}$ and $y \leq 4^{-l-1}$ respectively, we complete the estimate of $|f - g_{t^2}|$ by recalling

$$0 \leq \psi_k(x)\psi_m(y)(1 - \psi_{k+1}(x))(1 - \psi_{m+1}(y)) \leq 1$$

and only at most four of them are different from 0 at any point (x, y).

To estimate $\Phi_0(g_{t^2})$, we first estimate $x(\partial/\partial x)^2 g_{t^2}(x, y)$. (We should get $|x(\partial/\partial x)^2 g_{t^2}(x, y)| \leq Mt^{2\alpha-2}$.) We write

$$\begin{aligned} x\left(\frac{\partial}{\partial x}\right)^2 g_{t^2}(x, y) &= \sum_{k=0}^{l} \sum_{m=0}^{l} x\left(\frac{\partial}{\partial x}\right)^2 F_{t2^{-k},t2^{-m}}(x, y)\Psi_{k,m}(x, y) \\ &\quad + \sum_{k=0}^{l} \sum_{m=0}^{l} 2x\left(\frac{\partial}{\partial x}\right)^2 F_{t2^{-k},t2^{-m}}(x, y)\left(\frac{\partial}{\partial x}\right)\Psi_{k,m}(x, y) \\ &\quad + \sum_{k=0}^{l} \sum_{m=0}^{l} x(F_{t2^{-k},t2^{-m}}(x, y)\left(\frac{\partial}{\partial x}\right)^2 \Psi_{k,m}(x, y) \\ &\equiv J_1 + J_2 + J_3. \end{aligned}$$

First we estimate J_1. Recalling that at most four terms in the sum are different from zero, we have only to estimate a term of J_1. The function $\Psi_{k,m}(x, y)$ satisfies $0 \leq \Psi_{k,m} \leq 1$ and $\Psi_{k,m} \neq 0$ implies $x \sim 4^{-k}$ and $y \sim 4^{-m}$ unless k and/or m are equal to l, in which case $x \leq 3 \cdot 4^{-l-1}$ and/or $y \leq 3 \cdot 4^{-l-1}$ respectively. In both cases using (6.4) the term is smaller than $M_1 t^{2\alpha-2}$, as for $x \sim 4^{-k}$

$$Mx(t2^{-k})^{-2}\frac{(t2^{-k})^{2\alpha}}{x^\alpha} \leq M_1 t^{-2} t^{2\alpha}$$

and for $x \leq 3 \cdot 4^{-l-1}$

$$Mx(t2^{-l})^{-2}(t2^{-l})^\alpha \leq 3Mt^{-2} \cdot t^\alpha 2^{-l\alpha} \leq M_1 t^{2\alpha-2}.$$

Estimating J_2 and J_3, we have to distinguish between two situations: (A) $\Psi_{k,m}(x, y)$ is constant in x, for which points (x, y) the corresponding summands of J_2 and J_3 are equal to zero. (B) $4^{-k-1} < x < 34^{-k-1}$ in which case

$$\Psi_{k,m}(x, y) = \psi_k(x)\psi_m(y)(1 - \psi_{m+1}(y))$$

and

$$\Psi_{k-1,m}(x, y) = (1 - \psi_k(x))\psi_m(y)(1 - \psi_{m+1}(y))$$

for $m < l$, $k \le l$; and

$$\Psi_{k,l}(x, y) = \psi_k(x)\psi_l(y) \quad \text{and} \quad \Psi_{k-1,l}(x, y) = (1 - \psi_k(x))\psi_l(y).$$

Now we have to estimate for J_2 and J_3 their summands $J_2(k, m)$ and $J_3(k, m)$ given by

$$J_2(k, m) = 2x\left(\frac{\partial}{\partial x}\right)\{F_{t2^{-k},t2^{-m}}(x, y) - F_{t2^{-k+1},t2^{-m}}(x, y)\}$$
$$\times\{\psi_m(y)(1 - \psi_{m+1}(y))\}\frac{\partial}{\partial x}\psi_k(x)$$

and

$$J_3(k, m) = x\{F_{t2^{-k},t2^{-m}}(x, y) - F_{t2^{-k+1},t2^{-m}}(x, y)\}$$
$$\times\{\psi_m(y)(1 - \psi_{m+1}(y))\}\left(\frac{\partial}{\partial x}\right)^2\psi_k(x).$$

Since $|(\partial/\partial x)^i\psi_k(x)| \le M4^{ki}$ for $i = 1, 2$, we have

$$|J_2(k, m)| \le M\left|\frac{\partial}{\partial x}\{F_{t2^{-k},t2^{-m}}(x, y) - F_{t2^{-k+1},t2^{-m}}(x, y)\}\right|$$

and

$$|J_3(k, m)| \le M|\{F_{t2^{-k},t2^{-m}}(x, y) - F_{t2^{-k+1},t2^{-m}}(x, y)\}|4^k.$$

