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GENERALIZED s-NUMBERS OF τ-MEASURABLE OPERATORS

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We give a self-contained exposition on generalized s-numbers of τ-measurable operators affiliated with a semi-finite von Neumann algebra. As applications, dominated convergence theorems for a gage and convexity (or concavity) inequalities are investigated. In particular, relation between the classical L^p-norm inequalities and inequalities involving generalized s-numbers due to A. Grothendieck, J. von Neumann, H. Weyl and the first named author is clarified. Also, the Haagerup L^p-spaces (associated with a general von Neumann algebra) are considered.

0. Introduction. This article is devoted to a study of generalized s-numbers of τ-measurable operators affiliated with a semi-finite von Neumann algebra. Also dominated convergence theorems for a gage and convexity (or concavity) inequalities are investigated.

In the "hard" analysis of compact operators in Hilbert spaces, the notion of s-numbers (singular numbers) plays an important role as shown in [**10**], [**24**]. For a compact operator A, its nth s-number $\mu_n(A)$ is defined as the nth largest eigenvalue (with multiplicity counted) of $|A| = (A^*A)^{1/2}$. The following expression is classical:

$$\mu_n(A) = \inf\{\|AP_{\mathcal{K}}\|;\ \mathcal{K} \text{ is a closed subspace with } \dim \mathcal{K}^{\perp} \le n\},$$

where $P_{\mathcal{K}}$ denotes the projection onto $\mathcal{K}$.

In the present article, we will study the corresponding notion for a semi-finite von Neumann algebra. More precisely, let $\mathcal{M}$ be a semi-finite von Neumann algebra with a faithful trace τ. For an operator A in $\mathcal{M}$, the "tth" generalized s-number $\mu_t(A)$ is defined by

$$\mu_t(A) = \inf\{\|AE\|;\ E \text{ is a projection in } \mathcal{M} \text{ with } \tau(1-E) \le t\}, \quad t > 0.$$

Notice that the parameter t is no longer discrete corresponding to the fact that τ takes continuous values on the projection lattice. Actually this notion has already appeared in the literature in many contexts ([**8**], [**11**], [**25**], [33]). In fact, Murray and von Neumann used it (in the II_1-case), [**18**]. We will consider generalized s-numbers of τ-measurable operators in the sense of Nelson [**19**]. This is indeed a correct set-up to consider generalized s-numbers. In fact, the τ-measurability of an operator A

exactly corresponds to the property $\mu_t(A) < +\infty$, $t > 0$ and the measure topology ([**19**], [**27**]) can be easily and naturally expressed in terms of μ_t.

When $\mathscr{M}$ is commutative, $\mathscr{M} \cong L^\infty(X; m)$, $\tau(\cdot) = \int_X \cdot\, dm$, the generalized s-number $\mu_t(A)$ of $A \cong f$, a function on X, is exactly the non-increasing rearrangement $f^*(t)$ of f in classical analysis. (See [**26**] for example.) Therefore, through the use of μ_t, one can employ many classical analysis techniques (such as majorization arguments) in our non-commutative context (as shown in §4).

§1 consists of some preliminaries. §2 is expository and we give a self-contained and unified account on the theory of generalized s-numbers of τ-measurable operators. In §3, we prove certain dominated convergence theorems for a trace (i.e. gage, [**23**]). In the literature (see [**27**] for example), Fatou's lemma for a trace was emphasized. Instead, we will show Fatou's lemma for generalized s-numbers. Although its proof is simple, it will prove extremely useful. In fact, based on this, we will prove dominated convergence theorems unknown previously. Also, this Fatou's lemma is useful to extend known estimates (involving μ_t) for bounded operators to (unbounded) τ-measurable operators. In §4, we will study convexity (and concavity) inequalities involving μ_t. For applications and for the sake of completeness, we will prove classicial norm inequalities such as the Hölder and Minkowsky inequalities. However, our main emphasis here is to compare carefully the above classical norm inequalities with inequalities due to A. Grothendieck, J. von Neumann, H. Weyl and the first named author. We will also show that these "semi-finite techniques" are useful to derive the corresponding results for the Haagerup L^p-spaces, [**12**]. This is possible because $\mu_t(A)$ (with respect to the canonical trace on the crossed product, [**28**]) is particularly simple in this case. In the final §5, we prove the Clarkson-McCarthy inequalities for the Haagerup L^p-spaces "from scratch." Proofs are known, but it may not be without interest. In fact, some false proofs exist in the literature.

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1. Preliminaries. For the convenience of the reader, we will collect in this section some definitions and basic facts on the theory of non-commutative L^p-spaces associated with a von Neumann algebra. Our basic reference on the general theory of operator algebras is [**6**], [**29**].

