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ON CONTINUOUS APPROXIMATIONS FOR MULTIFUNCTIONS

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Problems concerning the approximation of convex valued multifunctions by continuous ones are considered. Approximation results of the type obtained by Gel'man, Cellina, and Hukuhara for Pompeiu-Hausdorff upper semicontinuous multifunctions are shown to hold for some larger classes of multifunctions. Moreover, it is proved that Pompeiu-Hausdorff semicontinuous multifunctions, with convex bounded values, are continuous almost everywhere (in the sense of the Baire category). As an application, an alternative proof is given of Kenderov's theorem stating that a maximal monotone operator is almost everywhere single-valued.

1. Introduction and preliminaries. Let X be a metric space. Let Y be a normed space. Denote by $\mathscr{C}(Y)$ (resp. $\mathscr{C}_b(Y)$, $\mathscr{C}_k(Y)$) the class of all nonempty subsets of Y which are convex (resp. convex bounded, convex compact). In any metric space, $S(u, r)$ stands for the open ball around u with radius $r > 0$.

We shall consider the following approximation problems (for the terminology see below):

I. Given a multifunction F: $X \to \mathscr{C}_b(Y)$ and an $\varepsilon > 0$, find an h-continuous multifunction G: $X \to \mathscr{C}_b(Y)$ such that $h(\text{graph } G, \text{graph } F) \leq \varepsilon$ (where h denotes the Pompeiu-Hausdorff pseudometric).

II. Given a multifunction F: $X \to \mathscr{C}(Y)$ and an $\varepsilon > 0$, find a continuous single-valued function g: $X \to Y$ such that $h^*(\text{graph } g, \text{graph } F) \leq \varepsilon$ (where h^* denotes the separation function).

III. Given a multifunction F: $X \to \mathscr{C}_b(Y)$, find a sequence $\{G_n\}$ of h-continuous multifunctions G_n: $X \to \mathscr{C}_b(Y)$ satisfying for each $x \in X$, $h(G_n(x), F(x)) \to 0$ as $n \to +\infty$, and $G_n(x) \supset F(S(x, \sigma_n(x)))$ for some $\sigma_n(x) > 0$.

Apparently, the idea of constructing continuous approximations for a multifunction goes back to Von Neumann [**29**]. When F is upper semicontinuous in the sense of the Pompeiu-Hausdorff separation h^* "h^*-u.s.c.", the approximation problems I, II, and III have been investigated by Gel'man (see references in [**2**]), Cellina [**5**, **6**], and Hukuhara [**17**], respectively. Further results can be found in [**23**], [**24**], [**13**], [**8**].

In this paper, following some ideas of Gel'man, Cellina, and Hukuhara, we shall consider the above approximation problems for more general classes of multifunctions. §2 is devoted to the approximation problem I. It is shown that such problem is solvable for F in the class of locally convexifying and locally bounded multifunctions. In view of Corollary 2.1, this class seems to be a natural setting for solving problem I. In §3 the approximation problem II is considered and some results of the type proved by Cellina [**5**, **6**], are obtained. In §4 we treat the problem III. Furthermore, we characterize some classes of semicontinuous multifunctions by the convergence properties of appropriate h-continuous approximations. In §5 we give some applications. For example, using an approximation result of §4, it is proved that each h^*-u.s.c. multifunction $F\colon X \to \mathscr{C}_b(Y)$ is h-continuous except at points of a Baire first category set (see [**15**], [**22**], [**7**], [**12**], [**20**], [**21**] for similar results). Also we deduce a theorem due to Kenderov [**18**] stating that a maximal monotone operator is almost everywhere single-valued.

Let us introduce, now, notation and terminology.

Throughout this paper X and Z denote metric spaces, while Y denotes a (real) normed space. The distance function in X, Z is denoted by d, and the norm of Y by $\|\cdot\|$. 2^Z (resp. 2^Y) stands for the family of all nonempty subsets of Z (resp. Y). We shall consider the following subsets of 2^Y:

$$\mathscr{C}(Y) = \{A \in 2^Y | A \text{ is convex}\}$$

$$\mathscr{C}_b(Y) = \{A \in 2^Y | A \text{ is convex bounded}\}$$

$$\mathscr{C}_k(Y) = \{A \in 2^Y | A \text{ is convex compact}\}.$$

For any $A \subset Y$, $\operatorname{co} A$ and $\overline{\operatorname{co}} A$ denote respectively the convex hull and the closed convex hull of A.

Let $\mathscr{X}$ be an arbitrary metric space, with distance d. By $S(x,\sigma)$ we denote the open ball in $\mathscr{X}$ with center at x and radius $\sigma > 0$. In a normed space, for notational convenience we set $S = S(0,1)$. For any set $A \subset \mathscr{X}$ we denote by $\bar{A}$ and $\operatorname{int} A$ respectively the closure and the interior of A.

Given a point $a \in \mathscr{X}$ and a nonempty set $B \subset \mathscr{X}$, we put

$$r(a,B) = \inf\{\, d(a,b) | b \in B\}.$$

If A and B are nonempty subsets of $\mathscr{X}$, we define

$$h^*(A,B) = \sup\{\, r(a,B) | a \in A\},$$

$$h(A,B) = \max\{\, h^*(A,B),\ h^*(B,A)\}.$$

$h^*(A, b)$ is called the (generalized) separation of A from B. As it is well known [**4**, p. 38], h is a (generalized) pseudometric in $2^{\mathscr{X}}$; when h is restricted to the space of the nonempty closed subsets of $\mathscr{X}$, it becomes the Pompeiu-Hausdorff (generalized) metric. In particular, on the space of the nonempty bounded (resp. bounded closed) subsets of a normed space, h is the usual Pompeiu-Hausdorff pseudometric (resp. metric).

For $A \subset \mathscr{X}$ ($A \neq \varnothing$) and $\varepsilon > 0$, we set $N_\varepsilon(A) = \{x \in \mathscr{X} | r(x, A) < \varepsilon\}$. Observe that if $h^*(A, B) < \varepsilon$ (resp. $A \subset N_\varepsilon(B)$), then $A \subset N_\varepsilon(B)$ (resp. $h^*(A, B) \leq \varepsilon$). From these it follows easily that $h^*(A, B) = \inf\{\varepsilon > 0 | A \subset N_\varepsilon(B)\}$. Moreover, if $\mathscr{X}$ is a normed space, we have

$$h^*(A, B) = \inf\{\varepsilon > 0 | A \subset B + \varepsilon S\},$$

$$h(A, B) = \inf\{\varepsilon > 0 | A \subset B + \varepsilon S,\ B \subset A + \varepsilon S\}.$$

Let $\mathscr{X}_1$, $\mathscr{X}_2$ be metric spaces with distance functions d_1, d_2. We always assume the Cartesian product $\mathscr{X}_1 \times \mathscr{X}_2$ to be endowed with the metric

$$e\big((x_1, x_2), (x_1', x_2')\big) = \max\{d_1(x_1, x_1'),\ d_2(x_2, x_2')\},$$

where $(x_1, x_2), (x_1', x_2') \in \mathscr{X}_1 \times \mathscr{X}_2$.

By a neighborhood of a point $x \in \mathscr{X}$ we mean an open subset of $\mathscr{X}$ containing x.

A family $\mathscr{P} = \{p_i\}_{i \in I}$ of continuous functions p_i: $X \to [0, 1]$ is called a partition of unity on X if the family $\{\operatorname{supp} p_i\}_{i \in I}$ consisting of the closed sets $\operatorname{supp} p_i = \overline{\{x \in X | p_i(x) > 0\}}$ is a neighborhood finite closed covering of X, and $\sum_{i \in I} p_i(x) = 1$ for each $x \in X$. We say that a partition $\mathscr{P}$ of unity is subordinated to a given open covering $\{U_i\}_{i \in I}$ of X if, for every $i \in I$, the support of each p_i lies in the corresponding U_i. It is well known [**9**, p. 170] that each open covering of a metric space admits a partition of unity subordinated to it. For any given partition $\mathscr{P} = \{p_i\}_{i \in I}$ of unity on X and any $A \subset X$ we set

$$D_{\mathscr{P}}(A) = \{i \in I | A \cap \operatorname{supp} p_i \neq \varnothing\}.$$

Observe that each $x \in X$ has a neighborhood V such that $D_{\mathscr{P}}(V)$ is finite.

By a multifunction F: $X \to 2^Z$ we mean a mapping F with domain X and range contained in 2^Z. The set

$$\operatorname{graph} F = \{(x, z) \in X \times Z | x \in X \text{ and } z \in F(x)\}$$

is called the graph of F. For arbitrary $A \subset X$ and $B \subset Z$ we set

$$F(A) = \bigcup\{F(x) | x \in A\}, \quad F^-(B) = \{x \in X | F(x) \cap B \neq \varnothing\}.$$

Note that $F(A)$ is a subset of Z.

We shall review some definitions of upper semicontinuity "u.s.c." and lower semicontinuity "l.s.c." for multifunctions. For the reader's convenience, the definitions are compared in a series of remarks and examples (for the proofs see **[1]**, **[16]**, **[26]**).

DEFINITION 1.1. F: $X \to 2^Z$ is called u.s.c. if for every set C closed in Z the set $F^-(C)$ is closed in X.

REMARK 1.1. F: $X \to 2^Z$ is u.s.c. if and only if for every set V open in Z the set $\{x \in X | F(x) \subset V\}$ is open in X.

DEFINITION 1.2. F: $X \to 2^Z$ is called h^*-u.s.c. if for every $x_0 \in X$ and for every $\varepsilon > 0$ there is a $\delta > 0$ such that $h^*(F(x), F(x_0)) < \varepsilon$ for every $x \in S(x_0, \delta)$.