The estimate of $J_3(k, m)$ follows immediately now from (6.3) as restrictions on m and k in relation to x and y imply

$$|F_{t2^{-k},t2^{-m}}(x, y) - F_{t2^{-k+1},t2^{-m}}| \le Mt^{2\alpha}$$

and therefore $|J_2(k, m)| \le M4^k t^{2\alpha} \le Mt^{2\alpha-2}$. The estimate of $J_2(k, m)$ though bit more complicated follows from

$$\|\varphi'(x)\|_{C[a,b]} \le M\left\{\frac{\|\varphi(x)\|_{C[a,b]}}{b - a} + (b - a)\|\varphi''(x)\|_{C[a,b]}\right\}$$

where M is independent of $[a, b]$ and φ (see [3, Lemma 3.1] for instance); we set $\varphi(x) = F_{t2^{-k},t2^{-m}}(x, y) - F_{t2^{-k+1},t2^{-m}}(x, y)$ and $[a, b] = [4^{-k-1}, 3 \cdot 4^{-k-1}]$ and the estimate of $J_2(k, m)$ reduces to one similar to J_1 and one similar to $J_3(k, m)$.

We now have to estimate

$$\left|y\left(\frac{\partial}{\partial x}\right)^2 g_{t^2}(x, y)\right| \quad \text{and} \quad \left|\sqrt{xy}\,\frac{\partial}{\partial x}\frac{\partial}{\partial y}g_{t^2}(x, y)\right|$$

but these estimates are similar to the above and are omitted. To get the estimates for the other regions, we transform the vertex of the simplex to $(0, 0)$ as prescribed earlier, construct the function with the correct estimate and take the inverse transformation after completion. (We will have a fixed constant multiplying our estimates as the transformation is not orthogonal.)

As a corollary from Theorem 6.1, we can state:

COROLLARY 6.3. *The function $f(x, y) \in (C, A)_\alpha$ implies that for $x + y \le 3/4$ and $e = \beta e_1 + \gamma e_2$ where $\gamma^2 + \beta^2 = 1$ we have*

$$|\Delta_{he}^2 f(x, y)| \le \frac{M}{x^\alpha}(\beta h)^{2\alpha} + \frac{M}{y^\alpha}(\gamma h)^{2\alpha} + \frac{M}{(xy)^{\alpha/2}}(\beta\gamma h^2)^\alpha \qquad ((x, y) \pm he \in S).$$

Proof. The result follows easily from the identity

$$\Delta_{he}^2 f(x, y) = \Delta_{\beta h e_1}^2 f(x, y - \gamma h) + \Delta_{\gamma h e_2}^2 f(x + \beta h, y) + 2\Delta_{\beta h e_1}\Delta_{\gamma h e_2} f(x + \tfrac{1}{2}\beta h, y - \tfrac{1}{2}\gamma h).$$

7. The difficulty in extending the Berens-Lorentz result. Berens and Lorentz proved for Bernstein polynomials on $C[0,1]$ and for $0 < \alpha < 2$ that

$$|B_n(f, x) - f(x)| \le M\left(\frac{x(1 - x)}{n}\right)^{\alpha/2}$$

if and only if $|\Delta_h^2 f(x)| \le M_1 h^\alpha$ for $(x - h, x + h) \subset [0,1]$. It would be nice to find a condition on $B_n(f, x, y) - f(x, y)$ that will be necessary and sufficient for the class of functions satisfying $|\Delta_{he}^2 f(v)| \le M_1 h^\alpha$ for $v - he$, $v + he \in S$. However, a condition of the type

$$|B_n(f, x, y) - f(x, y)| \le \left(\frac{\psi(x, y)}{n}\right)^{\alpha/2}$$

will fail (no matter what $\psi(x, y)$ is). Choosing the function $f(x, y) = x^\alpha$ for which

$$|B_n(f, x, y) - f(x, y)| = |B_n(f, x) - x^\alpha| \sim K\left(\frac{x}{n}\right)^{\alpha/2}$$

near $x = 0$ will imply that $\psi(x, y) \leq Kx$ near $x = 0$ regardless of y. However, for $f(x, y) = y^\alpha$ the behaviour near $y = 1/4$ regardless of x is

$$|B_n(f, x, y) - y^\alpha| \sim K\frac{1}{n^{\alpha/2}}.$$

Therefore in the neighbourhood of $(0, 1/4)$ we encounter a contradiction. (This contradiction can appear at any point of the boundary except at the vertices.)

