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# SUMS AND PRODUCTS OF $B_r$ SPACES

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**In reminiscence of Ptak's open mapping theorem, a topological space satisfying the open mapping theorem is called a $B_r$ space. This paper is devoted to the study of sums and products of $B_r$ spaces in the category of topological spaces. We prove that, in general, sums and products of even two $B_r$ spaces need no longer be $B_r$. On the other hand, for any $B_r$ space $E$ the sum $E \oplus E$ is again a $B_r$ space. Moreover, since Čech complete spaces are known to be $B_r$, we ask whether a sum $E \oplus F$ is $B_r$ provided that $E$ is Čech complete and $F$ is $B_r$. It turns out that, at least in the framework of complete regularity, the answer to this question is in the positive if and only if $F$ is a Baire space.**

**Introduction.** The notion of $B_r$ spaces has first been introduced by T. Husain in the categories of locally convex vector spaces (see [**Hu**$_1$]) and topological groups (see [**Hu**$_2$]). Originally, it goes back to V. Ptak's open mapping theorem (see [**Kö**], p. 35 ff).

A Hausdorff topological space $E$ is called a $B_r$ space if every continuous, nearly open bijection $f$ from $E$ onto any Hausdorff space $F$ is in fact open.

For a survey of the classical theory of $B_r$ spaces we refer the reader to Köthe's book [**Kö**], where the locally convex case is treated. The linear topological case is investigated in the lecture notes [**AEK**]. $B_r$ groups are considered by several authors. See for instance [**Hu**$_2$], [**Ba**], [**Gr**], [**Su**]. In a purely topological context, $B_r$ spaces have first been investigated by Weston in [**We**], although the term $B_r$ space is not used there. Translated into the $B_r$ terminology, Weston proved that every completely metrizable topological space is a $B_r$ space. His result has been generalized by Byczkowski and Pol in [**BP**], who proved that every Čech complete topological space is a $B_r$ space. In [**No**] we have further generalized this result proving that every Hausdorff, semi-regular topological space densely containing some Čech complete subspace is in fact a $B_r$ space. We just mention another generalization of Byczkowski and Pols' result into a somewhat different direction by Wilhelm (cf. [**Wi**]).

In the present paper we examine the invariance of the class of $B_r$ spaces under topological sums and products. It turns out that, in general, the sum of even two $B_r$ spaces need not be a $B_r$ space. A counterexample is given in §2. In §1 we obtain a positive result stating that the sum $E \oplus E$

is a $B_r$ space provided that $E$ is $B_r$. This result is then used to obtain a generalization of the Banach-Klee theorem. In §2 we investigate sums of $B_r$ spaces $E \oplus F$ in the specific case when $F$ is Čech complete. It turns out that for $E \oplus F$ to be a $B_r$ space in this situation one has to assume that $E$ be a Baire space. In §3 we present a special class of $B_r$ spaces which is of some interest in itself. This class is then used to give the promised counterexample. Finally, in §4 we deal with products of $B_r$ spaces. We prove that the product of two $B_r$ spaces need no longer be $B_r$, again using the spaces defined in §3. This negative answer is by no means surprising when we recall the situation in the classical theory. Indeed, counterexamples in the classical cases can be found in [**Kö**] and [**Gr**] for the locally convex and the case of groups respectively.

Our terminology is from the book [**E**]. To be somewhat shorter, a Hausdorff semi-regular space which densely contains some Čech complete subspace will be called almost Čech complete. Finally, a mapping $f\colon E \to F$ is called nearly open if for every $x \in E$ and every neighborhood $U$ of $x$ the set $\overline{f(U)}$ is a neighborhood of $f(x)$.

**1. Sums of $B_r$ spaces.** In this section we examine the problem of invariance of the class of $B_r$ spaces under topological sums. The following easily proved statement is the starting point of our investigation.

PROPOSITION 1. *Let $(E_\iota\colon \iota \in I)$ be a family of topological spaces whose sum $\oplus\{E_\iota\colon \iota \in I\}$ is a $B_r$ space. Then every summand $E_\iota$ must be $B_r$, too.* □

So far we know of a large class of $B_r$ spaces closed under arbitrary sums, the class of almost Čech complete spaces. Another example for this phenomenon is provided by the class of arbitrary sums of $H$-minimal ($= T_2$-minimal) spaces (see Proposition 5 below).

Our first step will be the following proposition which, in many cases, reduces the general problem to the case of the finite sums.

PROPOSITION 2. *Let $(E_\iota\colon \iota \in I)$ with* $\operatorname{card}(I) \geq 2$ *be a family of topological spaces such that $E_\iota \oplus E_\kappa$ is a $B_r$ space whenever $\iota, \kappa \in I$, $\iota \neq \kappa$. Then the sum $\oplus\{E_\iota\colon \iota \in I\}$ is a $B_r$ space as well.*

*Proof.* Let $f\colon \oplus_\iota E_\iota \to F$ be a continuous, nearly open bijection, $F$ a Hausdorff space. Since $\operatorname{card}(I) \geq 2$, every $E_\iota$ is $B_r$ by Proposition 1. Now $f \restriction E_\iota$ is a continuous, nearly open bijection $E_\iota \to f(E_\iota)$, hence we derive $E_\iota \approx f(E_\iota)$. We prove that $f(E_\iota)$ is closed in $F$. Assume not. Then there exists $\kappa \neq \iota$ in $I$ with $\overline{f(E_\iota)} \cap f(E_\kappa) \neq \varnothing$. But now $E_\iota \oplus E_\kappa$ is a $B_r$ space which implies $E_\iota \oplus E_\kappa \approx f(E_\iota \oplus E_\kappa) \approx f(E_\iota) \oplus f(E_\kappa)$, contradicting $\overline{f(E_\iota)} \cap f(E_\kappa) \neq \varnothing$. Hence $f(E_\iota)$ is closed in $F$ and since $f$ is nearly open this proves the result. □

With this observation in mind we shall now deal with the case of two-element families of $B_r$ spaces, i.e. given two $B_r$ spaces $E$ and $F$, we ask whether their sum $E \oplus F$ must be a $B_r$ space. It turns out that, in general, the answer is in the negative. An example will be given in §3. Consequently, we are now interested in situations in which, nevertheless, the answer is in the positive. We pose the following problems.

(1) Given a $B_r$ space $E$, must $E \oplus E$ be a $B_r$ space?

(2) Given a $B_r$ space $E$ and an (almost) Čech complete space $F$, under what conditions on $E$ will $E \oplus F$ be a $B_r$ space?

In this section we shall give a solution to the first problem. Problem (2) will be treated in §2.