1.1. Let $\mathscr{M}$ be a von Neumann algebra acting on a Hilbert space $\mathfrak{H}$.

Assume for a moment that $\mathscr{M}$ admits a faithful semi-finite normal trace τ. For a positive self-adjoint operator $T = \int_0^\infty \lambda\, dE_\lambda$ affiliated with $\mathscr{M}$, we set

$$\tau(T) = \sup_n \tau\left(\int_0^n \lambda\, dE_\lambda\right) = \int_0^\infty \lambda\, d\tau(E_\lambda).$$

For $0 < p < \infty$, $L^p(\mathscr{M};\ \tau)$ is defined as the set of all densely-defined closed operators T affiliated with $\mathscr{M}$ such that

$$\|T\|_p = \tau\left(|T|^p\right)^{1/p} < \infty.$$

In addition, we put $L^\infty(\mathscr{M};\ \tau) = \mathscr{M}$ and denote by $\|\ \|_\infty$ $(= \|\ \|)$ the usual operator norm. It is well-known ([**5**], [**16**], [**19**], [**23**]) that $L^p(\mathscr{M};\ \tau)$ is a Banach space under $\|\ \|_p$ $(1 \leq p \leq \infty)$ satisfying all the expected properties such as duality. From this definition, it is not obvious at all that for example the sum of two operators in $L^p(\mathscr{M};\ \tau)$ is well-defined and, a fortiori, belongs to the same space. In fact, as was shown at first by Segal [**23**], the sum is well-defined because involved operators are measurable. Instead of the notion of measurability introduced by Segal, we will use

1.2. DEFINITION ([**19**]). A densely-defined closed operator T (possibly unbounded) affiliated with $\mathscr{M}$ is said to be τ-measurable if for each $\varepsilon > 0$ there exists a projection E in $\mathscr{M}$ such that $E(\mathfrak{H}) \subseteq \mathscr{D}(T)$ and $\tau(1 - E) \leq \varepsilon$.

Let T be a (densely-defined closed) operator affiliated with $\mathscr{M}$. Let $T = U|T|$ be the polar decomposition and $|T| = \int_0^\infty \lambda\, dE_\lambda$ be the spectral decomposition. Then T is τ-measurable if and only if $\tau(1 - E_\lambda) < \infty$ for λ large enough, or equivalently, $\lim_{\lambda \to \infty} \tau(1 - E_\lambda) = 0$ due to the normality of τ (see [**19**] for example). Let us denote by $\overline{\mathscr{M}}$ the set of all τ-measurable operators. When $\mathscr{M} = L^\infty(X;\ m)$ and $\tau(f) = \int f\, dm$, where (X, m) is a measure space, a function on X is in $\overline{\mathscr{M}}$ if and only if it is a (finite m-a.e.) m-measurable function which is bounded except on a set of finite measure. Thus, $\overline{\mathscr{M}}$ is large enough to contain all L^p-spaces, $0 < p \leq \infty$. Also, $\overline{\mathscr{M}}$ is the closure of $\mathscr{M}$ with respect to the measure topology. All these facts remain valid for a von Neumann algebra $\mathscr{M}$ with semi-finite trace τ so that $\overline{\mathscr{M}}$ really appears as a "basis for investigations in non-commutative integration theory." The notion of τ-measurability does not appear in the classical theory of Schatten classes because $\overline{\mathscr{M}} = \mathscr{M}$ for the algebra $\mathscr{M}$ of all bounded operators on a Hilbert space (with the canonical trace). Note in contrast that $\overline{\mathscr{M}}$ is the set of all densely-defined closed operators affiliated with $\mathscr{M}$ when the trace is finite.

1.3. Definition. For a τ-measurable operator T, we define the distribution function of T by

$$\lambda_t(T) = \tau\big(E_{(t,\infty)}(|T|)\big), \qquad t \geq 0,$$

where $E_{(t,\infty)}(|T|)$ is the spectral projection of $|T|$ corresponding to the interval (t, ∞).

The operator T being τ-measurable, we have $\lambda_t(T) < \infty$ for t large enough and $\lim_{t\to\infty} \lambda_t(T) = 0$ as noted before. Moreover, the map: $t \in [0, \infty] \to \lambda_t(T)$ is non-increasing and continuous from the right (because τ is normal and $E_{(t_n,\infty)}(|T|) \nearrow E_{(t,\infty)}(|T|)$ strongly as $t_n \searrow t$). The reader may notice that $\lambda_t(T)$ is a non-commutative analogue of the distribution function in classical analysis. (cf. **[26]**).

1.4. Let T, S be τ-measurable operators. Then $T + S$, TS, and T^* are densely-defined and preclosed. Moreover, the closures $(T + S)^-$ (strong sum) $(TS)^-$ (strong product) and T^* are again τ-measurable, and $\overline{\mathscr{M}}$ is a $*$-algebra with respect to the strong sum, the strong product, and the adjoint operation. (See **[19]**, **[31]**.) In what follows, we will supress the closure sign.