The situation will not change if we treat the Bernstein polynomials on the square $[0, 1] \times [0, 1]$ given by

$$B^*_{n,m}(f; x, y) = \sum_{k=0}^{n} \sum_{l=0}^{m} P_{n,k}(x) P_{m,l}(y) f\left(\frac{k}{n}, \frac{l}{n}\right) \tag{7.1}$$

when $0 < K_1 < m/n < K_2 < \alpha$ in spite of the fact that at first glance (7.1) looks like a cartesian product of the one-dimensional Bernstein polynomial. The same functions as above would show a condition

$$|B^*_{n,m}(f, x, y) - f(x, y)| \leq M\left(\frac{\psi(x, y)}{n}\right)^{\alpha/2}$$

(or

$$|B^*_{n,m}(f, x, y) - f(x, y)| \leq M\left(\frac{\Phi_\alpha(x, y)}{n^{\alpha/2}}\right))$$

will fail.

8. The multidimensional Bernstein polynomials on a simplex. In this section we will generalize the results achieved in §§2–6 to the m-dimensional Bernstein polynomials on a simplex. As this is a more cumbersome situation, it would appear to be a very long task. However, the proofs are essentially the same as those for the two-dimensional Bernstein polynomials, which were treated first.

The m-dimensional Bernstein polynomial is given by (1.2) and (1.3) can be rewritten by

$$B_n(f, x) = \sum_{\nu_1=0}^{n} \sum_{\nu_2=0}^{n-\nu_1} \cdots \sum_{\nu_m=0}^{n-\nu_{m-1}} f\left(\frac{\nu_1}{n}, \ldots, \frac{\nu_m}{n}\right) P_{n,\nu}(x), \tag{8.1}$$

(where $P_{n,\nu}(x)$ is given by (1.3)). Recall that because of the symmetry in (8.1), and (1.2), we can consider any two variables to be either x_1 and x_2 or x_m and x_{m-1}, depending on what is advantageous at the time.

As in the two-dimensional case, we need a transformation that will carry the behaviour near $(0, \ldots, 0)$ to that near e_i.

This will be given in the following lemma that can be derived by simple computation.

LEMMA 8.1. *For the* $u = T_i x$ *given by* $u_j = x_j$ *for* $j \neq i$, $u_i = 1 - x_1 - \cdots - x_m$, *and* $f_i(x) = f(T_i x)$ *we have* $B_n(f, x) = B_n(f_i, u)$.

The subspace A of $C(S)$ is given by:

DEFINITION 8.1. $f(x_1, \ldots, x_m) \in C(S)$ belongs to class A if the semi-norm $\Phi(f) \equiv \max(\Phi_0(f), \Phi_1(f), \ldots, \Phi_m(f))$ is finite, where

$$\Phi_0(f) = \max_{\Sigma x_i \leq 1 - 1/2m} \left(\max_{i,j} \sqrt{x_i x_j} \left| \frac{\partial^2}{\partial x_i \partial x_j} f \right| \right),$$

$$\Phi_i(f) \equiv \Phi_0(f_i)$$

and $f_i(x) \equiv f(T_i x)$ where T_i is given in Lemma 8.1.

The domain $\Sigma x_i \leq 1 - 1/2m$ is chosen so that the domain satisfying $\Sigma x_i \leq 1 - \eta_m$, $\eta_m > 1/2m$ and its transformations by T_i still cover S.

The inverse theorem for m-dimensional Bernstein polynomials is given in the following two theorems.

THEOREM 8.2. $f \in (C(S), A)_\alpha$ *if and only if* $\|B_n(f, x) - f(x)\| = O(n^{-\alpha/2})$.

THEOREM 8.3. *For* $f \in C(S)$, $f \in (C(S), A)_\alpha$ *if and only if*
(a) *for* $\Sigma x_i \leq 1 - (1/2m)$ $|\Delta^2_{h\sqrt{x_i} e_i} f| \leq M h^{2\alpha}$, $x_i \geq h^2$,

$$\left| \Delta_{h\sqrt{x_i} e_i} \Delta_{k\sqrt{x_j} e_j} f \right| \leq M h^\alpha k^\alpha \quad \text{for } x_i \geq \tfrac{1}{4} h^2,\ x_j \geq \tfrac{1}{4} k^2 \text{ and } x_l \geq 0 \text{ for } l \neq i, j;$$

and for any i *we have condition* (a) *on* $f_i(x) = f(T_1 x)$.