Before giving an answer to problem (1), let us mention that every $B_r$ space must be semi-regular (cf. [E], p. 84). Indeed, if $E$ is a $B_r$ space and if $E_s$ denotes the semiregularization of $E$ (i.e. the set $E$ endowed with the topology generated by the regular-open subsets ( = open domains, see [E], p. 37) of the space $E$) then the mapping id: $E \to E_s$ is a continuous, nearly open bijection and, moreover, $E_s$ is a Hausdorff space. This implies $E = E_s$.

THEOREM 1. *Let $E$ be a $B_r$ space. Then $E \oplus E$ is a $B_r$ space as well.*

*Proof.* Let $f\colon E \oplus E \to F$ be a continuous, nearly open bijection onto the Hausdorff space $F$. Assume that the point set of $E \oplus E$ is $E \times \{1\} \cup E \times \{2\}$. Then we have $f(E \times \{1\}) \approx f(E \times \{2\}) \approx E$ since $E$ is a $B_r$ space. It remains to prove that $F = f(E \times \{1\}) \oplus f(E \times \{2\})$. This will be established in several steps.

(I) First we observe that $F$ may be assumed to be semi-regular. Indeed, if $F$ is not semi-regular then we regard $f$ as a mapping $f\colon E \oplus E \to F_s$, where $F_s$ denotes the semiregularization of $F$. Then $f$ is again a nearly open, continuous bijection and $F_s$ is Hausdorff. But if we can prove the openness of $f$ with target space $F_s$, then $f$ will be open with target space $F$, too.

(II) We prove that every nonempty open set $U \subset E$ contains some nonempty open subset $V$ with $\operatorname{int}\overline{f(V \times \{1\})} \cap f(V \times \{2\}) = \varnothing$. Indeed, if $x \in U$ then $f(x,1)$ and $f(x,2)$ can be separated in $F$ since $F$ is Hausdorff. Thus there exists a neighborhood $V$ of $x$ contained in $U$ with $\operatorname{int}\overline{f(V \times \{1\})} \cap \operatorname{int}\overline{f(V \times \{2\})} = \varnothing$.

(III) Now we prove that every nonempty open set $U \subset E$ contains some nonempty open subset $V$ with $\operatorname{int}\overline{f(V\times\{1\})} \cap \overline{f(E\times\{2\})} = \varnothing$. Assume that there exists a nonempty open set $U \subset E$ with $\operatorname{int}\overline{f(V\times\{1\})} \cap \overline{f(E\times\{2\})} \neq \varnothing$ for all nonempty open subsets $V$ of $U$. In view of (II) we may assume that $\operatorname{int}\overline{f(U\times\{1\})} \cap f(U\times\{2\})$ is empty. Moreover, since $E$ is semi-regular, we may assume that $U$ is a regular-open set.

For every open subset $V$ of $U$ there exists an open set $V^{\omega} \subset E$ such that $\operatorname{int}\overline{f(V\times\{1\})} \cap f(E\times\{2\}) = f(V^{\omega}\times\{2\})$. Let $U^* := \operatorname{int}\overline{U^{\omega}}$.

For every open subset $W$ of $U^*$ there exists an open set $W^{\rho} \subset E$ such that $\operatorname{int}\overline{f(W\times\{2\})} \cap f(E\times\{1\}) = f(W^{\rho}\times\{1\})$.

The mappings $V \to V^{\omega}$ and $W \to W^{\rho}$ are monotonic and moreover have the following properties which can be derived from the assumptions on $U$.

(i) $U \cap U^* = \varnothing$;

(ii) If $V \subset U$ then $V^{\omega} \subset U^*$ and if $V$ is nonempty, so is $V^{\omega}$. Moreover, if $V$ is regular-open, then $(V^{\omega})^{\rho} \subset V$.

(iii) If $W \subset U^*$ then $W^{\rho} \subset U$ and if $W$ is nonempty, so is $W^{\rho}$. Moreover, if $W$ is regular-open, then $(W^{\rho})^{\omega} \subset W$.

Now for every regular-open set $W$ in $E$ let $\tilde{W}$ denote the set $W \cup (W\cap U)^{\omega} \cup (W \cap U^*)^{\rho}$. Let $\mathfrak{B}$ be the set of all $\tilde{W}$, $W$ regular-open in $E$. Then $\mathscr{B}$ is a base for a new topology on $E$. In fact, let $\tilde{V}, \tilde{W} \in \mathfrak{B}$ and $x \in \tilde{V} \cap \tilde{W}$ be given. We have to find $\tilde{O}$ in $\mathscr{B}$ with $x \in \tilde{O} \subset \tilde{V} \cap \tilde{W}$. There are seven different cases to be considered. Let us give a proof for $x \in (V\cap U)^{\omega} \cap (W \cap U)^{\omega}$ exemplary. Choose a regular-open set $O$ with $x \in O \subset (V\cap U)^{\omega} \cap (W\cap U)^{\omega}$. Obviously, $O \subset \tilde{V}$. Since $O \subset (V \cap U)^{\omega} \subset U^*$ we have $O \cap U = \varnothing$, hence $(O\cap U)^{\omega} = \varnothing$, too. On the other hand $(O \cap U^*)^{\rho} = O^{\rho} \subset ((V\cap U)^{\omega})^{\rho} \subset V \subset \tilde{V}$. This proves $\tilde{O} \subset \tilde{V}$. Analogously, we obtain $\tilde{O} \subset \tilde{W}$.

Now let $\tau$ denote the original topology on $E$ and let $\sigma$ denote the topology generated by $\mathfrak{B}$. Then $\sigma$ is strictly coarser than $\tau$. We prove that $\sigma$ is Hausdorff. Let $x, y \in E$, $x \neq y$ be given. Choose regular-open neighborhoods $V_x$, $V_y$ of $x, y$ such that $\operatorname{int}\overline{f(V_x\times\{i\})} \cap \operatorname{int}\overline{f(V_y\times\{j\})} \neq \varnothing$ for all $i, j \in \{1,2\}$. This is possible since the points $f(x,1)$, $f(x,2)$, $f(y,1)$, $f(y,2)$ can be separated in $F$. But now we derive $\tilde{V}_x \cap \tilde{V}_y = \varnothing$. Hence $\sigma$ is Hausdorff.