1.5. The measure topology on $\overline{\mathscr{M}}$ is by definition the linear (Hausdorff) topology whose fundamental system of neighborhoods around 0 is given by

$$V(\varepsilon, \delta) = \big\{T \in \overline{\mathscr{M}};\ \text{there exists a projection } E \text{ in } \mathscr{M} \\ \text{such that } \|TE\| \leq \varepsilon \text{ and } \tau(1 - E) \leq \delta\big\}.$$

Here, ε, δ run over all strictly positive numbers. It is known (**[19]**, **[31]**) that $\overline{\mathscr{M}}$ is a complete topological $*$-algebra. Moreover, $\mathscr{M}$ is dense in $\overline{\mathscr{M}}$. In fact, if $T = U|T| \in \overline{\mathscr{M}}$ and $|T| = \int_0^\infty \lambda\, dE_\lambda$, then the sequence $\{U\int_0^n \lambda\, dE_\lambda\}_{n=1,2,\ldots}$ in $\mathscr{M}$ tends to T as $n \to \infty$ in the measure topology.

1.6. Now let $\mathscr{M}$ be a general (not necessarily semi-finite) von Neumann algebra. Let $\mathfrak{A}$ be the crossed product of $\mathscr{M}$ by the modular automorphism group $\{\sigma_t\}_{t\in\mathbf{R}}$ of a fixed weight on $\mathscr{M}$. By a result of Takesaki, **[28]**, $\mathfrak{A}$ admits the dual action $\{\theta_s\}_{s\in\mathbf{R}}$ and the faithful semi-finite normal trace τ satisfying $\tau \circ \theta_s = e^{-s}\tau$, $s \in \mathbf{R}$. The Haagerup L^p-spaces associated with $\mathscr{M}$, **[12]**, are defined by

$$L^p(\mathscr{M}) = \{T;\ T \text{ is a } \tau\text{-measurable operator (affiliated with } \mathfrak{A}) \\ \text{such that } \theta_s(T) = e^{-s/p}T, s \in \mathbf{R}\}, \qquad 0 < p \leq \infty.$$

It is known that $L^1(\mathcal{M})$ is order-isomorphic to the predual $\mathcal{M}_*$. We thus get a positive linear functional tr on $L^1(\mathcal{M})$ by the formula

$$\operatorname{tr}(H_\varphi) = \varphi(1).$$

Here H_φ is the element in $L^1(\mathcal{M})$ which corresponds to $\varphi \in \mathcal{M}_*$ by the order isomorphism. For T in $L^p(\mathcal{M})$, $0 < p < \infty$, we set

$$\|T\|_p = \operatorname{tr}(|T|^p)^{1/p}.$$

(As usual, $\| \ \|_\infty$ is the usual operator norm.) Some remarks are in order: (i) $L^p(\mathcal{M})$ does not depend on a choice of a weight on $\mathcal{M}$ (used to construct the crossed product), (ii) the functional tr on $L^1(\mathcal{M})$ and the canonical trace τ are quite different, (iii) when $\mathcal{M}$ is semi-finite, $L^p(\mathcal{M}; \tau)$ in 1.1 is isomorphic to $L^p(\mathcal{M})$.

Full details of Haagerup's theory can be found in **[31]**. For later reference, we record:

1.7. LEMMA. *Let T be an element in $L^1(\mathcal{M})$. For any $t > 0$, we have*

$$\lambda_t(T) = t^{-1}\|T\|_1.$$

Here, the distribution function λ_t is with respect to the canonical trace on the crossed product $\mathfrak{A} = \mathcal{M} x_\sigma \mathbf{R}$ explained in 1.6.

Its proof is found in **[31]**. Let us point out that this Lemma is crucial and indeed a starting point of Haagerup's theory. Because of this lemma, T is τ-measurable (with respect to the trace τ on $\mathfrak{A}$) and additions and multiplications are justified.

2. Generalized s-numbers. The notion of generalized s-numbers for bounded operators was carefully developed in **[8]** by the first named author. On the other hand, its generalization to τ-measurable operators has fruitful applications as indicated in **[15]** by the second named author. In this section, we will give a self-contained and unified account on generalized s-numbers of τ-measurable operators.

Throughout the section, let $\mathcal{M}$ be a von Neumann algebra on a Hilbert space $\mathfrak{H}$ with a faithful semi-finite normal trace τ.