DEFINITION 1.3. F: $X \to 2^Z$ is called K-closed if graph F is a closed set in $X \times Z$.

In the abbreviation "h^*-u.s.c.", h^* is written to emphasize the role of the Pompeiu-Hausdorff (generalized) separation h^*. In Definition 1.3, K stands for Kuratowski.

REMARK 1.2. F: $X \to 2^Z$ is K-closed if and only if, given any two sequences $\{x_n\} \subset X$ and $\{z_n\} \subset Z$ such that $z_n \in F(x_n)$, $x_n \to x$, and $z_n \to z$ then we have $z \in F(x)$.

REMARK 1.3. Each u.s.c. multifunction F: $X \to 2^Z$ is h^*-u.s.c.

EXAMPLE 1.1. Let F: $\mathbf{R} \to 2^{\mathbf{R}^2}$ be defined by $F(x) = \{(x, y) \in \mathbf{R}^2 | 0 \le y \le 1/|x|\}$ if $x \neq 0$, and $F(0) = \{(0, y) \in \mathbf{R}^2 | y \ge 0\}$. Clearly F is h^*-u.s.c. However F is not u.s.c. because while $C = \{(1/n, n) \in \mathbf{R}^2 | n \in \mathbf{N}\}$ is closed in $\mathbf{R}^2$, the set $F^-(C)$ is not closed in $\mathbf{R}$.

REMARK 1.4. Each closed valued h^*-u.s.c. multifunction F: $X \to 2^Z$ is K-closed.

EXAMPLE 1.2. The multifunction F: $\mathbf{R} \to 2^{\mathbf{R}^2}$ defined by $F(x) = \{(t, xt) \in \mathbf{R}^2 | t \in \mathbf{R}\}$ is K-closed but not h^*-u.s.c.

REMARK 1.5. Let F: $X \to 2^Z$ be compact valued. Then F is u.s.c. if and only if F is h^*-u.s.c.

REMARK 1.6. Let F: $X \to 2^Z$ be compact valued. Let $\overline{F(X)} \subset Z$ be compact. Then F is h^*-u.s.c. if and only if F is K-closed.

DEFINITION 1.4. F: $X \to 2^Z$ is called l.s.c. if for every set V open in Z the set $F^-(V)$ is open in X.

DEFINITION 1.5. F: $X \to 2^Z$ is called h^*-l.s.c. if for every $x_0 \in X$ and for every $\varepsilon > 0$ there is a $\delta > 0$ such that $h^*(F(x_0), F(x)) < \varepsilon$ for every $x \in S(x_0, \delta)$.

REMARK 1.7. Each h^*-l.s.c. multifunction F: $X \to 2^Z$ is l.s.c.

The multifunction F defined in Example 1.2 is l.s.c. but not h^*-l.s.c.

REMARK 1.8. Let F: $X \to 2^Z$ be compact valued. Then F is l.s.c. if and only if F is h^*-l.s.c.

DEFINITION 1.6. A multifunction F: $X \to 2^Z$ which is both u.s.c. (resp. h^*-u.s.c.) and l.s.c. (resp. h^*-l.s.c.) is called continuous (resp. h-continuous).

REMARK 1.9. Let F: $X \to 2^Z$ be compact valued. In view of Remarks 1.5 and 1.8, F is continuous if and only if F is h-continuous. In general, neither of these statements implies the other as it is shown in the examples below.

EXAMPLE 1.3. The multifunction F: $\mathbf{R} \to 2^{\mathbf{R}}$ defined by $F(x) = [-1/|x|, 1/|x|]$ if $x \neq 0$, and $f(0) = \mathbf{R}$, is continuous but not h-continuous.

EXAMPLE 1.4. Let c_0 be the Banach space of all real sequences $z = (z_1, z_2, \ldots)$ such that $z_n \to 0$, with norm $\|z\| = \sup\{|z_n| \; |n \in \mathbf{N}\}$. Denote by $A \subset c_0$ a closed bounded convex body with the following property: there exists a point $\tilde{z} \in c_0 \setminus A$ such that $\|\tilde{z} - a\| > r(\tilde{z}, A)$ for each $a \in A$. The existence of such a set A follows from **[11]**. Define F: $\mathbf{R} \to 2^{c_0}$ by $F(x) = A$ if $x \leq 0$, and $F(x) = \overline{A + xS}$ if $x > 0$. Clearly F is a (closed valued) h-continuous multifunction. On the other hand for $B = \{z \in c_0 | \; \|z - \tilde{z}\| \leq r(\tilde{z}, A)\}$, a closed subset of c_0, we have $F^-(B) = (0, +\infty)$, which is not closed in $\mathbf{R}$. Hence F is not u.s.c. and so not even continuous.

DEFINITION 1.7. F: $X \to 2^Z$ is called weakly h^*-u.s.c. at x_0 if for every $\varepsilon > 0$ and $\eta > 0$ there is a δ $(0 < \delta \leq \eta)$ and there is a point

$x' \in S(x_0, \delta)$ such that $h^*(F(x), F(x')) < \varepsilon$ for every $x \in S(x_0, \delta)$. F is called weakly h^*-u.s.c. if it is weakly h^*-u.s.c. at each $x_0 \in X$.

If $x' = x_0$ the above definition reduces to that of an h^*-u.s.c. multifunction. While each h^*-u.s.c. multifunction is weakly h^*-u.s.c., the converse is not true in general.

It is easy to see that a weakly h^*-u.s.c. multifunction is not necessarily K-closed. The next example shows that a K-closed multifunction can fail to be weakly h^*-u.s.c.

EXAMPLE 1.5. Let X be the set of the rational numbers with the usual metric. Order X into a sequence $\{x_1, x_2, \dots\}$. Define F: $X \to 2^{\mathbf{R}}$ by $F(x_n) = \{n\}$. Clearly F is K-closed but not weakly h^*-u.s.c.

Let f: $X \to Z$ be a single-valued function. Then f is u.s.c. (resp. l.s.c., h^*-u.s.c., h^*-l.s.c., weakly h^*-u.s.c.) if and only if f is continuous.

DEFINITION 1.8. F: $X \to 2^Z$ is called locally bounded if for each $x \in X$ there exists a $\delta > 0$ such that $F(S(x, \delta))$ is a bounded subset of Z.

DEFINITION 1.9. F: $X \to 2^Y$ is called locally convexifying if for every $x \in X$ and for every $\varepsilon > 0$ and $\eta > 0$, there is a $\sigma(x)$, $0 < \sigma(x) \leq \eta$, such that $\operatorname{co} F(S(x, \sigma(x))) \subset F(S(x, \sigma(x))) + \varepsilon S$.

Each weakly h^*-u.s.c. multifunction F: $X \to \mathscr{C}_b(Y)$ is locally convexifying and locally bounded. The converse is not true in general. Moreover a locally convexifying and locally bounded multifunction is not necessarily K-closed.

REMARK. 1.10. Some of the above definitions and remarks, in particular Definitions 1.1, 1.3, 1.4, and Remark 1.1, are meaningful also for multifunctions from a topological space to the nonempty subsets of another topological space.

2. Continuous multi-valued approximations for multifunctions. In this section we consider the problem of approximating a multifunction with convex bounded values by another one which is h-continuous.

THEOREM 2.1. *Let F: $X \to \mathscr{C}_b(Y)$ be locally convexifying and locally bounded. Then for every $\varepsilon > 0$ there is an h-continuous multifunction G: $X \to \mathscr{C}_b(Y)$ with the following properties*:

(i) *for each $x \in X$ there is a $\sigma(x) > 0$ such that $F(S(x, \sigma(x))) \subset G(x)$*;

(ii) $h(\operatorname{graph} G, \operatorname{graph} F) \leq \varepsilon$.

Proof. Let $\varepsilon > 0$. Let $z \in X$. Since F is locally bounded and locally convexifying, there is a $\delta_z = \delta(z, \varepsilon)$ $(0 < \delta_z < \varepsilon/2)$ such that $F(S(z, \delta_z))$ is bounded and

$$\operatorname{co} F(S(z, \delta_z)) \subset F(S(z, \delta_z)) + \varepsilon S.$$

Evidently $\mathscr{V} = \{S(z, \delta_z/3)\}_{z \in X}$ is an open covering of X. Denote by $\mathscr{P} = \{p_z\}_{z \in X}$ a partition of unity subordinated to $\mathscr{V}$. For each $x \in X$, set

$$G(x) = \sum_{z \in X} p_z(x) G_z \quad \text{where } G_z = \operatorname{co} F(S(z, \delta_z/3)). \tag{2.1}$$

It is routine to verify that (2.1) defines an h-continuous multifunction G: $X \to \mathscr{C}_b(Y)$.

Let $\tilde{x} \in X$. Let $D_{\mathscr{P}}(\tilde{x}) = \{z_1, z_2, \ldots, z_k\}$. We have

$$G(\tilde{x}) = \sum_{i=1}^{k} p_{z_i}(\tilde{x}) G_{z_i}.$$

Since $\tilde{x} \in \operatorname{supp} p_{z_i} \subset S(z_i, \delta_{z_i}/3)$ $(i = 1, 2, \ldots, k)$, there exists a $\sigma = \sigma(\tilde{x}) > 0$ such that $S(\tilde{x}, \sigma) \subset S(z_i, \delta_{z_i}/3)$ for each $i = 1, 2, \ldots, k$. Then we have

$$F(S(\tilde{x}, \sigma)) \subset \sum_{i=1}^{k} p_{z_i}(\tilde{x}) F(S(\tilde{x}, \sigma)) \subset \sum_{i=1}^{k} p_{z_i}(\tilde{x}) G_{z_i} = G(\tilde{x})$$

and so, since $\tilde{x}$ is arbitrary in X, (i) is fulfilled.