Finally, we prove that $\mathrm{id}_E\colon (E,\tau) \to (E,\sigma)$ is nearly open. We have to prove $\tilde{V} \subset \mathrm{cl}_{\sigma}(V)$ for regular-open $V$. Since $V \subset \mathrm{cl}_{\sigma}(V)$ is clear we prove $(V\cap U)^{\omega} \subset \mathrm{cl}_{\sigma}(V)$. The proof for $(V \cap U^*)^{\rho} \subset \mathrm{cl}_{\sigma}(V)$ is similar. Let $y \in (V\cap U)^{\omega}$. Let $\tilde{O}$ be a basic neighborhood of $y$. It is easy to see that we may assume $y \in O \subset (V \cap U)^{\omega}$. Thus $(O \cap U^*)^{\rho} \subset (V \cap U)^{\omega\rho} \subset V$ proving $\tilde{O} \cap V \neq \varnothing$. But now we have arrived at a contradiction since $E$ is a $B_r$ space. This proves (III).

(IV) Now every nonempty open $U$ contains some nonempty open $V_U$ such that the statement of (III) holds. Hence $G := \bigcup\{\operatorname{int}\overline{f(V_U \times \{1\})}: U$ open in $E\}$ is an open subset of $F$ densely contained in $f(E \times \{1\})$. Since $f$ is nearly open this proves the theorem. □

Thus our first problem has been solved in the positive. In view of Proposition 2 we obtain the following

COROLLARY 1. *Let $E$ be a $B_r$ space and let $(E_\iota: \iota \in I)$ be a family of copies of $E$* (i.e. *$E_\iota \approx E$ for all $\iota$). Then $\oplus\{E_\iota: \iota \in I\}$ is a $B_r$ space.* □

The following consequence of Theorem 1 is somewhat more interesting.

COROLLARY 2. *Let $G$ be a topological group which is a $B_r$ space. Then $G$ is complete with respect to its two-sided uniformity.*

*Proof.* Let $\tilde{G}$ be the completion of $G$ with respect to the two-sided uniformity. Suppose that there exists some $\tilde{x} \in \tilde{G} \setminus G$. Let $H := G \cup \tilde{x}G$ be endowed with the trace of the topology of $\tilde{G}$. Let $G \oplus G = G \times \{1\} \cup G \times \{2\}$ and define the mapping $f\colon G \oplus G \to H$ by $f(x,1) = x$ and $f(x,2) = \tilde{x}x$. Then $f$ is continuous and bijective, the latter since $G \cap \tilde{x}G = \varnothing$. Moreover, since $G$ and $\tilde{x}G$ are both dense in $H, f$ is nearly open. Hence it must be open and this yields the openness of $G$ in $H$, a contradiction. □

From this result we may derive the Banach-Klee theorem stating that every completely metrizable topological vector space is complete with respect to its natural uniformly (cf. **[K1]**, **[We]**).

Let us mention another point of view from which Corollary 2 seems to be of some interest. It is well-known that a $B_r$ group need not be complete with respect to its two-sided uniformity. Indeed, there even exist $H$-minimal abelian groups which are incomplete. Take for instance the group $Q = \{e^{2\pi i q}: q \in \mathbf{Q}\}$. (Cf. **[Gr]**). Now, in view of Corollary 2 above, it is clear that $Q$ cannot be a $B_r$ space in the topological sense.

**2. Absolute Baire spaces.** In this section we continue our investigation of finite sums of $B_r$ spaces with problem (2). We obtain a rather satisfactory answer to problem (2). It turns out that a necessary and sufficient condition on a $B_r$ space $E$ can be given to assure that the sum $E \oplus F$ is $B_r$ for every (almost) Čech complete space $F$, at least when $E$ is assumed to be completely regular: $E$ has to be a Baire space. In the absence of complete regularity, however, this condition is not really necessary and has to be replaced by the following notion.

DEFINITION 1. (a) Let $\tilde{E}$ be a Hausdorff topological space and let $E$ be a dense subspace of $\tilde{E}$. Then $E$ is called Baire-embedded in $\tilde{E}$ if every $G_\delta$ subset $G$ of $\tilde{E}$ with $G \cap E = \varnothing$ is nowhere dense in $\tilde{E}$.

(b) A Hausdorff topological space $E$ is called an absolute Baire space if $E$ is Baire-embedded in every Hausdorff topological space $\tilde{E}$ containing $E$ as a dense subspace. □

REMARKS. (1) The reader might consult here Aarts and Lutzers' paper [**AL**], where the concept of an $A$-embedded subspace is introduced. If $E$ is dense and co-dense in $\tilde{E}$ then $E$ is Baire-embedded in $\tilde{E}$ if and only if $\tilde{E} \setminus E$ is $A$-embedded in $\tilde{E}$.

(2) Every Hausdorff Baire space is an absolute Baire space. Indeed, if $\tilde{E}$ is a Hausdorff space densely containing the Baire space $E$ and if $G$ is a $G_\delta$ set in $\tilde{E}$ with $G \cap E = \varnothing$ and $G$ dense in the nonempty open set $W$ then $W \cap G$ is a dense $G_\delta$ subset of $W$ and $W \cap E$ is a dense second category subset of $W$, but both sets have empty intersection, a contradiction.

(3) Let $\tilde{E}$ be a Hausdorff Baire space and let $E$ be a dense Baire-embedded subspace of $\tilde{E}$. Then $E$ is a Baire space. For let $U \subset E$ be nonempty and open. Choose $\tilde{U}$ open in $\tilde{E}$ with $\tilde{U} \cap E = U$. Assume that $U$ is of the first category, say $U \subset \bigcup\{C_n : n \in \mathbf{N}\}$ where the $C_n$ are closed and nowhere dense in $\tilde{E}$. Now let $G := \bigcap\{\tilde{E} \setminus C_n : n \in \mathbf{N}\}$, then $G$ is a dense $G_\delta$ in $\tilde{E}$, hence $\tilde{U} \cap G$ is a somewhere dense $G_\delta$. Since $(\tilde{U} \cap G) \cap E = \varnothing$ this is a contradiction.

(4) If $E$ is $H$-closed, then $E$ is an absolute Baire space. Consequently, an absoluite Baire space need not be Baire since Herrlich asserts the existence of first category $H$-minimal spaces (see [**He**]). However, in view of (3) we know that an absolute Baire space is Baire provided that it admits some Hausdorff Baire extension. In [**Ca**] it is proved that a Hausdorff space $E$ admits some Baire Hausdorff extension if and only if its Fomin extension $\sigma E$ is a Baire space.