2.1. DEFINITION. Let T be a τ-measurable operator and $t > 0$. The "tth singular number of T" $\mu_t(T)$ is

$$\mu_t(T) = \inf\{\|TE\|;\ E \text{ is a projection in } \mathcal{M} \text{ with } \tau(1-E) \le t\}.$$

It follows from 1.3 that $\mu_t(T) < \infty$, $t > 0$. The reader may notice that $\mu_t(T)$ is indeed a generalization of the classical s-number for a compact operator. (Recall the first part of §0.) Obviously, $\mu_t(A)$ admits the following "minimax" representation:

$$\mu_t(T) = \inf_{\substack{E \text{ is a projection in } \mathcal{M} \\ \text{with } \tau(1-E) \leq t}} \left[\operatorname*{Sup}_{\substack{\xi \in E(\mathfrak{H}) \\ \|\xi\|=1}} \|T\xi\| \right].$$

As shown shortly, we have $\mu_t(T) = \mu_t(T^2)^{1/2}$ wen T is positive. Therefore, for a positive T, this expression reads

$$\mu_t(T) = \inf_{\substack{E \text{ is a projection in } \mathcal{M} \\ \text{with } \tau(1-E) \leq t}} \left[\sup_{\substack{\xi \in E(\mathfrak{H}) \\ \|\xi\|=1}} (T\xi \mid \xi) \right]$$

2.2. PROPOSITION. *Let T be a τ-measurable operator. For any $t > 0$, we have*

$$\mu_t(T) = \inf\{s \geq 0;\ \lambda_s(T) \leq t\},$$

where $\lambda_s(T)$ is the distribution function in 1.3. *Moreover, the infimum is attained and $\lambda_{\mu_t(T)}(T) \leq t$, $t > 0$.*

Proof. As the map: $s \to \lambda_s(T)$ is continuous from the right, the second statement is obvious. Let us denote the infimum in the proposition by a. The inequality $\lambda_a(T) \leq t$ means $\tau(1 - E) \leq t$ with $E = E_{[0,a]}(|T|)$. But, $\|TE\| = \||T|E\| \leq a$ and $\mu_t(T) \leq a$.

On the other hand, let $\varepsilon > 0$ and take a projection E in $\mathcal{M}$ such that $\tau(1 - E) \leq t$ and

$$\|TE\| < \mu_t(T) + \varepsilon = \alpha.$$

If $\xi \in E(\mathfrak{H}) \cap E_{(\alpha,\infty)}(|T|)(\mathfrak{H})$, $\|\xi\| = 1$, then we have

$$(T^*T\xi \mid \xi) \geq \alpha^2, \quad (T^*T\xi \mid \xi) < \alpha^2.$$

Therefore $E \wedge E_{(\alpha,\infty)}(|T|) = 0$ and

$$\begin{aligned} E_{(\alpha,\infty)}(|T|) &= E_{(\alpha,\infty)}(|T|) - E \wedge E_{(\alpha,\infty)}(|T|) \\ &\sim E \vee E_{(\alpha,\infty)}(|T|) - E \leq 1 - E \end{aligned}$$

in the Murray-von Neumann sense. We thus get

$$\lambda_\alpha(T) \leq \tau(1 - E) \leq t, \quad a \leq \alpha = \mu_t(T) + \varepsilon.$$

The proof is complete since ε is arbitrary. □

2.3. REMARKS.

2.3.1. Let $\mathfrak{A}$ be any von Neumann subalgebra of $\mathcal{M}$ containing the spectral projections of $|T|$. The above proof actually shows that

$$\mu_t(T) = \inf\{\|TE\|;\ E \text{ is a projection in } \mathfrak{A} \text{ with } \tau(1 - E) \le t\}.$$

2.3.2. When $\mathcal{M} = L^\infty(X; m)$ and $\tau(f) = \int f\,dm$, we get

$$\mu_t(f) = \inf\{s \ge 0;\ m(\{x \in X;\ |f(x)| > s\}) \le t\}.$$

Hence, $\mu_t(f)$ is exactly the classical non-increasing rearrangement $f^*(t)$. (cf. **[26]**.) These two remarks are useful tricks to reduce the analysis of s-numbers for a single operator to the classical commutative situation.

The next proposition gives a more geometrical interpretation of the generalized s-numbers.

2.4. PROPOSITION. *For each t, Let $\mathcal{R}_t$ be the set of all τ-measurable operators S such that* $\tau(\mathrm{supp}(|S|)) \le t$. *(Here,* $\mathrm{supp}(|S|)$ *denotes the support projection of $|S|$.) Then, for a τ-measurable operator T, $\mu_t(T)$ is exactly the "approximation number"*

$$d(T, \mathcal{R}_t) = \inf\{\|T - S\|;\ S \in \mathcal{R}_t\}.$$

Proof. Let $T = U|T|$ be the polar decomposition and

$$S = U\int_{(\alpha,\infty)} \lambda\, dE_\lambda(|T|)$$

with $\alpha = \mu_t(T)$. We have

$$\|T - S\| \le \alpha = \mu_t(T), \quad \tau(\mathrm{supp}(|S|)) = \lambda_\alpha(T) \le t \text{ (by 2.2)}.$$

Hence we get $d(T, \mathcal{R}_t) \le \mu_t(T)$.