From (i) it follows that graph $F \subset$ graph G. Therefore to prove (ii) it is sufficient to show that h^*(graph G, graph $F) \leq \varepsilon$. To this end, let $(\tilde{x}, \tilde{y}) \in$ graph G. Set $D_{\mathscr{P}}(\tilde{x}) = \{z_1, z_2, \ldots, z_k\}$ and $\delta_{z_{i_0}} = \max\{\delta_{z_1}, \delta_{z_2}, \ldots, \delta_{z_k}\}$. Since $S(z_i, \delta_{z_i}/3) \subset S(z_{i_0}, \delta_{z_{i_0}})$, we have $G_{z_i} \subset \operatorname{co} F(S(z_{i_0}, \delta_{z_{i_0}}))$ and so

$$\begin{aligned} G(\tilde{x}) &= \sum_{i=1}^{k} p_{z_i}(\tilde{x}) G_{z_i} \subset \sum_{i=1}^{k} p_{z_i}(\tilde{x}) \operatorname{co} F\big(S\big(z_{i_0}, \delta_{z_{i_0}}\big)\big) \\ &= \operatorname{co} F\big(S\big(z_{i_0}, \delta_{z_{i_0}}\big)\big) \subset F\big(S\big(z_{i_0}, \delta_{z_{i_0}}\big)\big) + \varepsilon S. \end{aligned}$$

But $\tilde{y} \in G(\tilde{x})$ thus, for some $x' \in S(z_{i_0}, \delta_{z_{i_0}})$ and some $u \in \varepsilon S$, we have $\tilde{y} = y' + u$, where $y' \in F(x')$. Clearly,

$$d(\tilde{x}, x') \leq d(\tilde{x}, z_{i_0}) + d\big(z_{i_0}, x'\big) < \delta_{z_{i_0}}/3 + \delta_{z_{i_0}} < \varepsilon$$

and $\|\tilde{y} - y'\| < \varepsilon$. Therefore $r((\tilde{x}, \tilde{y})$, graph $F) < \varepsilon$, for (x', y') lies in graph F. Since $(\tilde{x}, \tilde{y})$ is arbitrary in graph G, it follows that h^*(graph G, graph $F) \leq \varepsilon$. This completes the proof.

REMARK 2.1. In addition to the hypotheses of Theorem 2.1, suppose that F is compact (that is whenever $B \subset X$ is bounded, the set $F(B)$ is contained in a compact convex subset of Y). Then, arguing as before one can prove the existence of a compact h-continuous multifunction G: $X \to \mathscr{C}_k(Y)$ satisfying the properties (i) and (ii) of Theorem 2.1.

THEOREM 2.2. *Let* F: $X \to \mathscr{C}_k(Y)$ *satisfy at each point* $x \in X$ *the condition*:

$$(2.2) \quad \lim_{\delta \to 0+} h(F(S(x,\delta)), A(x)) = 0, \quad \text{where } A(x) = \bigcap_{\rho > 0} \overline{F(S(x,\rho))}.$$

In addition, suppose that for every $\varepsilon > 0$ *there exists an h-continuous multifunction* G: $X \to \mathscr{C}_k(Y)$ *satisfying the conditions* (i) *and* (ii) *of Theorem* 2.1. *Then F is locally convexifying.*

Proof. For a contradiction, suppose that F is not locally convexifying. There exist then an $\tilde{x} \in X$, an $\varepsilon > 0$, and a decreasing sequence $\{\delta_n\}$ of positive numbers δ_n converging to zero, such that

$$(2.3) \quad \operatorname{co} F(S(\tilde{x},\delta_n)) \not\subset F(S(\tilde{x},\delta_n)) + 3\varepsilon S, \qquad n = 1,2,\dots .$$

Set $\tilde{A} = A(\tilde{x})$. By (2.2) we have that $h(F(S(\tilde{x},\delta_n)), \tilde{A}) \to 0$ as $n \to +\infty$, and so also $h(\operatorname{co} F(S(\tilde{x},\delta_n)), \operatorname{co}\tilde{A}) \to 0$ as $n \to +\infty$. Hence, $\tilde{A} \subset F(S(\tilde{x},\delta_n)) + (\varepsilon/2)S$ and $\operatorname{co} F(S(\tilde{x},\delta_n)) \subset \operatorname{co}\tilde{A} + (\varepsilon/2)S$ are satisfied for some n large enough. We have

$$\operatorname{co}\tilde{A} \not\subset \tilde{A} + 2\varepsilon S.$$

In fact, in the contrary case $\operatorname{co}\tilde{A} \subset \tilde{A} + 2\varepsilon S$, and thus

$$\operatorname{co} F(S(\tilde{x},\delta_n)) \subset \operatorname{co}\tilde{A} + (\varepsilon/2)S \subset \tilde{A} + (5/2)\varepsilon S \subset F(S(\tilde{x},\delta_n)) + 3\varepsilon S,$$

a contradiction to (2.3). Let $\tilde{y} \in \operatorname{co}\tilde{A}$ be such that $\tilde{y} \notin \tilde{A} + 2\varepsilon S$. Evidently, $S(\tilde{y},\varepsilon) \cap (\tilde{A} + \varepsilon S) = \varnothing$. Fix $n_0 \in \mathbf{N}$ such that $F(S(\tilde{x},\delta_{n_0})) \subset \tilde{A} + \varepsilon S$ and observe that

$$S(\tilde{y},\varepsilon) \cap F\big(S(\tilde{x},\delta_{n_0})\big) = \varnothing .$$

Set $\theta = \min\{\varepsilon, \delta_{n_0}\}$. Let (x,y) be an arbitrary point in graph F. Since $e((\tilde{x},\tilde{y}),(x,y)) \geq \|\tilde{y} - y\| \geq \varepsilon \geq \theta$ if $x \in S(\tilde{x},\delta_{n_0})$, and $e((\tilde{x},\tilde{y}),(x,y)) \geq d(\tilde{x},x) \geq \delta_{n_0} \geq \theta$ if $x \notin S(\tilde{x},\delta_{n_0})$, we have

$$(2.4) \quad r((\tilde{x},\tilde{y}), \operatorname{graph} F) \geq \theta.$$

On the other hand, by hypothesis there is an h-continuous multifunction G: $X \to \mathscr{C}_k(Y)$ which satisfies the condition (i) of Theorem 2.1, and is such that

$$(2.5) \quad h(\operatorname{graph} G, \operatorname{graph} F) \leq \theta/2.$$

For some $\sigma(\tilde{x}) > 0$ we have $F(S(\tilde{x}, \sigma(\tilde{x}))) \subset G(\tilde{x})$ and so $\overline{F(S(\tilde{x}, \delta_n))} \subset G(\tilde{x})$, provided n is large enough. Since $\tilde{y} \in \operatorname{co} \tilde{A} \subset \operatorname{co} \overline{F(S(\tilde{x}, \delta_n))} \subset G(\tilde{x})$, it follows that $(\tilde{x}, \tilde{y}) \in \text{graph } G$. Hence by (2.4) we obtain $h^*(\text{graph } G, \text{graph } F) \geq \theta$, and thus $h(\text{graph } G, \text{graph } F) \geq \theta$, a contradiction to (2.5). Therefore F is locally convexifying. This completes the proof.

Proposition 2.1. *Each compact multifunction F: $X \to \mathscr{C}_k(Y)$ satisfies the condition* (2.2).

Proof. Let $x_0 \in X$. It suffices to prove that $h^*(F(S(x_0, \delta)), A_0) \to 0$ as $\delta \to 0$, where $A_0 = A(x_0)$. If this is not true, there exists an $\varepsilon > 0$ and two sequences $\{x_n\} \subset X$ and $\{y_n\} \subset Y$ such that $x_n \to x_0$, $y_n \in F(x_n)$, and $r(y_n, A_0) > \varepsilon$ $(n \in \mathbf{N})$. By the compactness of F we assume (without loss of generality) that $y_n \to y_0 \in Y$, and so $r(y_0, A_0) \geq \varepsilon$. On the other hand, given any $\rho > 0$, for $n \in \mathbf{N}$ large enough we have $x_n \in S(x_0, \rho)$, which implies that $y_n \in F(S(x_0, \rho))$. Consequently $y_0 \in \overline{F(S(x_0, \rho))}$. Since $\rho > 0$ is arbitrary, it follows that $y_0 \in A_0$, a contradiction. This completes the proof.

From Remark 2.1, Proposition 2.1, and Theorem 2.2 we have:

Corollary 2.1. *Let F: $X \to \mathscr{C}_k(Y)$ be compact. Then the following statements are equivalent*:

(a) *F is locally convexifying.*

(b) *For every $\varepsilon > 0$ there exists a* (*compact*) *h-continuous multifunction G: $X \to \mathscr{C}_k(Y)$ satisfying the conditions* (i) *and* (ii) *of Theorem* 2.1.

For the next approximation theorem we need the following

Lemma 2.1. *Let F: $X \to \mathscr{C}_b(Y)$ be locally convexifying. Suppose that G_1: $X \to \mathscr{C}_b(Y)$ is a closed valued h-continuous multifunction satisfying the property*: *for each $x \in X$ there exists a $\sigma_1(x) > 0$ and a $\theta_1(x) > 0$ such that*

$$\operatorname{co} F(S(x, \sigma_1(x))) + \theta_1(x) S \subset G_1(x).$$

Then for every $\varepsilon > 0$ there exists a closed valued h-continuous multifunction G_2: $X \to \mathscr{C}_b(Y)$ with the following properties:

(i) *for every $x \in X$ there exists a $\sigma_2(x) > 0$ and a $\theta_2(x) > 0$ such that*

$$\operatorname{co} F(S(x, \sigma_2(x))) + \theta_2(x) S \subset G_2(x);$$

(ii) *$G_2(x) \subset G_1(x)$ for each $x \in X$*;

(iii) $h(\text{graph } G_2, \text{graph } F) \leq \varepsilon$.