PROPOSITION 3. *Let $E$ be a topological space such that the sum $E \oplus F$ is a $B_r$ space whenever $F$ is (almost) Čech complete. Then $E$ is Baire-embedded in any (almost) Čech complete space $\tilde{E}$ containing $E$ as a dense subspace. Consequently, if $E$ is completely regular, it is a Baire space.*

*Proof*. Let $\tilde{E}$ be an (almost) Čech complete space containing $E$ as a dense subspace. Let $G$ be a $G_\delta$ subset of $\tilde{E}$ with $G \cap E = \varnothing$ and assume that $W := \operatorname{int} \overline{G}$ is nonempty. Let $F := W \cap G$. Thus $F$ is a dense $G_\delta$ subset of an open subset $W$ of $\tilde{E}$ and therefore is (almost) Čech complete. As far as the inheritance of semi-regularity is concerned we refer the reader to [**E**], p. 154. Moreover, since Čech completeness is hereditary with respect to $G_\delta$ subsets, we derive that $F$ is an (almost) Čech complete space. By assumption, $E \oplus F$ is a $B_r$ space. Let $H := E \cup F$ be endowed with the trace of the topology of $\tilde{E}$ and define the mapping $f\colon E \oplus F \to H$ in the natural way. Then $f$ is a continuous bijection. But $f$ is also nearly open since $\overline{f(E)} = H$ and $\overline{f(F)} = \overline{W \cap H}$. Hence we conclude that $f$ is an open mapping. This, however, contradicts the fact that $f(E)$ is dense. This proves the first part of the proposition. The second part is a consequence of remark (3) and the fact that $E$ is now Baire-embedded in its Stone-Čech compactification. □

We shall see in the following that if $E$ is assumed to be an absolute Baire space then $E \oplus F$ will in fact be $B_r$ whenever $E$ is $B_r$ and $F$ is almost Čech complete. However, our method of proof is valid for a much larger class of spaces $F$ than the class of almost Čech complete spaces.

Definition 2. A Hausdorff topological space $E$ is called a $(\beta)$-space if $E$ is a set with the Baire property in every Hausdorff space $\tilde{E}$ containing $E$ as a dense subspace (i.e. there exists an open set $G \subset \tilde{E}$ and a first category set $P \subset \tilde{E}$ with $E = G \triangle P$). □

Remarks. (1) Every almost Čech complete space is a $(\beta)$-space. Indeed, if $E$ is almost Čech complete and if $C$ is a dense Čech complete subspace of $E$, then $E = C \cup (E \setminus C)$ is a set with the Baire property in any Hausdorff extension $\tilde{E}$ of $E$ since $C$ is a dense $G_\delta$ in $\tilde{E}$. (Cf. [**Ox**], p. 19).

(2) If $E$ is completely regular and analytic in the sense of Frolík [**F**], then $E$ is a $(\beta)$-space.

(3) Every first category space is a $(\beta)$-space.

(4) Every $H$-closed space is a $(\beta)$-space.

Proposition 4. *Let $E$ be a $B_r$ space which is absolute Baire and let $F$ be a $B_r$ space which is a Baire $(\beta)$-space. Then $E \oplus F$ is a $B_r$ space.*

*Proof*. Let $f\colon E \oplus F \to G$ be a continuous, nearly open bijection onto the Hausdorff space $G$. Since $E$ and $F$ both are $B_r$ spaces we conclude

$E \approx f(E)$ and $F \approx f(F)$. It remains to prove that $f(F)$ is closed in $G$. Assume that $\overline{f(F)} \cap f(E) \neq \varnothing$. Since $f(F)$ is a Baire space, $\operatorname{int}\overline{f(E)} \cap f(F)$ is Baire, too. Thus $\operatorname{int}\overline{f(E)} \cap f(F)$ is a second category subset of $H := \operatorname{int}\overline{f(E)} \cap \operatorname{int}\overline{f(F)}$. Now $f(F)$ is a $(\beta)$-space, so $f(F)$ is a set with the Baire property in its Hausdorff extension $\operatorname{int}\overline{f(F)}$, say $f(F) = G \triangle P$ for $G$ open and $P$ of the first category in $\operatorname{int}\overline{f(F)}$. We claim that $f(E) \cap G$ is of the second category in $H$. Assume not. Then there exist an $F_\sigma$ set $R \subset H$ with $f(E) \cap G \subset R$ and $R$ is of the first category in $H$. Hence $\operatorname{int}\overline{f(E)} \cap (G \setminus R)$ is a $G_\delta$ set in $H$, hence in $\operatorname{int}\overline{f(F)}$, which does not intersect $f(E) \approx E$. Consequently, $\operatorname{int}\overline{f(E)} \cap G \setminus R)$ is nowhere dense in $\operatorname{int}\overline{f(E)}$, hence is nowhere dense in $H$. But now we have

$$\begin{aligned}\operatorname{int}\overline{f(E)} \cap G &= \left(\operatorname{int}\overline{f(E)} \cap G \cap R\right) \cup \left(\operatorname{int}\overline{f(E)} \cap (G \setminus R)\right)\\ &\subset R \cup \left(\operatorname{int}\overline{f(E)} \cap (G \setminus R)\right)\end{aligned}$$

contradicting the fact that $\operatorname{int}\overline{f(E)} \cap G$ is an open subset of $H$ and therefore has to be of the second category in $H$. This proves the claim. But now recall that $f(F)$ differs from $G$ only by a set of the first category. Hence also $f(E) \cap f(F)$ must be of the second category in $H$, which is absurd since it is empty. This proves our proposition. □

THEOREM 2. *Let $E$ be a completely regular $B_r$ space. The following statements are equivalent*:

(1) *$E$ is a Baire space*;

(2) *For every Čech complete space $F$ the topological sum $E \oplus F$ is a $B_r$ space*;

(3) *For every Baire $(\beta)$-space $F$ which is a $B_r$ space the sum $E \oplus F$ is a $B_r$ space.* □

Combining Proposition 4 with Proposition 2 we obtain the following

COROLLARY 1. *Let $(E_\iota\colon \iota \in I)$ be a family of Baire $(\beta)$-spaces which are $B_r$ spaces. Then their topological sum $\oplus\{E_\iota\colon \iota \in I\}$ is a $B_r$ space as well.* □

We conclude with

PROPOSITION 5. *Let $(E_\iota\colon \iota \in I)$ be a family of $H$-minimal spaces. Then $\oplus\{E_\iota\colon \iota \in I\}$ is a $B_r$ space.*

*Proof*. This follows again from Proposition 2 since the sum of two $H$-minimal spaces is $H$-minimal hence $B_r$. □

**3. Stationary sets and $B_r$ spaces.** We have promised to give an example of two $B_r$ spaces whose topological sum is no longer a $B_r$ space. Such an example will be constructed in this section. To this end we shall present an interesting class of $B_r$ spaces defined by means of stationary sets of ordinals.