On the other hand, let $S \in \mathcal{R}_t$ and set

$$E = 1 - \mathrm{supp}(|S|).$$

As $TE = (T - S)E$, we get

$$\|TE\| \le \|T - S\|.$$

But $\tau(1 - E) \le t$ so that $\mu_t(T) \le \|T - S\|$ and $\mu_t(T) \le d(T, \mathcal{R}_t)$. □

2.5. LEMMA. *Let T, S be τ-measurable operators.*

(i) *The map: $t \in (0, \infty) \to \mu_t(T)$ is non-increasing and continuous from the right. Moreover,*

$$\lim_{t \downarrow 0} \mu_t(T) = \|T\| \in [0,\infty].$$

(ii) $\mu_t(T) = \mu_t(|T|) = \mu_t(T^*)$ *and*

$$\mu_t(\alpha T) = |\alpha|\mu_t(T) \quad \text{for } t > 0 \text{ and } \alpha \in \mathbf{C}.$$

(iii) $\mu_t(T) \leq \mu_t(S)$, $t > 0$, *if* $0 \leq T \leq S$.

(iv) $\mu_t(f(|T|)) = f(\mu_t(|T|))$, $t > 0$ *for any continuous increasing function f on* $[0, \infty)$ *with* $f(0) \geq 0$

(v) $\mu_{t+s}(T + S) \leq \mu_t(T) + \mu_s(S)$, t, $s > 0$.

(vi) $\mu_t(STR) \leq \|S\|\,\|R\|\mu_t(T)$, $t > 0$.

(vii) $\mu_{t+s}(TS) \leq \mu_t(T)\mu_s(S)$, t, $s > 0$.

Proof. (i) The monotone property is evident. If it were not continuous from the right at t, we would get

$$\mu_t(T) > \alpha \geq \mu_{t+\varepsilon}(T) \quad \text{for all } \varepsilon > 0.$$

Then $\lambda_\alpha(T) \leq \lambda_{\mu_{t+\varepsilon}(T)}(T) \leq t + \varepsilon$ (by 2.2) and $\lambda_\alpha(T) \leq t$. It follows that $\mu_t(T) \leq \alpha$, a contradiction.

From the definition, we obviously have $\mu_t(T) \leq \|T\|$.

If we had $\|T\| > \alpha \geq \mu_\varepsilon(T)$, $\varepsilon > 0$, then we would get $\lambda_\alpha(T) = 0$ as before and $\|T\| \leq \alpha$, a contradiction.

(ii) follows immediately from Proposition 2.2 or 2.4.

(iii) From $0 \leq T \leq S$, we get

$$E_{(s,\infty)}(T) \wedge E_{[0,s]}(S) = 0, \quad E_{(s,\infty)}(T) \lesssim E_{(s,\infty)}(S)$$

in the Murray-von Neumann sense (as in the proof of Proposition 2.2). Therefore, $\lambda_s(T) \leq \lambda_s(S)$, $s \geq 0$, and the result follows.

(iv) Let $\mathfrak{A}$ be the von Neumann subalgebra of $\mathcal{M}$ generated by the spectral projections of $|T|$. It obviously contains the spectral projections of $f(|T|)$. For any projection E in $\mathfrak{A}$, we get

$$f(\|\,|T|E\,\|) = \|f(|T|)E\|$$

since f is continuous increasing and $f(0) \geq 0$. Using Remark 2.3.1, we get

$$f\big(\mu_t(|T|)\big) = \mu_t\big(f(|T|)\big).$$

(v), (vi), and (vii) follow from Proposition 2.4. We will just prove (vii). ((v) and (vi) are easier.) Choose and fix $\varepsilon > 0$. Proposition 2.4 guarantees the existence of τ-measurable A, B such that

$$\|T - A\| \leq \mu_t(T) + \varepsilon, \quad \tau\big(\mathrm{supp}(|A|)\big) \leq t,$$
$$\|S - B\| \leq \mu_s(S) + \varepsilon, \quad \tau\big(\mathrm{supp}(|B|)\big) \leq s,$$

With $C = (T - A)B + AS$, we get

$$\|TS - C\| = \|(T - A)(S - B)\| \leq \big(\mu_t(T) + \varepsilon\big)\big(\mu_s(S) + \varepsilon\big).$$

We have

$$\tau\big(\operatorname{supp}(|C|)\big) \le t + s$$

thanks to the following three facts:

$$\operatorname{supp}(|C|) \le \operatorname{supp}(|(T-A)B|) \vee \operatorname{supp}(|AS|),$$
$$\operatorname{supp}(|(T-A)B|) \le \operatorname{supp}(|B|),$$
$$\operatorname{supp}(|AS|) \sim \operatorname{supp}(|S^*A^*|) \le \operatorname{supp}(|A^*|) \sim \operatorname{supp}(|A|).$$

Therefore, Proposition 2.4 implies

$$\mu_{t+s}(TS) \le \big(\mu_t(T) + \varepsilon\big)\big(\mu_s(S) + \varepsilon\big). \qquad \square$$