Proof. Let $\varepsilon > 0$. Let $z \in X$. From the hypothesis, there exists a $\sigma_1(z) > 0$ and $\theta_1(z) > 0$ such that $\operatorname{co} F(S(z, \sigma_1(z))) + \theta_1(z)S \subset G_1(z)$. By the h-continuity of G_1, there exists a ρ_z, $0 < \rho_z \leq \sigma_1(z)$, such that $G_1(z) \subset G_1(x) + \frac{1}{2}\theta_1(z)S$ for every $x \in S(z, \rho_z)$. Hence, $\operatorname{co} F(S(z, \sigma_1(z))) + \theta_1(z)S \subset G_1(x) + \frac{1}{2}\theta_1(z)S$ and so, since $G_1(x)$ is a convex closed set, by Rådström lemma **[27]** we have

$$\operatorname{co} F(S(z, \sigma_1(z))) + \tfrac{1}{2}\theta_1(z)S \subset G_1(x) \quad \text{for every } x \in S(z, \rho_z). \tag{2.6}$$

On the other hand F is locally convexifying, thus there is a δ_z, $0 < \delta_z \leq \min\{\rho_z, \varepsilon/2\}$, such that

$$\operatorname{co} F(S(z, \delta_z)) \subset F(S(z, \delta_z)) + (\varepsilon/2)S. \tag{2.7}$$

Set $\theta_z = \min\{\frac{1}{2}\theta_1(z), \varepsilon\}$. From (2.6) it follows that

$$\operatorname{co} F(S(z, \delta_z)) + \theta_z S \subset G_1(z) \quad \text{for every } x \in S(z, \delta_z), \tag{2.8}$$

because $\delta_z \leq \rho_z \leq \sigma_1(z)$ and $\theta_z \leq \frac{1}{2}\theta_1(z)$. Let $\mathscr{P} = \{p_z\}_{z \in X}$ be a partition of unity subordinated to the open covering $\mathscr{V} = \{S(z, \delta_z/3)\}_{z \in X}$ of X. For each $x \in X$, we put

$$G_2(x) = \overline{\sum_{z \in X} p_z(x) G_z} \quad \text{where } G_z = \operatorname{co} F\left(S\left(z, \frac{\delta_z}{3}\right)\right) + \frac{1}{4}\theta_z S$$

$$\theta_2(x) = \frac{1}{4} \sum_{z \in X} p_z(x)\theta_z.$$

It is routine to see that the above formulas define respectively a closed valued h-continuous multifunction G_2: $X \to \mathscr{C}_b(Y)$ and a positive continuous function θ_2: $X \to [0, \varepsilon/4]$.

Let us show that G_2 satisfies (i)–(iii). Let $\tilde{x} \in X$. Let $D_{\mathscr{P}}(\tilde{x}) = \{z_1, z_2, \ldots, z_k\}$. We have

$$G_2(\tilde{x}) = \overline{\sum_{i=1}^{k} p_{z_i}(\tilde{x}) G_{z_i}}, \qquad \theta_2(\tilde{x}) = \frac{1}{4}\sum_{i=1}^{k} p_{z_i}(\tilde{x})\theta_{z_i}. \tag{2.9}$$

(i) Indeed, $\tilde{x} \in \operatorname{supp} p_{z_i}(\tilde{x}) \subset S(z_i, \delta_{z_i}/3)$ $(i = 1, 2, \ldots, k)$, thus there is a $\sigma_2(\tilde{x}) > 0$ such that $S(\tilde{x}, \sigma_2(\tilde{x})) \subset S(z_i, \delta_{z_i}/3)$ for each $i = 1, 2, \ldots, k$. From this it follows

$$\operatorname{co} F(S(\tilde{x}, \sigma_2(\tilde{x}))) \subset \sum_{i=1}^{k} p_{z_i}(\tilde{x}) \operatorname{co} F\left(S\left(z_i, \frac{\delta_{z_i}}{3}\right)\right)$$

and so

$$\begin{aligned}\operatorname{co} F(S(\tilde{x}, \sigma_2(\tilde{x}))) + \theta_2(\tilde{x})S &\subset \sum_{i=1}^{k} p_{z_i}(\tilde{x})\left[\operatorname{co} F\left(S\left(z_i, \frac{\delta_{z_i}}{3}\right)\right) + \frac{1}{4}\theta_{z_i} S\right] \\ &\subset G_2(\tilde{x}).\end{aligned}$$

Since $\tilde{x}$ is arbitrary in X, (i) is proved.

(ii) For each $i = 1, 2, \ldots, k$ we have $\tilde{x} \in S(z_i, \delta_{z_i}/3)$ and hence, by (2.8), $G_{z_i} \subset G_1(\tilde{x})$. Therefore

$$\sum_{i=1}^{k} p_{z_i}(\tilde{x}) G_{z_i} \subset \sum_{i=1}^{k} p_{z_i}(\tilde{x}) G_1(\tilde{x}) = G_1(\tilde{x}),$$

from which $G_2(\tilde{x}) \subset G_1(\tilde{x})$ follows at once.

(iii) In view of (i), it suffices to show that $h^*(\text{graph } G_2, \text{graph } F) \leq \varepsilon$. Indeed, let $(\tilde{x}, \tilde{y}) \in \text{graph } G_2$. Set $D_{\mathscr{P}}(\tilde{x}) = \{z_1, z_2, \ldots, z_k\}$ and let $\delta_{z_{i_0}} = \max\{\delta_{z_1}, \delta_{z_2}, \ldots, \delta_{z_k}\}$. Since $S(z_i, \delta_{z_i}/3) \subset S(z_{i_0}, \delta_{z_{i_0}})$, we have $\operatorname{co} F(S(z_i, \delta_{z_i}/3)) \subset \operatorname{co} F(S(z_{i_0}, \delta_{z_{i_0}}))$. Using this inclusion in the formula giving $G_2(\tilde{x})$ (see (2.9)), one easily finds

$$G_2(\tilde{x}) \subset \overline{\operatorname{co} F\big(S\big(z_{i_0}, \delta_{z_{i_0}}\big)\big) + \theta_2(\tilde{x}) S}.$$

Hence, by (2.7),

$$G_2(\tilde{x}) \subset \overline{F\big(S\big(z_{i_0}, \delta_{z_{i_0}}\big)\big) + (\varepsilon/2) S + \theta_2(\tilde{x}) S}$$

and so $G_2(\tilde{x}) \subset F(S(z_{i_0}, \delta_{z_{i_0}})) + \varepsilon S$, because $\theta_2(\tilde{x}) \leq \varepsilon/4$. Then, as in the proof of Theorem 2.1, one shows that $r((\tilde{x}, \tilde{y}), \text{graph } F) < \varepsilon$. Since $(\tilde{x}, \tilde{y})$ is arbitrary in graph G_2, we infer that $h^*(\text{graph } G_2, \text{graph } F) \leq \varepsilon$. This completes the proof.

THEOREM 2.3. *Let F: $X \to \mathscr{C}_b(Y)$ be locally convexifying and locally bounded. Then there is a sequence $\{G_n\}$ of closed valued h-continuous multifunctions G_n: $X \to \mathscr{C}_b(Y)$ with the following properties*:

(i) *for every $n \in \mathbf{N}$ and $x \in X$ there is a $\sigma_n(x) > 0$ such that $F(S(x, \sigma_n(x))) \subset G_n(x)$;*

(ii) $G_1(x) \supset G_2(x) \supset \ldots$, *for each $x \in X$;*

(iii) $h(\text{graph } G_n, \text{graph } F) \to 0$ *as $n \to +\infty$.*

Proof. Set $\varepsilon_n = 1/n$, $n \in \mathbf{N}$. By Theorem 2.1 there is an h-continuous multifunction G: $X \to \mathscr{C}_b(Y)$ with the following two properties: $h(\text{graph } G, \text{graph } F) \leq 1/2$; for each $x \in X$ there is a $\sigma_1(x) > 0$ such that $F(S(x, \sigma_1(x))) \subset G(x)$. Define G_1: $X \to \mathscr{C}_b(Y)$ by $G_1(x) = \overline{G(x) + (1/2) S}$. Observe that: G_1 is closed valued and h-continuous; $h(\text{graph } G_1, \text{graph } F) \leq \varepsilon_1$; and $\operatorname{co} F(S(x, \sigma_1(x))) + (1/2) S \subset G_1(x)$ for each $x \in X$. Since F and G_1 satisfy the hypotheses of Lemma 2.1 (with $\theta_1(x) \equiv 1/2$), there is a closed valued h-continuous multifunction G_2: $X \to \mathscr{C}_b(Y)$ with the following properties: $h(\text{graph } G_2, \text{graph } F) \leq \varepsilon_2$; $G_2(x) \subset G_1(x)$ for each $x \in X$; for every $x \in X$ there exist a $\sigma_2(x) > 0$ and a $\theta_2(x) > 0$ such that $\operatorname{co} F(S(x, \sigma_2(x))) + \theta_2(x)S \subset G_2(x)$. By induction, using Lemma 2.1, one constructs a sequence $\{G_n\}$ of closed valued h-continuous multifunctions G_n: $X \to \mathscr{C}_b(Y)$ enjoying the properties (i)–(iii) of the theorem. This completes the proof.