Let $\kappa$ be a regular uncountable cardinal. A subset $S$ of $\kappa$ is called cofinal if $\operatorname{card}(S) = \kappa$. A subset $C$ of $\kappa$ is called $\omega$-closed (resp. closed) if $C$ is sequentially closed (resp. closed) in $\kappa$ for the order topology on $\kappa$. A cofinal subset $S$ of $\kappa$ is called $\omega$-stationary (resp. stationary) if it intersects every $\omega$-closed (resp. closed) cofinal subset of $\kappa$. Now let $\kappa$ be endowed with the discrete topology and let $\kappa^\omega$ have the product topology. If $S \subset \kappa$ is cofinal then let $S^*$ denote the set of all mappings $f \in \kappa^\omega$ with $f^* := \sup\{f(n) : n \in \omega\} \in S$. Let $S^*$ be endowed with the trace of the product topology. A base for the topology of $S^*$ is formed by the sets $B_S(\alpha_0, \alpha_1, \ldots, \alpha_n) = \{f \in S^*\colon\ f(i) = \alpha_i \text{ for } i \leq n\}$, where the $\alpha_i$ vary over $\kappa$. If no confusion may occur, we omit the subscript $S$.

The class of spaces $S^*$ with $\kappa = \omega_1$ has been used by Fleissner and Kunen in **[FK]** to give counterexamples in the problem of invariance of the class of Baire spaces under products.

THEOREM 3. *Let $S \subset \kappa$ be cofinal. The following statements are equivalent*:

(1) *$S$ is $\omega$-stationary*;

(2) *$S^*$ is a Baire space*;

(3) *$S^*$ is a $B_r$ space*;

(4) *For every almost Čech complete space $F$ the sum $S^* \oplus F$ is $B_r$*.

*Proof.* (1) implies (2). This is proved for the case $\kappa = \omega_1$ in **[FK]**. ex. 1. The proof in the present case, however, is completely analogous.

(1) implies (3). Let $f\colon S^* \to F$ be a continuous, nearly open bijection onto the Hausdorff space $F$. Fix $x \in S^*$ and a neighborhood $U = B(\alpha_0, \ldots, \alpha_n)$ of $x$. We prove that $f(U)$ is closed in $F$. Let $y \in \overline{f(U)}$, $y = f(z)$. Since $U$ is closed it remains to prove $z \in \overline{U}$. Let $V = B(\gamma_0, \ldots, \gamma_m)$ be a neighborhood of $z$. We have to prove $U \cap V \neq \varnothing$. Let $FS$ denote the set of all finite sequences of ordinals less than $\kappa$. We define a mapping $\Theta\colon FS \times FS \times \omega \to FS$ such that $\Theta(\mathfrak{a}, \mathfrak{b}, n) \supset \mathfrak{a}$ whenever $n$ is even and $\Theta(\mathfrak{a}, \mathfrak{b}, n) \supset \mathfrak{b}$ whenever $n$ is odd.

(1) $\Theta(\mathfrak{a}, \mathfrak{b}, 0)$ has to be defined. If $\mathfrak{a} = \mathfrak{b} = \varnothing$ then we proceed as follows. The set $\operatorname{int}\overline{f(V)} \cap f(U)$ is nonempty and contains some $x^1$. Define $\Theta(\varnothing, \varnothing, 0) := (x^1(0), \ldots, x^1(n), x^1(n+1))$. In all other cases define $\Theta(\mathfrak{a}, \mathfrak{b}, 0) = \mathfrak{a}$.

(2) $\Theta(\mathfrak{a}, \mathfrak{b}, 1)$ has to be defined. If $\mathfrak{b} = \varnothing$ and if $(x^1(0), \ldots, x^1(n+1))$ is an initial segment of $\mathfrak{a}$ then define as follows. Choose $y^1 \in \operatorname{int}\overline{f(B(\mathfrak{a}))} \cap f(V)$. This is possible since $\mathfrak{a}(i) = x^1(i)$ for $i \leq n+1$. Now define $\Theta(\mathfrak{a}, \varnothing, 1) = (y^1(0), \ldots, y^1(m),\ y^1(m+1))$. In all other cases define $\Theta(\mathfrak{a}, \mathfrak{b}, 1) = \mathfrak{b}$.

(3) $\Theta(\mathfrak{a}, \mathfrak{b}, k)$ for $k$ even, $k \geq 2$ has to be defined. If $\operatorname{int}\overline{f(B(\mathfrak{a}))} \cap f(B(\mathfrak{b}))$ is empty then define $\Theta(\mathfrak{a}, \mathfrak{b}, k) = \mathfrak{a}$. Otherwise choose $x^{k+1}$ in this intersection and define $\Theta(\mathfrak{a}, \mathfrak{b}, k) = (x^{k+1}(0), \ldots, x^{k+1}(s), x^{k+1}(s+1))$ where $\mathfrak{a} = (x^{k+1}(0), \ldots, x^{k+1}(s))$.

(4) $\Theta(\mathfrak{a}, \mathfrak{b}, k)$ for $k$ odd, $k \geq 3$ has to be defined. If $\operatorname{int}\overline{f(B(\mathfrak{b}))} \cap f(B(\mathfrak{a}))$ is empty then define $\Theta(\mathfrak{a}, \mathfrak{b}, k) = \mathfrak{b}$. Otherwise choose $y^k$ within this intersection and define $\Theta(\mathfrak{a}, \mathfrak{b}, k) = (y^k(0), \ldots, y^k(r), y^k(r+1))$, where $\mathfrak{b} = (y^k(0), \ldots, y^k(r))$.

Suppose now that $\Theta$ has been defined. An ordinal $\eta < \kappa$ is called a fixed-point of $\Theta$ if the following condition is satisfied: Whenever $\Theta(\mathfrak{a}, \mathfrak{b}, k) = \mathfrak{c}$ holds for some $k$ and any $\mathfrak{a}, \mathfrak{b} \in FS$ with $\alpha < \eta$, $\beta < \eta$ for all $\alpha$ in $\mathfrak{a}$, $\beta$ in $\mathfrak{b}$, then also $\gamma < \eta$ for all $\gamma$ in $\mathfrak{c}$. It is easy to see that the set $\Phi$ of all fixed-points of $\Theta$ is closed and cofinal. Let $C$ denote the set of all limit ordinals in $\kappa$ with cofinality $\omega$. Then $\Phi \cap C$ is $\omega$-closed and cofinal and consequently there exists $\eta \in \Phi \cap C \cap S$ and a sequence $(\eta_n)$ with $\eta_0 < \eta_1 < \cdots \nearrow \eta$. Now define two sequences $(\mathfrak{a}_n)$ and $(\mathfrak{b}_n)$ in $FS$ with $\mathfrak{a}_0 \subset \mathfrak{a}_1 \subset \cdots$ and $\mathfrak{b}_0 \subset \mathfrak{b}_1 \subset \cdots$ as follows.