Outline of proof of Theorem 8.2 *and* 8.3. We will just indicate some of the needed modifications to the proofs in two dimensions. For the direct result we essentially have to prove the inequality

$$\|B_n(f) - f\| \leq L\Phi(f) \quad \text{for } f \in A. \tag{8.2}$$

Here we have to use for the definition of a corresponding $\Phi^*(f)$ the domain $\Sigma x_i \leq 1 - 1/(m+1)$ and its transformations by T_i. We also observe that (2.9) and (2.10) are valid with $\phi_i(x_1, \ldots, x_m) \equiv x_i$ for $i \leq m$

and $\phi_{m+1}(x_1,\dots,x_m) \equiv 1-\sum_{i=1}^m x_i$. For the converse result we need the inequalities

$$\Phi(B_n(f)) \le Ln\|f\| \tag{8.3}$$

and

$$\Phi(B_n(f)) \le L\Phi(F) \quad \text{for } f\in A. \tag{8.4}$$

The proof follows the proof in earlier sections. For (8.3) we need to replace (2.2), (2.3) and (2.4) by

$$\begin{aligned}\frac{\partial}{\partial x_i}B_n(f;x_1\cdots x_m) &= \sum_{\nu/n\in S} f\left(\frac{\nu}{n}\right)P_{\nu,n}(x)\frac{1}{x_i(1-\sum_{s=1}^m x_s)}\\ &\quad\times\left[\nu_i\left(1-\sum_{s=1}^n x_s\right)-\left(n-\sum_{s=1}^n \nu_s\right)x_i\right],\end{aligned} \tag{8.5}$$

$$\begin{aligned}&\left(\frac{\partial}{\partial x_i}\right)^2 B_n(f;x_1\cdots x_m)\\ &\quad= \sum_{\nu/n\in S} f\left(\frac{\nu}{n}\right)P_{\nu,n}(x)\frac{1}{x_i^2(1-\sum_{s=1}^m x_s)^2}\\ &\quad\quad\times\left[\nu_i(\nu_i-1)\left(1-\sum_{s=1}^m x_s\right)^2\right.\\ &\quad\quad\quad -2\nu_i\left(n-\sum_{s=1}^m \nu_s\right)x_i\left(1-\sum_{s=1}^m x_s\right)\\ &\quad\quad\quad\left.+\left(n-\sum_{s=1}^m \nu_s\right)\left(n-1-\sum_{s=1}^m \nu_s\right)x_i^2\right]\end{aligned} \tag{8.6}$$

and

$$\begin{aligned}&\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j}B_n(f;x_1,\dots,x_m)\\ &\quad= \sum_{\nu/n\in S} f\left(\frac{\nu}{n}\right)P_{\nu,n}(x)\frac{1}{x_ix_j(1-\sum_{s=1}^m x_s)^2}\\ &\quad\times\left[\nu_i\nu_j\left(1-\sum_{s=1}^m x_s\right)^2-(\nu_ix_j+\nu_jx_i)\left(n-\sum_{s=1}^m \nu_s\right)\left(1-\sum_{s=1}^m x_s\right)\right.\\ &\quad\quad\left.+\left(n-\sum_{s=1}^m \nu_i\right)\left(n-1-\sum_{s=1}^m \nu_i\right)x_ix_j\right].\end{aligned} \tag{8.7}$$

The construction of g_t in Theorem 8.3 follows §6 with $F_{t_1,\dots,t_m}(x_1,\dots,x_m)$ replacing $F_{t_1,t_2}(x_1x_2)$ (using $2m$ iterated integral).

9. Multidimensional Bernstein polynomials on $[0,1] \times \cdots \times [0,1]$. In this section we will generalize the result to Bernstein polynomials on the box B, $B \equiv [0,1] \times \cdots \times [0,1]$ given by (1.4) for which the inverse result is the following theorem for $B_n^*(f, x) \equiv B_{n_1,\dots,n_m}^*(f, x_1, \dots, x_m)$.

THEOREM 9.1. *For $f \in C(B)$, $B_n^*(f, x)$ given by* (1.4), *$n = (n_1, \dots, n_m)$ and $n_i/n_j \le K$ for all i and j the following are equivalent for $0 < \alpha < 1$:*

(a) $\|B_{n_1,\dots,n_m}^*(f, x_1, \dots, x_m) - f(x_1, \dots, x_m)\|_{C(B)} = O(n_i^{-\alpha})$ (*for any i*).

(b) *$f \in (C(B), A)_\alpha$ where $A \equiv \{f; x_i(1 - x_i)(\partial^2/\partial x_i^2)f \in C(B)$ and $(\partial/\partial x_i)f$ is locally the integral of $(\partial^2/\partial x_i^2)f\}$.*

(c) *For all i $|(x_i(1 - x_i))^\alpha \Delta_{he_i}^2 f(x)| \le Mh^{2\alpha}$ if $x \pm he_i \in B$.*

(d) *For all i $|\Delta_{h\sqrt{x_i(1-x_i)}\,e_i}^2 f(x)| \le M_1 h^{2\alpha}$ if $x \pm h\sqrt{x_i(1 - x_i)}\, e_i \in B$.*

The proof of the above theorem, while not trivial, is made redundant by the fact that at every step it is simpler than the earlier proofs in this paper and therefore will be omitted.

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