3. Continuous single-valued approximations for multifunctions. In this section we treat the problem of approximating a multifunction with convex values by a single-valued continuous function.

LEMMA 3.1. *Let F: $X \to 2^Y$ be an arbitrary multifunction. Let δ: $X \to \mathbf{R}$ be a l.s.c. positive function. Then the multifunction G: $X \to \mathscr{C}(Y)$ defined by $G(x) = \operatorname{co} F(S(x, \delta(x)))$ is l.s.c.*

Proof. Let V be open in Y. We claim that the set $U = \{x \in X | G(x) \cap V \neq \varnothing\}$ is open in X. In fact, let $x_0 \in U$ (the case $U = \varnothing$ is trivial). Fix any point $y_0 \in G(x_0) \cap V$. Evidently $y_0 = \lambda_1 y_1 + \lambda_2 y_2 + \cdots + \lambda_k y_k$ for suitable points $y_i \in F(x_i)$, $x_i \in S(x_0, \delta(x_0))$, and $\lambda_i > 0$ with $\lambda_1 + \lambda_2 + \cdots + \lambda_k = 1$. Set

$$\eta = \min\{\delta(x_0) - d(x_i, x_0) | i = 1, 2, \dots, k\},$$

and observe that $\eta > 0$. Since δ is l.s.c. there is a ρ, $0 < \rho \le \eta/2$, such that $\delta(x) > \delta(x_0) - \eta/2$ for every $x \in S(x_0, \rho)$. Let $x \in S(x_0, \rho)$ be arbitrary. For each $i = 1, 2, \dots, k$ we have $d(x_i, x_0) \le \delta(x_0) - \eta$ and so

$$\begin{aligned} d(x_i, x) &\le d(x_i, x_0) + d(x_0, x) < \delta(x_0) - \eta + \rho \\ &\le \delta(x_0) - \eta/2 < \delta(x), \end{aligned}$$

that is $x_i \in S(x, \delta(x))$. Therefore $y_i \in F(x_i) \subset \operatorname{co} F(S(x, \delta(x)))$ and consequently $y_0 \in G(x)$. But y_0 lies in V too, thus $G(x) \cap V \neq \varnothing$. Hence $x \in U$. Since x is arbitrary in $S(x_0, \rho)$, it follows that $S(x_0, \rho) \subset U$. This implies that U is an open set, for x_0 also is arbitrary in U. This completes the proof.

Let F: $X \to 2^Y$. Let $\varepsilon > 0$. Define $\Delta(x) = \{0 < \rho \le \varepsilon |$there is an $x' \in S(x, \rho)$ such that $h^*(F(z), F(x')) < \varepsilon$ for every $z \in S(x, \rho)\}$. Observe that $\Delta(x)$ is certainly nonempty if F is weakly h^*-u.s.c.

LEMMA 3.2. *Let F: $X \to 2^Y$ be weakly h^*-u.s.c. Let $\varepsilon > 0$. Then the function δ: $X \to [0, \varepsilon]$ given by $\delta(x) = \sup \Delta(x)$ is positive and l.s.c.*

Proof. It is only needed to show that δ is l.s.c. To see that, fix an $x_0 \in X$ and let η, $0 < \eta < \delta(x_0)$, be any. Take $\rho \in \Delta(x_0)$ such that

$\delta(x_0) - \eta/2 < \rho \leq \delta(x_0)$. Then there is an $x' \in S(x_0, \rho)$ such that

$$h^*(F(z), F(x')) < \varepsilon \quad \text{for each } z \in S(x_0, \rho). \tag{3.1}$$

Set $\sigma = \min\{(\rho - d(x', x_0))/2, \eta/2\}$. For each $x \in S(x_0, \sigma)$ we have

$$x' \in S(x, \rho - \sigma) \subset S(x_0, \rho). \tag{3.2}$$

Indeed, let $x \in S(x_0, \sigma)$. Since $d(x', x) \leq d(x', x_0) + d(x_0, x) < d(x', x_0) + \sigma$ and $d(x', x_0) \leq \rho - 2\sigma$, it follows that $d(x', x) < \rho - \sigma$ and so $x' \in S(x, \rho - \sigma)$. Furthermore each $z \in S(x, \rho - \sigma)$ satisfies

$$d(z, x_0) \leq d(z, x) + d(x, x_0) < (\rho - \sigma) + \sigma = \rho,$$

and hence $z \in S(x_0, \rho)$ which completes the proof of (3.2).

Now take any $x \in S(x_0, \sigma)$. Evidently by (3.2) it follows that (3.1) is satisfied with $S(x, \rho - \sigma)$ in the place of $S(x_0, \rho)$ (x' unchanged). Hence $\rho - \sigma \in \Delta(x)$. Then $\delta(x) \geq \rho - \sigma \geq \rho - \eta/2$ and so $\delta(x) > \delta(x_0) - \eta$, because $\rho > \delta(x_0) - \eta/2$. Since $x \in S(x_0, \sigma)$ is arbitrary, the function δ is l.s.c. This completes the proof.

THEOREM 3.1. *Let F: $X \to \mathscr{C}(Y)$ be weakly h^*-u.s.c. Then for every $\varepsilon > 0$ there exists a l.s.c. multifunction G: $X \to \mathscr{C}(Y)$ such that* $h(\text{graph } G, \text{graph } F) \leq \varepsilon$.

Proof. Let F: $X \to \mathscr{C}(Y)$ be weakly h^*-u.s.c. Let $\varepsilon > 0$. Denote by δ the corresponding l.s.c. function defined in Lemma 3.2. Let G: $X \to \mathscr{C}(Y)$ be defined by $G(x) = \operatorname{co} F(S(x, \delta(x)))$. By Lemma 3.1, G is l.s.c.

Since graph $F \subset$ graph G, we only need to show that $h^*(\text{graph } G, \text{graph } F) \leq \varepsilon$. To this end, let $(\tilde{x}, \tilde{y}) \in$ graph G. Then $\tilde{y} \in G(\tilde{x})$ and so, for some $y_i \in F(x_i)$, $x_i \in S(\tilde{x}, \delta(\tilde{x}))$, and $\lambda_i > 0$ (with $\lambda_1 + \lambda_2 + \cdots + \lambda_k = 1$), we have $\tilde{y} = \lambda_1 y_1 + \lambda_2 y_2 + \cdots + \lambda_k y_k$. Since $\max\{d(x_i, \tilde{x}) | i = 1, 2, \ldots, k\} < \delta(\tilde{x})$, there is a $\rho \in \Delta(\tilde{x})$ with $\max\{d(x_i, \tilde{x}) | i = 1, 2, \ldots, k\} < \rho \leq \delta(\tilde{x})$, and there is an $x' \in S(\tilde{x}, \rho)$ such that $h^*(F(z), F(x')) < \varepsilon$ for every $z \in S(\tilde{x}, \rho)$. Thus $F(z) \subset F(x') + \varepsilon S$ for every $z \in S(\tilde{x}, \rho)$. In particular, $F(x_i) \subset F(x') + \varepsilon S$ for $i = 1, 2, \ldots, k$ and consequently, $\tilde{y} \in F(x') + \varepsilon S$. Thus $\tilde{y} = y' + v$ for some $y' \in F(x')$ and some $v \in \varepsilon S$. Clearly $(x', y') \in$ graph F and

$$e((\tilde{x}, \tilde{y}), (x', y')) = \max\{d(\tilde{x}, x'), \|\tilde{y} - y'\|\} < \max\{\rho, \varepsilon\} = \varepsilon.$$

This implies $r((\tilde{x}, \tilde{y}), \text{graph } F) < \varepsilon$. Since $(\tilde{x}, \tilde{y})$ is arbitrary in graph G we have $h^*(\text{graph } G, \text{graph } F) \leq \varepsilon$. This completes the proof.

THEOREM 3.2. *Let F: $X \to \mathscr{C}(Y)$ be weakly h^*-u.s.c. Then for every $\varepsilon > 0$ there is a continuous single-valued function g: $X \to Y$ such that* $h^*(\text{graph } g, \text{graph } F) \leq \varepsilon$.

Proof. By Theorem 3.1 there is a l.s.c. multifunction $G\colon X \to \mathscr{C}(Y)$ such that $h(\text{graph } G, \text{graph } F) \le \varepsilon/2$. By Michael's lemma [**25**, p. 368] there is a continuous function $g\colon X \to Y$ satisfying $r(g(x), G(x)) < \varepsilon/2$ for every $x \in X$. Evidently $h^*(\text{graph } g, \text{graph } G) \le \varepsilon/2$. Hence,

$$h^*(\text{graph } g, \text{graph } F) \le h^*(\text{graph } g, \text{graph } G) + h^*(\text{graph } G, \text{graph } F) \le \varepsilon,$$

which completes the proof.

Arguing as in the proof of Theorem 2.1 one can prove the following

THEOREM 3.3. *Let $F\colon X \to \mathscr{C}(Y)$ be locally convexifying. Then for every $\varepsilon > 0$ there exists a continuous single-valued function $g\colon X \to Y$ such that* $h^*(\text{graph } g, \text{graph } F) \le \varepsilon$.

4. Characterizations of some classes of multifunctions. In this section we present some results concerning the characterization of certain classes of multifunctions by means of h-continuous approximation.

Let Y be a real normed space. We suppose Y to be endowed with a (translation) invariant metric γ (that is $\gamma(u + w, v + w) = \gamma(u, v)$ for every u, v, and w in Y) satisfying the condition:

$$\gamma(y, 0) \le \|y\| \quad \text{for every } y \in Y. \tag{4.1}$$

This guarantees that the topology on Y induced by γ is weaker than the topology induced by the norm. We shall denote by γ_0 the particular invariant metric γ on Y which is generated by the norm of Y.