2.6. LEMMA. *Let T be a τ-measurable operator. For a projection E in $\mathscr{M}$, we get*

$$\mu_t(TE) = 0 \quad \textit{for } t \ge \tau(E).$$

In particular, if $\tau(1) = \alpha$ is finite, then

$$\mu_t(T) = 0 \quad \textit{for } t \ge \alpha.$$

Proof. It follows from Proposition 2.4. $\square$

2.7. PROPOSITION. *Let T be a positive τ-measurable operator. Then we have*

$$\tau(T) = \int_0^\infty \mu_t(T)\, dt.$$

Proof. When T is of the form

$$\sum_{i=1}^{n} \alpha_i E_i \tag{1}$$

($\alpha_i > 0$ and E_i are mutually orthogonal projections in $\mathscr{M}$), the result is proved in p. 190–191, **[26]**. (In **[26]**, the extra assumption $\tau(E_i) < +\infty$ is posed. But, if $\tau(E_i) = +\infty$ for some i, then we obviously get $\infty = \infty$.)

At first we claim that the equality remains valid for any positive bounded T. Using the spectral projections, one can write T as the norm limit of an increasing sequence $\{T_n\}$ of operators having the form (1). We have

$$\mu_t(T_n) \nearrow \mu_t(T).$$

(In fact, it follows immediately from Lemma 2.5, (i), (v) that $|\mu_t(S_1) - \mu_t(S_2)| \le \|S_1 - S_2\|$ for any operators S_i.) Therefore, $\int_0^\infty \mu_t(T_n)\, dt \nearrow \int_0^\infty \mu_t(T)\, dt$ as $n \nearrow \infty$ by the monotone convergence theorem. On the other hand, we obviously have $\tau(T_n) \nearrow \tau(T)$ as $n \to \infty$.

Let us now assume that T is a general positive τ-measurable operator. For each n, we set

$$E_n = E_{[0,n]}(T).$$

By Lemma 2.5, (iii),

$$\mu_t(TE_1) \le \mu_t(TE_2) \le \cdots \le \mu_t(T).$$

We claim that $\lim_{n\to\infty} \mu_t(TE_n) = \mu_t(T)$. Assume that $s = \lim_{n\to\infty} \mu_t(TE_n) < \mu_t(T)$. Then $E_{(\mu_t(TE_n),\infty)}(TE_n) = E_{(\mu_t(TE_n),n]}(T)$ converges to $E_{[s,\infty)}(T)$ strongly. Since $\tau(E_{(\mu_t(TE_n),\infty)}(TE_n)) \le t$, the lower semi-continuity of τ implies

$$\tau\big(E_{(s,\infty)}(T)\big) \le \tau\big(E_{[s\,,\,\infty)}(T)\big) \le \liminf_{n\to\infty} \tau\Big(E_{(\mu_t(TE_n),\infty)}(TE_n)\Big) \le t,$$

which contradicts $s < \mu_t(T)$. Therefore we have

$$\mu_t(TE_n) \nearrow \mu_t(T) \quad \text{and} \quad \int_0^\infty \mu_t(TE_n)\,dt \nearrow \int_0^\infty \mu_t(T)\,dt$$

as $n \to \infty$ by the monotone convergence theorem. On the other hand, $\tau(TE_n) \nearrow \tau(T)$ as $n \to \infty$ (from the definition). Therefore, the general case is reduced to the bounded case. □

2.8. COROLLARY. *Let f be a continuous increasing function on $[0,\infty)$ with $f(0) = 0$. For each τ-measurable operator T, we have*

$$\tau\big(f(|T|)\big) = \int_0^\infty f(\mu_t(T))\,dt.$$

In particular,

$$\|T\|_p = \left(\int_0^\infty \mu_t(T)^p\,dt\right)^{1/p} \quad \text{for } 0 < p < \infty.$$

Proof. Apply Lemma 2.5, (iv), and Proposition 2.7. □

We now characterize spectral dominance (see **[1]**, **[3]**) in terms of generalized s-numbers. The next result was stated in Remark 5, **[3]** without proof.

2.9. COROLLARY. *Let T, S be positive τ-measurable operators. The following conditions are equivalent*:

(i) $\mu_t(T) \le \mu_t(S)$, $t > 0$,

(ii) $\lambda_s(T) \le \lambda_s(S)$, $s \ge 0$,

(iii) $\tau(f(T)) \leq \tau(f(S))$ *for any continuous increasing function* f *on* $[0, \infty)$ *with* $f(0) = 0$.

Proof. (i) $\Rightarrow$ (iii) follows from Corollary 2.8.