$$\begin{aligned}
&\Theta(\varnothing, \varnothing, 0) = \mathfrak{a}_0,\ \mathfrak{a}_1 = \mathfrak{a}_0^\frown \eta_0,\\
&\Theta(\mathfrak{a}_1, \varnothing, 1) = \mathfrak{b}_0,\ \mathfrak{b}_1 = \mathfrak{b}_0^\frown \eta_1,\\
&\Theta(\mathfrak{a}_1, \mathfrak{b}_1, 2) = \mathfrak{a}_2,\ \mathfrak{a}_3 = \mathfrak{a}_2^\frown \eta_2,\\
&\Theta(\mathfrak{a}_3, \mathfrak{b}_1, 3) = \mathfrak{b}_2,\ \mathfrak{b}_3 = \mathfrak{b}_2^\frown \eta_3, \text{etc.}
\end{aligned}$$

Now there exist mappings $u$, $v \in \kappa^\omega$ with $u = \bigcup\{\mathfrak{a}_n : n \in \omega\}$, $v = \bigcup\{\mathfrak{b}_n : n \in \omega\}$. Since $\eta$ is a fixed point of $\Theta$ we have $u^*$, $v^* \leq \eta$. But on the other hand the $\eta_n$ converge to $\eta$ and this yields $u^* = v^* = \eta \in S$, hence $u$, $v \in S^*$. Note that $u \in U$ and $v \in V$. Furthermore, observe that $\operatorname{int}\overline{f(B(\mathfrak{a}_n))} \cap f(B(\mathfrak{b}_n)) \neq \varnothing$ holds for all $n$. Since $(B(\mathfrak{a}_n) : n \in \omega)$ is a neighborhood base of $u$ and $(B(\mathfrak{b}_n) : n \in \omega)$ is a neighborhood base of $v$ this implies $u = v$, hence $U \cap V \neq \varnothing$. This proves (3).

Now it is clear that (1) implies (4) since (2) and (3) together imply (4). Obviously (4) implies both, (2) and (3). We prove that (2) implies (1). Let $C$ be $\omega$-closed and cofinal. Define open dense subsets $G_n$ of $S^*$ by $G_n = \{f \in S^*\colon\ \exists k \geq n \sup_{i \leq k} f(i) \in C\}$. Then $\bigcap\{G_n\colon\ n \in \omega\} \neq \varnothing$. Choose $f$ herein, then $f^* \in S \cap C$. This proves (1). It remains to prove that (3) implies (1). Assume that $S$ is not $\omega$-stationary and choose an

$\omega$-closed unbounded set $C$ with $S \cap C = \varnothing$. We define a bijective mapping $*\colon \kappa \to \kappa$ with the following properties:

(i) $\alpha^{**} = \alpha$ for all $\alpha < \kappa$;

(ii) If $\alpha \notin C$, then $\alpha < \alpha^*$ and $\alpha^* \in C$.

We proceed by induction. Assume that $\gamma^*$ has been defined for all $\gamma < \alpha$. If there exists $\gamma < \alpha$ such that $\gamma^* = \alpha$ then define $\alpha^* = \gamma$. If such a $\gamma$ does not exist then let $\alpha^*$ be the first element of $C$ greater than $\alpha$ with $\alpha^* \notin \{\gamma^*\colon \gamma < \alpha\}$.

We claim that $*$ has the following property. If $f \in S^*$ and if $g$ is defined by $g(n) = f(n)^*$, then $g$ does not belong to $S^*$.

Indeed, let $n_1, n_2, \ldots$ be those indices with $f(n_i) < f(n_i)^*$ and let $m_1, m_2, \ldots$ be the remaining indices with $f(m_i) > f(m_i)^*$. By the construction of $*$ we must have $f(m_i) \in C$, hence $f \in S^*$ implies $\sup_i f(n_i) > \sup_i f(m_i)$. On the other hand we have $g(n_i) = f(n_i)^* \in C$ by the construction of $*$. Now $f(n_i)^* > f(n_i) > f(m_i) > f(m_i)^*$ yields $g^* = \sup_i f(n_i)^* \in C$, hence $g \notin S^*$.

We finish the proof by showing that $S^*$ is not a $B_r$ space. Let $\tau$ denote the original topology on $S^*$. For $\alpha_i \in \kappa$, $i \leq n$ define $B^*(\alpha_1, \ldots, \alpha_n) := B(\alpha_1, \ldots, \alpha_n) \cup B(\alpha_1^*, \ldots, \alpha_n^*)$. The set $\mathfrak{B}$ of the $B^*(\mathfrak{a})$ with $\mathfrak{a} \in FS$ is a base for a new topology $\tau^*$ on $S^*$ which is strictly coarser than $\tau$. We prove that $\tau^*$ is Hausdorff. Let $f,\ g \in S^*$, $f \neq g$ be given. Assume that $B^*(f(0), \ldots, f(n)) \cap B^*(g(0), \ldots, g(n)) \neq \varnothing$ holds for all $n$. Since $\tau$ is Hausdorff we may assume that $B(f(0), \ldots, f(n)) \cap B(g(0)^*, \ldots, g(n)^*) \neq \varnothing$ holds for all $n$. But by the claim proved above this implies $g \notin S^*$, a contradiction. Hence $\tau^*$ is Hausdorff. It remains to prove that the mapping id: $(S^*, \tau) \to (S^*, \tau^*)$ is nearly open. To this end we prove $B^*(\alpha_0, \ldots, \alpha_n) \subset \mathrm{cl}_{\tau^*}(B(\alpha_0, \ldots, \alpha_n))$. Let $f$ be in the left hand side. We may assume that $f \in B(\alpha_0^*, \ldots, \alpha_n^*)$. We have to prove that for all $m \geq n$ $B^*(f(0) \cdots f(m)) \cap B(\alpha_0 \cdots \alpha_n) \neq \varnothing$. Choose $\alpha_{n+1}, \ldots, \alpha_m$ with $\alpha_i^* = f(i)$ and define $h \in S^*$ such that $h(i) = \alpha_i$ for all $i \leq m$. Then $h$ obviously belongs to both $B(\alpha_0 \cdots \alpha_n)$ and $B(f(0)^* \cdots f(m)^*)$. This proves that the identity mapping is nearly open. But now we have arrived at a contradiction and this proves the statement. $\square$

We can now give the promised counterexample. Let $\kappa = \omega_1$, then stationary and $\omega$-stationary sets coincide. Choose two stationary sets $S, T$ in $\omega_1$ with $S \cap T = \varnothing$. (For the existence of such sets see **[So]**.) Then $S^*$ and $T^*$ are both $B_r$ spaces but their sum $S^* \oplus T^*$ is not a $B_r$ space. To see this regard the natural mapping $f\colon S^* \oplus T^* \to \kappa^\omega$. $f$ is a continuous, nearly open bijection onto $f(S^* \oplus T^*)$ but $S^*$ and $T^*$ both being dense in $\kappa^\omega$, it cannot be an open map.