For A, B in 2^Y we set

$$h_\gamma^*(A, B) = \sup\{\, r_\gamma(a, B) \mid a \in A\},$$

$$h_\gamma(A, B) = \max\{\, h_\gamma^*(A, B),\ h_\gamma^*(B, A)\}.$$

Here $r_\gamma(a, B) = \inf\{\gamma(a, b) \mid b \in B\}$. When $\gamma = \gamma_0$ we denote h_γ^*, h_γ by h^*, h respectively. For A in 2^Y and $\varepsilon > 0$ we set $N_\varepsilon^\gamma(A) = \{y \in Y \mid r_\gamma(y, A) < \varepsilon\}$.

It is easy to verify that for A, B, C, D in 2^Y and α, $\alpha_0 \in \mathbf{R}$ we have:

$$h_\gamma(A + B, C + D) \le h_\gamma(A, C) + h_\gamma(B, D), \tag{4.2}$$

$$h_\gamma(\alpha A, \alpha_0 A) \to 0 \quad \text{as} \quad \alpha \to \alpha_0 \quad (A \text{ bounded in norm}). \tag{4.3}$$

DEFINITION 4.1. A multifunction $F\colon X \to 2^Y$ is called strictly h^*_γ-u.s.c. if for every $x \in X$ and for every $\varepsilon > 0$ there is a $\delta > 0$ such that $h^*_\gamma(\operatorname{co} F(S(x,\delta)), F(x)) < \varepsilon$.

REMARK 4.1. Evidently, each strictly h^*_γ-u.s.c. multifunction is also h^*_γ-u.s.c. Moreover, a multifunction $F\colon X \to \mathscr{C}(Y)$ is strictly h^*-u.s.c. if and only if F is h^*-u.s.c.

The symbols X, Y, $\mathscr{C}_b(Y)$, $\mathscr{C}_k(Y)$ retain the meaning stated in §1.

THEOREM 4.1. *Let Y be a real normed space with an invariant metric γ satisfying* (4.1). *Let $F\colon X \to \mathscr{C}_b(Y)$. Then the following statements are equivalent*:

(a) *F is locally bounded (with respect to the norm of Y), and strictly h^*_γ-u.s.c.*

(b) *There exists a sequence $\{G_n\}$ of h_γ-continuous multifunctions $G_n\colon X \to \mathscr{C}_b(Y)$ such that*: (i) *for each $n \in \mathbf{N}$ and for every $x \in X$ there is a $\sigma_n(x) > 0$ such that $F(S(x, \sigma_n(x))) \subset G_n(x)$, and* (ii) $h_\gamma(G_n(x), F(x)) \to 0$ *as $n \to +\infty$.*

Proof. (b) ⇒ (a). Let $x \in X$. Let $\varepsilon > 0$. By (ii) there is an $n_0 \in \mathbf{N}$ such that $h_\gamma(G_{n_0}(x), F(x)) < \varepsilon$. By (i) there is a $\sigma = \sigma_{n_0}(x) > 0$ such that $F(S(x,\sigma)) \subset G_{n_0}(x)$. Hence $\operatorname{co} F(S(x,\sigma)) \subset G_{n_0}(x)$ and so $h^*_\gamma(\operatorname{co} F(S(x,\sigma)), F(x)) \le h^*_\gamma(G_{n_0}(x), F(x)) < \varepsilon$, which implies that F is strictly h^*_γ-u.s.c. Trivially F is locally bounded. Hence (a) is satisfied.

(a) ⇒ (b). Let $n \in \mathbf{N}$. Since F satisfies (a), for every $z \in X$ there is $\delta^n_z = \delta(z,n)$, $0 < \delta^n_z < 1/n$, such that the set $F(S(z,\delta^n_z))$ is bounded (in the norm of Y) and so $\operatorname{co} F(S(z,\delta^n_z)) \in \mathscr{C}_b(Y)$. Let $\mathscr{P}_n = \{p^n_z\}_{z\in X}$ be a partition of unity subordinated to the open covering $\{S(z,\delta^n_z/3)\}_{z\in X}$ of X. Let $G_n\colon X \to \mathscr{C}_B(Y)$ be given by

$$G_n(x) = \sum_{z\in X} p^n_z(x) G^n_z, \quad \text{where } G^n_z = \operatorname{co} F\big(S(z,\delta^n_z/3)\big).$$

Clearly G_n is well defined. We shall show that G_n is h_γ-continuous.

In fact, let $\tilde{x} \in X$ and let V be a neighborhood of $\tilde{x}$ which meets only a finite number of the sets $\operatorname{supp} p^n_z$. Let $D_{\mathscr{P}_n}(V) = \{z_1, z_2, \ldots, z_k\}$ (the points z_i depend on n). Then for each $x \in V$ we have

$$h_\gamma(G_n(x), G_n(\tilde{x})) = h_\gamma\left(\sum_{i=1}^k p^n_i(x) G^n_{z_i}, \sum_{i=1}^k p^n_i(\tilde{x}) G^n_{z_i}\right).$$

From this, by virtue of (4.2), (4.3) and the continuity of the functions $p^n_{z_i}$, it follows that $h_\gamma(G_n(x), G_n(\tilde{x})) \to 0$ as $x \to \tilde{x}$. Thus G_n is h_γ-continuous.

G_n satisfies (i). In fact, let $\tilde{x} \in X$ and let $D_{\mathscr{P}_n}(\tilde{x}) = \{z_1, z_2, \dots, z_k\}$. Since $\tilde{x} \in \operatorname{supp} p^n_{z_i} \subset S(z_i, \delta^n_{z_i}/3)$, there is a $\sigma_n(\tilde{x}) > 0$ such that $S(\tilde{x}, \sigma_n(\tilde{x})) \subset S(z_i, \delta^n_{z_i}/3)$ for each $i = 1, 2, \dots, k$. Then

$$F(S(\tilde{x}, \sigma_n(\tilde{x}))) \subset \sum_{i=1}^{k} p^n_{z_i}(\tilde{x}) F(S(\tilde{x}, \sigma_n(\tilde{x}))) \subset \sum_{i=1}^{k} p^n_{z_i}(\tilde{x}) G^n_{z_i} = G_n(\tilde{x}),$$

and so (i) is fulfilled.

The sequence $\{G_n\}$ satisfies (ii). To see this, let $\tilde{x} \in X$ and let $\varepsilon > 0$. Since F is strictly h^*_γ-u.s.c., there is a $\sigma = \sigma(\tilde{x}, \varepsilon) > 0$ such that

$$h^*_\gamma(\operatorname{co} F(S(\tilde{x}, \sigma)),\ F(\tilde{x})) < \varepsilon. \tag{4.4}$$

Fix an integer $n_0 \geq 1/\sigma$ and let $n \geq n_0$ be any. Let $D_{\mathscr{P}_n}(\tilde{x}) = \{z_1, z_2, \dots, z_k\}$ (the points z_i depend on n). For each $u \in S(z_i, \delta^n_{z_i}/3)$ we have

$$d(u, \tilde{x}) \leq d(u, z_i) + d(z_i, \tilde{x}) < \delta^n_{z_i}/3 + \delta^n_{z_i}/3 < 2/(3n) < \sigma,$$

and so $S(z_i, \delta^n_{z_i}/3) \subset S(\tilde{x}, \sigma)$. Then, we have

$$G_n(\tilde{x}) = \sum_{i=1}^{k} p^n_{z_i}(\tilde{x}) G^n_{z_i} \subset \sum_{i=1}^{k} p^n_{z_i}(\tilde{x}) \operatorname{co} F(S(\tilde{x}, \sigma)) = \operatorname{co} F(S(\tilde{x}, \sigma)).$$

The above inclusion and (4.4) imply $h^*_\gamma(G_n(\tilde{x}),\ F(\tilde{x})) < \varepsilon$, if $n \geq n_0$. By (i), $h^*_\gamma(F(\tilde{x}), G_n(\tilde{x})) = 0$. Thus $h_\gamma(G_n(\tilde{x}), F(\tilde{x})) < \varepsilon$, if $n \geq n_0$. This completes the proof.

THEOREM 4.2. *Let F: $X \to \mathscr{C}_k(\mathbf{R}^q)$ be K-closed. Then the following statements are equivalent*:

(a) *F is h^*-u.s.c.*

(b) *F is locally bounded and locally convexifying.*

(c) *There is a sequence $\{G_n\}$ of h-continuous multifunctions G_n: $X \to \mathscr{C}_k(\mathbf{R}^q)$ satisfying the conditions* (i), (ii), *and* (iii) *of Theorem* 2.3.