(iii) $\Rightarrow$ (ii) One can approximate the characteristic function $\chi_{(s,\infty)}$ from below by a sequence $\{f_n\}$ of continuous increasing functions on $[0, \infty)$ with $f_n(0) = 0$, and we have $\tau(f_n(T)) \leq \tau(f_n(S))$ by the assumption. It follows from Lebesgue's dominated convergence theorem (applied to $d\|E_s(T)\xi\|^2$, ξ is a vector) that $f_n(T) \nearrow E_{(s,\infty)}(T)$ and $f_n(S) \nearrow E_{(s,\infty)}(S)$ strongly as $n \to \infty$. Thus, the normality of τ implies $\lambda_s(T) \leq \lambda_s(S)$. (ii) $\Rightarrow$ (i) follows from Proposition 2.2. □

When $\mathcal{M}$ is a factor, the above three conditions are of course equivalent to the spectral dominance $E_{(s,\infty)}(T) \lesssim E_{(s,\infty)}(S)$, $s \geq 0$.

3. Convergence theorems for gages. Throughout the section, let $\mathcal{M}$ be a von Neumann algebra with a faithful semi-finite normal trace τ (i.e. τ is a gage in the sense of Segal, **[23]**). As a direct analogue of the measure theory, on expects for example the following "dominated convergence theorem": if a sequence $\{T_n\}$ of τ-measurable operators tends to T in the measure topology and if there is a positive integrable operator S with $|T_n| \leq S$, then one would get

$$\lim_{n\to\infty} \tau(T_n) = \tau(T). \tag{2}$$

However, in the literature (**[20]**, **[27]**) the theorem in the above expected form has not been proved. (As explained in p. 29, **[27]**, difficulty comes from failure of the operator inequality $|T + S| \leq |T| + |S|$). In fact, (2) was proved under slightly unnatural conditions such as

$$-S \leq \operatorname{Re} T_n \leq S \quad \text{and} \quad -S \leq \operatorname{Im} T_n \leq S.$$

In this section, among other things, we will prove the "correct" dominated convergence theorem based on the tools developed in §2. We will also prove "Fatou's lemma" for generalized s-numbers. As mentioned in §0, this viewpoint is new, and will prove very useful.

We begin by characterizing the convergence in the measure topology (recall 1.5) in terms of generalized s-numbers.

3.1. Lemma. *Let* T_n, $n = 1, 2, \ldots$, *and* T *be* τ*-measurable operators. Then* $\{T_n\}$ *converges to* T *in the measure topology if and only if*

$$\lim_{n\to\infty} \mu_t(T - T_n) = 0 \quad \textit{for each } t > 0.$$

Proof. We will prove that $S \in V(\varepsilon, \delta)$ (1.5) if and only if $\mu_\delta(S) \le \varepsilon$. When $S \in V(\varepsilon, \delta)$, we clearly have $\mu_\delta(S) \le \varepsilon$ (Definition 2.1). Conversely, if $\mu_\delta(S) \le \varepsilon$, then $E = E_{[0,\mu_\delta(S)]}(|S|)$ satisfies $\tau(1-E) = \lambda_{\mu_\delta(S)}(S) \le \delta$ (Proposition 2.2). Since

$$\|SE\| = \| \, |S| E \| \le \mu_\delta(S) \le \varepsilon,$$

we get $S \in V(\varepsilon, \delta)$. □

The next result was stated in [**3**] (Proposition 13) without proof.

3.2. PROPOSITION. *The following three conditions are equivalent for a τ-measurable operator T:*

(i) $\lambda_\varepsilon(T) < +\infty$ *for all* $\varepsilon > 0$,

(ii) $\lim_{t \to \infty} \mu_t(T) = 0$,

(iii) *there exists a sequence* $\{T_n\}$ *of τ-measurable operators* (*bounded if wished*) *converging to T in the measure topology such that* $\tau(\mathrm{supp}|T_n|) < \infty$ *for each n.*

Proof. (iii) $\Rightarrow$ (ii) For any $\varepsilon > 0$, pick up an integer n_0 with $\mu_1(T - T_{n_0}) \le \varepsilon$ (Lemma 3.1). Since $\mu_t(T_{n_0}) = 0$ for $t \ge \tau(\mathrm{supp}(|T_{n_0}|))$ (Lemma 2.6), Lemma 2.5, (v) implies

$$\mu_t(T) \le \mu_{t-1}(T_{n_0}) + \mu_1(T - T_{n_0}) \le \varepsilon$$

for $t \ge \tau(\mathrm{supp}(|T_{n_0}|)) + 1$.

(ii) $\Rightarrow$ (i) For any $\varepsilon > 0$, pick up $t_0 > 0$ such that $\mu_{t_0}(T) \le \varepsilon$. We then get

$$\infty > t_0 \ge \lambda_{\mu_{t_0}(T)}(T) \ge \lambda_\varepsilon(T)$$

(Proposition 2.2).