THEOREM 4. *Let $S$ be a cofinal subset of $\kappa$. The following statements are equivalent*:

(1) *$S$ contains some $\omega$-closed cofinal subset*;

(2) *$S^*$ contains a dense completely metrizable subspace (and therefore is almost Čech complete).*

(3) *$S^*$ is a Baire $(\beta)$-space.*

(4) *For every Baire $B_r$ space $E$ the sum $E \oplus S^*$ is a $B_r$ space.*

*Proof.* (1) implies (2) for if $C \subset S$ is $\omega$-closed and cofinal then $C^*$ is a dense completely metrizable subspace of $S^*$. (2) obviously implies (3). (3) implies (4) by Proposition 4. Finally, (4) implies (1) for if $S$ does not contain any $\omega$-closed cofinal set then there exists an $\omega$-stationary set $T$ with $S \cap T = \varnothing$. But then the sum $T^* \oplus S^*$ is not a $B_r$ space, a contradiction with (4). □

**4. Products of $B_r$ spaces.** In this section we examine products of $B_r$ spaces. We start with a negative result stating that even the product of two $B_r$ spaces need not be a $B_r$ space.

PROPOSITION 6. *Let $S$, $T \subset \kappa$ be $\omega$-stationary subsets. The following statements are equivalent*:

(1) *$S \cap T$ is $\omega$-stationary*;

(2) *$S^* \otimes T^*$ is a Baire space*;

(3) *$S^* \otimes T^*$ is a $B_r$ space.*

*Proof.* (1) implies (2). This may be established using the method of proof in **[FK]**, ex. 1. On the other hand, if $S \cap T$ is not $\omega$-stationary then choose an $\omega$-closed cofinal set $C$ with $S \cap T \cap C = \varnothing$. Using again ex. 1 in **[FK]** one derives that $S^* \otimes T^*$ is not a Baire space. (1) implies (3). This may be obtained by a slight modification of the proof of Theorem 3, (1) $\Rightarrow$ (3). It remains to prove that (3) implies (1). Assume that $S \cap T$ is not $\omega$-stationary and choose an $\omega$-closed cofinal set $C$ with $S \cap T \cap C = \varnothing$. We construct a bijection $*$: $\kappa \times \kappa \to \kappa \times \kappa$ with the following properties:

(a) $(\alpha, \beta)^{**} = (\alpha, \beta)$ for all $(\alpha, \beta) \in \kappa \times \kappa$;

(b) If $(\alpha, \beta)^* = (\gamma, \delta)$, then either $\alpha = \beta \in C$, $\alpha = \beta > \max\{\gamma, \delta\}$ or $\gamma = \delta \in C$, $\gamma = \delta > \max\{\alpha, \beta\}$.

We define $*$ by induction. Let $\rho$: $\kappa \to \kappa \times \kappa$ be some bijection. Assume that $\rho(\gamma)^*$ has been defined for all $\gamma < \alpha$. If there exists $\gamma < \alpha$ with $\rho(\gamma)^* = \rho(\alpha)$ then define $\rho(\alpha)^* = \rho(\gamma)$. Otherwise choose $\eta \in C$ such that $(\eta,\eta) \notin \{\rho(\gamma), \rho(\gamma)^*\colon \gamma < \alpha\}$ and such that $\eta > \max\{\varepsilon, \zeta\}$, where $\rho(\alpha) = (\varepsilon, \zeta)$. Define $\rho(\alpha)^* = (\eta,\eta)$.

Assume that * has been defined. For $\alpha_0, \dots, \alpha_n$, $\beta_0, \dots, \beta_n$ in $\kappa$ let

$$\begin{aligned} &B^*((\alpha_0,\beta_0),\dots,(\alpha_n,\beta_n)) \\ &\quad := B((\alpha_0,\beta_0),\dots,(\alpha_n,\beta_n)) \cup B((\alpha_0,\beta_0)^*,\dots,(\alpha_n,\beta_n)^*), \end{aligned}$$

where

$$B((\gamma_0,\delta_0),\dots,(\gamma_n,\delta_n)) := B_S(\gamma_0,\dots,\gamma_n) \times B_T(\delta_0,\dots,\delta_n).$$

The sets $B^*(\mathrm{a})$ define a topology $\tau^*$ on $S^* \times T^*$ which is strictly coarser than the original topology $\tau$. We prove that $\tau^*$ is Hausdorff. Let $(f_1, g_1)$, $(f_2, g_2) \in S^* \times T^*$, $(f_1, g_1) \neq (f_2, g_2)$ be given. Assume that for all $n \in \omega$ we have

$$\begin{aligned} &B^*((f_1(0),g_1(0)),\dots,(f_1(n),g_1(n))) \\ &\qquad \cap B^*((f_2(0),g_2(0)),\dots,(f_2(n),g_2(n))) \neq \varnothing. \end{aligned}$$

Since $\tau$ is Hausdorff we may assume that for all $n \in \omega$

$$\begin{aligned} &B((f_1(0),g_1(0)),\dots,(f_1(n),g_1(n))) \\ &\qquad \cap B((f_2(0)g_2(0))^*,\dots,(f_2(n)g_2(n))^*) \end{aligned}$$

is nonempty. This yields $(f_1(n), g_1(n))^* = (f_2(n), g_2(n))$ for all $n$. Now let $n_1$, $n_2 \cdots$ denote those indices with $\max\{f_2(n_i), g_2(n_i)\} < f_1(n_i) = g_1(n_i) \in C$ and let $m_1$, $m_2, \cdots$ be the remaining indices which consequently satisfy $\max\{f_1(m_i), g_1(m_i)\} < f_2(m_i) = g_2(m_i) \in C$. Since $S \cap T \cap C = \varnothing$ we have $\sup_i g_1(n_i) < \sup_i g_1(m_i)$. By the definition of the sequences $(n_i)$, $(m_i)$ we obtain the following inequalities:

$$\begin{aligned} \sup_i f_2(m_i) = \sup_i g_2(m_i) &\geq \sup_i \max\{f_1(m_i), g_1(m_i)\} \\ &\geq \sup_i g_1(m_i) = g^*_1 > \sup_i g_1(n_i) = \sup_i f_1(n_i) \\ &\geq \sup_i \max\{f_2(n_i), g_2(n_i)\} \end{aligned}$$

hence $f_2^* = g_2^* = \sup_i f_2(m_i) = \sup_i g_2(m_i) \in C$, contradicting the fact that $S \cap T \cap C = \varnothing$. Hence $\tau^*$ is a Hausdorff topology.