(d) *There is a sequence $\{G_n\}$ of h-continuous multifunctions G_n: $X \to \mathscr{C}_k(\mathbf{R}^q)$ satisfying, in addition to the conditions* (i) *and* (ii) *of Theorem* 2.3, *the following one*: (iii)′ *for each $x \in X$, $h(G_n(x), F(x)) \to 0$ as $n \to +\infty$.*

Proof. Indeed, (a) ⇒ (b) is obvious, (b) ⇒ (c) follows from Theorem 2.3, while (d) ⇒ (a) is shown in **[8]** (see the proof of Theorem 4.5). Let us prove that (c) ⇒ (d). To this end, suppose that $\{G_n\}$ satisfies the conditions stated in (c). We claim that $\{G_n\}$ satisfies also (iii′). Suppose the contrary. Then there exist $\tilde{x} \in X$, $\varepsilon > 0$, and a subsequence, $\{\tilde{G}_n\}$ say, of $\{G_n\}$ such that $h(\tilde{G}_n(\tilde{x}), F(\tilde{x})) > \varepsilon$ for every $n \in \mathbf{N}$. For each $n \in \mathbf{N}$ take $y_n \in \tilde{G}_n(\tilde{x})$ such that $r(y_n, F(\tilde{x})) > \varepsilon$. Since $\{y_n\}$ is contained in the compact set $\tilde{G}_1(\tilde{x})$, we assume (without loss of generality) that $y_n \to \tilde{y} \in \tilde{G}_1(\tilde{x})$. Thus we have $r(\tilde{y}, F(\tilde{x})) \geq \varepsilon$. Since $(\tilde{x}, y_n) \in$ graph $\tilde{G}_n$ and (from the hypothesis) $h(\text{graph } \tilde{G}_n, \text{graph } F) \to 0$ as $n \to +\infty$, there is a sequence $\{(x_n', y_n')\} \subset$ graph F such that $x_n' \to \tilde{x}$ and $y_n' \to \tilde{y}$. But graph F is closed, and so $\tilde{y} \in F(\tilde{x})$, a contradiction. This completes the proof.

5. Some applications. In this section we show that an h^*-u.s.c. (resp. h^*-l.s.c.) multifunction $F\colon X \to \mathscr{C}_b(Y)$ is h-continuous except at points of a first category set. This is proved using h-continuous approximations to F. Results of such type have been established, under different hypotheses on F, by Hill **[15]**, Kuratowski **[22]**, Choquet **[7]**, Fort **[12]**, and Kenderov **[20, 21]**. As an application of Theorem 5.1 below, we present an alternative proof of Kenderov's theorem **[18]** stating that a maximal monotone operator is almost everywhere single-valued. For related results and further generalizations see **[19]**, **[14]**, **[30]**.

THEOREM 5.1. *Let X be a metric space. Let Y be a real normed space with an invariant metric γ satisfying* (4.1). *Let $F\colon X \to \mathscr{C}_b(Y)$ be locally bounded* (*with respect to the norm of Y*) *and strictly h_γ^*-u.s.c. Then there exists a Baire first category set $X_0 \subset X$ such that F restricted to $X \setminus X_0$ is h_γ-continuous.*

Proof. By Theorem 4.1, there is a sequence $\{G_n\}$ of h_γ-continuous multifunctions $G_n\colon X \to \mathscr{C}_b(Y)$ satisfying at each $x \in X$ the properties: $G_n(x) \supset F(x)$ ($n \in \mathbf{N}$), and $h_\gamma(G_n(x), F(x)) \to 0$ as $n \to \infty$. For $n \in \mathbf{N}$, define $\lambda_n\colon X \to \mathbf{R}$ by $\lambda_n(x) = h_\gamma(G_n(x), F(x))$. We claim that λ_n is a l.s.c. function. Evidently $\lambda_n(x) = h_\gamma^*(G_n(x), F(x))$, because $F(x) \subset G_n(x)$. Let $x_0 \in X$ and let $\eta > 0$. Taking into account the h_γ-continuity of G_n, the fact that $G_n(x) \subset N^\gamma_{\lambda_n(x)+\eta}(F(x))$, and the h_γ^*-u.s.c. of F, we have

$$G_n(x_0) \subset N_\eta^\gamma(G_n(x)) \subset N^\gamma_{\lambda_n(X)+2\eta}(F(x)) \subset N^\gamma_{\lambda_n(x)+3\eta}(F(x_0)),$$

for each x in some neighborhood U of x_0. Consequently, $\lambda_n(x_0) \leq \lambda_n(x) + 3\eta$ for each $x \in U$. Hence λ_n is l.s.c. Thus there is a Baire first category set $X_n \subset X$, such that λ_n restricted to $X \setminus X_n$, is continuous. Set $X_0 = \bigcup_{n=1}^\infty X_n$. Evidently X_0 is of the first category in X. Suppose $X \setminus X_0 \neq \varnothing$ (otherwise there is nothing to prove). Clearly each λ_n is continuous on $X \setminus X_0$, and at each point $x \in X \setminus X_0$ we have $\lambda_n(x) \to 0$ as $n \to +\infty$.

We are going to see that F restricted to $X \setminus X_0$ is h_γ-continuous. In fact, let $x_0 \in X \setminus X_0$ and let $\varepsilon > 0$. Choose $n_0 \in \mathbf{N}$ such that $\lambda_{n_0}(x_0) < \varepsilon/4$. Take $\delta > 0$ so that $x \in S(x_0, \delta) \setminus X_0$ implies $\lambda_{n_0}(x) < \varepsilon/2$ and $h_\gamma(G_{n_0}(x), G_{n_0}(x_0)) < \varepsilon/4$. Then for each $x \in S(x_0, \delta) \setminus X_0$ we have

$$\begin{aligned} h_\gamma(F(x), F(x_0)) &\leq h_\gamma\big(F(x), G_{n_0}(x)\big) + h_\gamma\big(G_{n_0}(x), G_{n_0}(x_0)\big) \\ &\quad + h_\gamma\big(G_{n_0}(x_0), F(x_0)\big) < \lambda_{n_0}(x) + \varepsilon/4 + \lambda_{n_0}(x_0) \\ &< \varepsilon/2 + \varepsilon/4 + \varepsilon/4 = \varepsilon, \end{aligned}$$

and so the statement of the theorem is proved.

REMARK 5.1. Let $F\colon X \to \mathscr{C}_b(Y)$ be h^*-u.s.c. By Remark 4.1 and Theorem 5.1 it follows that there exists a Baire first category set $X_0 \subset X$ such that F restricted to $X \setminus X_0$ is h-continuous. Observe that the set $X \setminus X_0$ is certainly nonempty and dense in X, if X is a Baire space.

REMARK 5.2. Let $F\colon X \to \mathscr{C}_b(Y)$. Define $\tilde{F}\colon X \to \mathscr{C}_b(Y)$ by $\tilde{F}(x) = \overline{F(x) + S}$. Then F is h^*-l.s.c. (resp. h^*-u.s.c.) if and only if $\tilde{F}$ is h^*-l.s.c. (resp. h^*-u.s.c.). Evidently, $\tilde{F}$ is h^*-l.s.c. if F is so. Conversely, suppose that $\tilde{F}$ is h^*-l.s.c. Let $x_0 \in X$ and $\varepsilon > 0$. Then for each x in a neighborhood U of x_0 we have

$$F(x_0) + S \subset \tilde{F}(x_0) \subset \tilde{F}(x) + \varepsilon S = \overline{F(x) + \varepsilon S} + S.$$

Hence, by Rådström's lemma [27], $F(x_0) \subset \overline{F(x) + \varepsilon S}$ for each $x \in U$, from which the h^*-l.s.c. of F follows at once. By a similar argument one shows that F is h^*-u.s.c. if and only if $\tilde{F}$ is so.

THEOREM 5.2. *Let $F\colon X \to \mathscr{C}_b(Y)$ be h^*-l.s.c. Then there exists a Baire first category set $X_0 \subset X$ such that F restricted to $X \setminus X_0$ is h-continuous.*

Proof. In view of Remark 5.2, it is sufficient to show that there is a set $X_0 \subset X$ of the Baire first category such that $\tilde{F}$ restricted to $X \setminus X_0$ is h-continuous. Indeed, by [8, Proposition 3.4] there is a sequence $\{G_n\}$ of h-continuous multifunctions $G_n\colon X \to \mathscr{C}_b(Y)$ satisfying at each $x \in X$ the properties: $G_n(x) \subset \tilde{F}(x)$ ($n \in \mathbf{N}$), and $h(G_n(x), \tilde{F}(x)) \to 0$ as $n \to +\infty$. For each $n \in \mathbf{N}$ define $\lambda_n\colon X \to \mathbf{R}$ by $\lambda_n(x) = h(\tilde{F}(x), G_n(x))$. We claim that λ_n is a l.s.c. function. Evidently $\lambda_n(x) = h^*(\tilde{F}(x), G_n(x))$, because

$G_n(x) \subset \tilde{F}(x)$. Let $x_0 \in X$ and let $\eta > 0$. By virtue of the h^*-l.s.c. of $\tilde{F}$, the fact that $\tilde{F}(x) \subset G_n(x) + (\lambda_n(x) + \eta)S$, and the h-continuity of G_n we have

$$\tilde{F}(x_0) \subset \tilde{F}(x) + \eta S \subset G_n(x) + (\lambda_n(x) + 2\eta)S$$
$$\subset G_n(x_0) + (\lambda_n(x) + 3\eta)S$$

for every x in some neighborhood U of x_0. Consequently $\lambda_n(x_0) \leq \lambda_n(x) + 3\eta$ for every $x \in U$. Hence λ_n is l.s.c. The end of the proof is like that of Theorem 5.1.

From now on, Y will denote a separable real Banach space and Y^* its (topological) conjugate. We denote the pairing between y^* in Y^* and y in Y by $\langle y, y^* \rangle$. We suppose that Y is endowed with the norm topology.

A multi-valued mapping F: $D(F) \to 2^{Y^*}$ with domain $D(F) \subset Y$ ($D(F) \neq \varnothing$) is said to be a monotone operator if $\langle x - y, x^* - y^* \rangle \geq 0$ for all $x, y \in D(F)$ and all $x^* \in F(x)$ and $y^* \in F(y)$. It is called a maximal monotone operator if, in addition, the graph of F is not properly contained in the graph of any other monotone operator G: $D(G) \to 2^{Y^*}$ with $D(G) \subset Y$.

Let $\{x_n\}$ be a countable dense subset of Y. For $x^*, y^* \in Y^*$ we set

$$\gamma(x^*, y^*) = \sum_{n=1}^{\infty} \frac{1}{2^{n+\|x_n\|}} \frac{|\langle x_n, x^* - y^* \rangle|}{1 + |\langle x_n, x^* - y^* \rangle|}.$$

It is not difficult to prove that γ is an invariant metric on Y^*. Moreover,

$$\gamma(y^*, 0) \leq \sum_{n=1}^{\infty} \frac{1}{2^{n+\|x_n\|}} \frac{\|x_n\| \, \|y^*\|}{1 + \|x_n\| \, \|y^*\|} \leq \sum_{n=1}^{\infty} \frac{\|x_n\| \, \|y^*\|}{2^{n+\|x_n\|}} \leq \|y^*\|,$$

thus γ satisfies (4.1) (with Y^* in the place of Y).

Let $B^* = \{y^* \in Y^* | \, \|y^*\| \leq 1\}$. Let $r > 0$. On the set rB^* we shall consider the topology, denoted τ_γ, which is generated by the metric γ, and the relative $\sigma(Y^*, Y)$ topology, denoted τ_σ. Using the argument of [**10**, Theorem V. 5.1, p. 426] one can prove that the topologies τ_γ and τ_σ for rB^* are identical.

LEMMA 5.1. *Let G: $X \to 2^{Y^*}$ (X a metric space) satisfy the hypotheses*: (i) $G(x) \subset rB^*$ ($r > 0$) *for every* $x \in X$; (ii) $G(x)$ *is convex and* $\sigma(Y^*, Y)$ *closed, for every* $x \in X$; (iii) G *is u.s.c. from* X *to* 2^{Y^*}, *where* Y^* *is given the* $\sigma(Y^*, Y)$ *topology. Then G is strictly h^*_γ-u.s.c.*

Proof. By Alaoglu's theorem the space rB^* is τ_σ compact. Since $\tau_\sigma = \tau_\gamma$ it follows that rB^* is a τ_γ compact metric space. Suppose that G satisfies the hypotheses of the lemma but is not strictly h^*_γ-u.s.c. Then there exist an $x_0 \in X$, an $\varepsilon > 0$, and a sequence $\{\delta_n\}$ of positive numbers δ_n converging to zero such that $h^*_\gamma(\operatorname{co} G(S(x_0, \delta_n)), G(x_0)) > \varepsilon$ for every $n \in \mathbf{N}$. Take $y^*_n \in \operatorname{co} G(S(x_0, \delta_n))$ such that $r_\gamma(y^*_n, G(x_0)) > \varepsilon$, $n \in \mathbf{N}$. Passing to a subsequence, we suppose that $y^*_n \to y^* \in rB^*$ (in the metric γ). Hence $r_\gamma(y^*, G(x_0)) \geq \varepsilon$. But in the space Y^* with the $\sigma(Y^*, Y)$ topology, the set $G(x_0)$ is closed convex and $y^*_0 \notin G(x_0)$. Then by Hahn-Banach's theorem, there is a continuous linear functional φ: $Y^* \to \mathbf{R}$ and there exist constants c and θ, $\theta > 0$, such that

$$y^*_0 \in U = \left\{ y^* \in Y^* | \langle y^*, \varphi \rangle > c \right\},$$
$$G(x_0) \subset V = \left\{ y^* \in Y^* | \langle y^*, \varphi \rangle < c - \theta \right\}.$$

Evidently $y^*_n \in U \cap rB^*$, provided n is large enough, say $n \geq k$. On the other hand, the set $\{ x \in X | G(x) \subset V \}$ is a neighborhood of x_0, because V is $\sigma(Y^*, Y)$ open and G is u.s.c. (see Remark 1.10). Hence, for some $n \geq k$ we have $G(S(x_0, \delta_n)) \subset V$. Therefore $y^*_n \in U \cap V$, a contradiction. This completes the proof.

LEMMA 5.2. *Let G: $U \to 2^{Y^*}$ be a monotone operator defined on a nonempty open subset U of Y. Suppose that*: (i) $G(x) \subset rB^*$ $(r > 0)$ *for every $x \in U$*; (ii) *$G(x)$ is $\sigma(Y^*, Y)$ closed for every $x \in U$*; (iii) *there is a dense subset $\tilde{U}$ of U such that the restriction $\tilde{G}$ of G to $\tilde{U}$ is h_γ-continuous. Then $\tilde{G}$ is single-valued and demicontinuous* (*i.e. continuous as a single-valued mapping from $\tilde{U}$ with the relative norm topology, to Y^* with the $\sigma(Y^*, Y)$ topology*).

Proof. Observe that the h_γ-continuous multifunction $\tilde{G}$ from $\tilde{U}$ to 2^{rB^*} takes on τ_γ compact values. Consequently by Remark 1.9, $\tilde{G}$ is continuous if rB^* is assigned the τ_γ topology or, equivalently, the τ_σ topology.

Arguing as in **[18]** we shall prove that $\tilde{G}$ is single-valued. In the contrary case, there exist an $x_0 \in \tilde{U}$ and $u^*_0, v^*_0 \in \tilde{G}(x_0)$ such that $u_0 \neq v^*_0$. Clearly for some $c \in U$ we have $|\langle c, v^*_0 - u^*_0 \rangle| > 0$. Without loss of generality we can assume that $\eta = \langle c, v^*_0 - u^*_0 \rangle > 0$. For n large enough, say $n \geq n_0$, $c_n = x_0 + (1/n)c$ lies in U. Since $\tilde{U}$ is a dense subset of U, there is a sequence $\{x_n\} \subset \tilde{U}$ satisfying $\|x_n - c_n\| < 1/n^2$, $n \geq n_0$.

Let $V = rB^* \cap \{y^* \in Y^* | |\langle c, y^* - u_0^* \rangle| < \eta/2\}$ and observe that V is τ_σ open, and $\tilde{G}(x_0) \cap V \neq \varnothing$. By the continuity of $\tilde{G}$ the set $\{x \in \tilde{U} | \tilde{G}(x) \cap V \neq \varnothing\}$ is a neighborhood of x_0 in $\tilde{U}$. Since $x_n \to x_0$, there is an $n_1 \geq n_0$ such that $\tilde{G}(x_n) \cap V \neq \varnothing$ for every $n \geq n_1$. Let $y_n^* \in \tilde{G}(x_n) \cap V$ $(n \geq n_1)$. We have

$$\begin{aligned} 0 &\leq \langle x_n - x_0, y_n^* - v_0^* \rangle \\ &= \langle x_n - c_n, y_n^* - v_0^* \rangle + \frac{1}{n}\langle c, y_n^* - u_0^* \rangle + \frac{1}{n}\langle c, u_0^* - v_0^* \rangle \\ &\leq \|x_n - c_n\| \, \|y_n^* - v_0^*\| + \frac{1}{n}\frac{\eta}{2} - \frac{1}{n}\eta < \frac{1}{n}\left[\frac{2r}{n} - \frac{\eta}{2}\right]. \end{aligned}$$

From this, taking n sufficiently large, a contradiction follows. Hence $\tilde{G}$ is single-valued. It is obvious that $\tilde{G}$ is demicontinuous. This completes the proof.

THEOREM 5.3 [**18**]. *Let Y be a separable real Banach space and let Y^* be its conjugate. Let F: $D(F) \to 2^{Y^*}$ be a maximal monotone operator. Suppose that* $\operatorname{int}(\operatorname{co} D(F)) \neq \varnothing$. *Then there exists a residual subset $\tilde{D}$ of* $\operatorname{int} D(F)$ *such that F restricted to $\tilde{D}$ is single-valued and demicontinuous.*

Proof. Let F satisfy the hyootheses of the theorem. It is well known [**28**] that $\operatorname{int} D(F)$ is a nonempty convex set whose closure is $\overline{D(F)}$, and F restricted to $\operatorname{int} D(F)$ is locally bounded (in the norm of Y^*). Furthermore (see [**3**], [**18**]) for every $x \in \operatorname{int} D(F)$, $F(x)$ is convex and $\sigma(Y^*, Y)$ closed, and F restricted to $\operatorname{int} D(F)$ is u.s.c. as a multifunction from $\operatorname{int} D(F)$ to 2^{Y^*}, where Y^* is assigned the $\sigma(Y^*, Y)$ topology.

For each $n \in \mathbf{N}$, set $U_n = \{x \in \operatorname{int} D(F) | F(S(x, \sigma(x))) \subset nB^*$, for some $\sigma(x) > 0\}$. Clearly the sets U_n are open and $U_1 \subset U_2 \subset \cdots$. Moreover for n large enough, say $n \geq k$, each U_n is nonempty and $\bigcup_{n \geq k} U_n = \operatorname{int} D(F)$. The restriction of F to U_n satisfies the hypotheses of Lemma 5.1 (with $X = U_n$ and $r = n$) and so is strictly h_Y^*-u.s.c. By Theorem 5.1 (with Y^* in the place of Y) there is a set $Z_n \subset U_n$ of the first Baire category in U_n (and so with dense complement $\tilde{U}_n = U_n \setminus Z_n$) such that the restriction of F to $\tilde{U}_n$ is h_Y-continuous. By virtue of Lemma 5.2, F restricted to U_n is single-valued and demicontinuous. Set $Z_0 = \bigcup_{n \geq k} Z_n$, and $\tilde{D} = \operatorname{int} D(F) \setminus Z_0$. Evidently $\tilde{D}$ is a residual subset of $\operatorname{int} D(F)$ and F restricted to $\tilde{D}$ is single-valued and demicontinuous. This completes the proof.

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