It remains to prove that the mapping

$$\mathrm{id}\colon (S^* \times T^*, \tau) \to (S^* \times T^*, \tau^*)$$

is nearly open. We prove

$$B^*((\alpha_0,\beta_0),\dots,(\alpha_n,\beta_n)) \subset \mathrm{cl}_{\tau*} B((\alpha_0,\beta_0),\dots,(\alpha_n,\beta_n)).$$

Let $(f, g) \in B^*((\alpha_0, \beta_0), \dots, (\alpha_n, \beta_n))$. We may assume $(f, g) \in B((\alpha_0, \beta_0)^*, \dots, (\alpha_n, \beta_n)^*)$. Let $m \geq n$. We have to prove that

$$B^*((f(0), g(0)), \dots, (f(m), g(m))) \cap B((\alpha_0, \beta_0), \dots, (\alpha_n, \beta_n)) \neq \varnothing .$$

We choose pairs $(\alpha_{n+1}, \beta_{n+1}), \dots, (\alpha_m, \beta_m)$ such that $(\alpha_\iota, \beta_i)^* = (f(i), g(i))$. Since $S^*$ and $T^*$ are both dense in $\kappa^\omega$ we can find $f_1 \in S^*$, $g_1 \in T^*$ with $f_1(i) = \alpha_\iota$, $g_1(i) = \beta_i$ for $i \leq m$. But then we have

$$(f_1, g_1) \in B((\alpha_0, \beta_0), \dots, (\alpha_n, \beta_n)) \cap B((f(0), g(0))^*, \dots, (f(m), g(m))^*).$$

This proves our claim. But now we have proved that (3) does not hold, a contradiction. □

Now it is clear how to give an example of two $B_r$ spaces whose product is no longer a $B_r$ space. Take disjoint stationary subsets $S$,$T$ of $\omega_1$, then $S^*$, $T^*$ are $B_r$ spaces but $S^* \otimes T^*$ is not.

So far we have only obtained a negative result. And in fact, it seems very difficult to obtain positive answers for the problem of invariance of the class of $B_r$ spaces even under finite products. This is corroborated by the fact that even if one of the factor spaces is compact we do not know whether the product of two $B_r$ spaces is again $B_r$. A special positive answer is provided by the class of almost Čech complete spaces, which is closed under countable products. Another special situation in which positive answers occur is provided by the results in §1. Indeed, if $(E_\iota : \iota \in I)$ is a family of copies of the space $E$, then $\oplus \{ E_\iota : \iota \in I \}$ is homeomorphic with the product $E \otimes I$ where $I$ is endowed with the discrete topology. Hence in view of Theorem 1 and Proposition 2 we may state

Proposition 7. *If $E$ is a $B_r$ space and if $D$ is a discrete space, then $E \otimes D$ is again a $B_r$ space.* □

In the remainder of this section we will apply the idea, which lead to this observation in the category of topological groups.

Theorem 5. *Let $G$ be an abelian $B_r$ group. The following statements are equivalent*:

(1) *Whenever $D$ is a discrete abelian topological group, then the product group $G \otimes D$ is again a $B_r$ group.*

(2) *Whenever $H$ is a subgroup of the completion $\tilde{G}$ of $G$ with $G \cap H = \{ e \}$, then $H = \{ e \}$ must hold.*

*Proof*. Assume (1). Let $H$ be a subgroup of $\tilde{G}$ with $G \cap H = \{e\}$. Let $f$: $G \otimes H_d \to \tilde{G}$ be the homomorphism defined by $f(x, y) = xy$. Thus $f$ is a continuous bijection onto $f(G \times H)$. Here $H_d$ denotes $H$ with discrete topology. $f$ is nearly open since $G$ is dense in $\tilde{G}$. Hence (1) implies that $f$ is open. This proves $H = \{e\}$. On the other hand, assume that (2) holds. Let $f$: $G \otimes D \to H$ be a bijective, continuous and nearly open homomorphisms with $D$ a discrete abelian group and $H$ a Hausdorff topological group. Since $G \times \{e\}$ is open in $G \times D$, the restriction $f \restriction G \times \{e\}$ is again a continuous, nearly open, bijective homomorphism onto $f(G \times \{e\})$ and therefore is in fact open. It remains to prove that $f(G \times \{e\})$ is closed in $H$. Let $\tilde{H}$ be the completion of $H$. (Note that $H$ is abelian since $G$ and $D$ are.) Let $\tilde{G}$ be the closure of $f(G \times \{e\}) \approx G$ in $\tilde{H}$. Thus $\tilde{G}$ is the completion of $G$. Let $F := f(\{e\} \times D) \cap \tilde{G}$. Thus $F$ is a subgroup of $\tilde{G}$ with $G \cap F = \{e\}$. In view of (1) this implies $F = \{e\}$ in $\tilde{G}$. Hence in $H$ we have $\overline{f(G \times \{e\})} \cap f(\{e\} \times D) = \{e\}$. But this implies that $f(G \times \{e\})$ is closed in $H$. □

Following Banaschewski, (cf. [**Bn**]), a Hausdorff abelian topological group $G$ is called essentially embedded into its completion $\tilde{G}$ if for every closed subgroup $C$ of $\tilde{G}$ the relation $G \cap C = \{e\}$ implies $C = \{e\}$. Hence a group $G$ satisfying (2) above might be called strongly essentially embedded into $\tilde{G}$. Now using the ideas from [**Bn**], one can easily prove that a Hausdorff abelian topological group $G$ is a $B_r$ group if and only if its completion $\tilde{G}$ is $B_r$ and the embedding $G \to \tilde{G}$ is essential (see for instance [**Gr**]). Now if $Q$ is the group defined at the end of §1 then $Q$ is essentially embedded in its completion $S^1$ and therefore is a $B_r$ group. But obviously $Q$ is not strongly essentially embedded into $S^1$, hence there exists a discrete abelian topological group $D$ such that $Q \otimes D$ is no longer a $B_r$ group.

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