(i) $\Rightarrow$ (iii) If $T = U\int_0^\infty \lambda \, dE_\lambda(|T|)$, then the sequence

$$\left\{ U\int_{n^{-1}}^{n} \lambda \, dE_\lambda(|T|) \right\}_{n=1,2,\ldots}$$

does a job. □

3.3. REMARK. For a τ-measurable operator in the class characterized in Proposition 3.2, we can strengthen Corollary 2.8. Namely, if T is in this class and g is a non-negative Borel function on $[0, \infty)$ with $g(0) = 0$ (not necessarily increasing), we get

$$\tau(g(|T|)) = \int_0^\infty g(\mu_t(T)) \, dt. \tag{3}$$

To prove this, at first we note

$$\tau\big(E_{(s,\infty)}(S)\big) = \int_0^\infty \chi_{(s,\infty)}\big(\mu_t(S)\big)\,dt, \qquad s \geq 0,$$

for any positive τ-measurable S. This is obtained by approximating the characteristic function $\chi_{(s,\infty)}$ from below by functions described in Corollary 2.8. Similarly we get

$$\tau\big(E_{[s,\infty]}(S)\big) = \int_0^\infty \chi_{[s\,,\,\infty)}\big(\mu_t(S)\big)\,dt, \qquad s \geq 0.$$

Therefore, if T is in the above mentioned class and I is an interval in $(0, \infty)$, we get

$$\tau\big(E_I(|T|)\big) = \int_0^\infty \chi_I\big(\mu_t(T)\big)\,dt.$$

The class of measurable subsets I in the interval (ε, ∞) ($\varepsilon > 0$ fixed) for which the above equality holds is closed under countable disjoint union and complement and hence includes all Borel subsets in (ε, ∞). ($\infty - \infty$ does not occur when one takes a complement thanks to $\lim_{t\to\infty}\mu_t(T) = 0$.) Since $g\chi_{(\varepsilon,\infty)} \uparrow g$ as $\varepsilon \to 0$, (3) follows.

We also remark that all $L^p(\mathcal{M};\ \tau)$, $0 < p < \infty$, are included in this class. In fact, if T is in $L^p(\mathcal{M};\ \tau)$ then

$$\mu_t(T)^p \leq t^{-1}\int_0^t \mu_s(T)^p\,ds \leq t^{-1}\int_0^\infty \mu_s(T)^p\,ds$$

$$= t^{-1}\|T\|_p^p \to 0 \quad \text{as } t \to \infty.$$

The following "Fatou's lemma" is very useful:

3.4. LEMMA (*cf. Appendix of* **[15]**). *Let* $\{T_n\}$ *be a sequence of* τ*-measurable operators converging to* T *in the measure topology.*

(i) $\mu_t(T) \leq \liminf\limits_{n\to\infty} \mu_t(T_n)$ *for each* $t > 0$.

(ii) $\mu_t(T) = \lim\limits_{n\to\infty} \mu_t(T_n)$ *if* $s \to \mu_s(T)$ *is continuous at* $s = t$, *or if* $\mu_t(T_n) \leq \mu_t(T)$.

Proof. (i) For each $\varepsilon > 0$, Lemma 2.5(v) implies that

$$\mu_{t+\varepsilon}(T) - \mu_\varepsilon(T - T_n) \leq \mu_t(T_n).$$

Taking the $\liminf_{n\to\infty}$ of the both sides and using Lemma 3.1, we get

$$\mu_{t+\varepsilon}(T) \leq \liminf_{n\to\infty} \mu_t(T_n).$$

Letting $\varepsilon \downarrow 0$, we get by Lemma 2.5, (i),

$$\mu_t(T) \leq \liminf_{n\to\infty} \mu_t(T_n).$$

(ii) Picking up a small $\varepsilon > 0$ $(0 < \varepsilon < t)$, we get as before

$$\mu_t(T_n) \leq \mu_{t-\varepsilon}(T) + \mu_\varepsilon(T_n - T),$$

$$\limsup_{n\to\infty} \mu_t(T_n) \leq \mu_{t-\varepsilon}(T).$$

If $s \to \mu_s(T)$ is continuous at $s = t$, letting $\varepsilon \downarrow 0$ we get

$$\limsup_{n\to\infty} \mu_t(T_n) \leq \mu_t(T)\Big(\leq \liminf_{n\to\infty} \mu_t(T_n)\Big).$$

When $\mu_t(T_n) \leq \mu_t(T)$, the result is clear. □

The proof of the next (known) result should be compared with that in **[27]**.

3.5. THEOREM. *Let $\{T_n\}$ be a sequence of positive τ-measurable operators converging to T in the measure topology.*

(i) (*Fatou's lemma*) $\tau(T) \leq \liminf_{n\to\infty} \tau(T_n)$.

(ii) (*Monotone convergence theorem.*) *If $T_n \leq T$ (or even if $\mu_t(T_n) \leq \mu_t(T)$, $t > 0$), then*

$$\tau(T) = \lim_{n\to\infty} \tau(T_n).$$

Proof. (i) We estimate:

$$\begin{aligned}
\tau(T) &= \int_0^\infty \mu_t(T)\,dt \quad (\textit{Proposition } 2.7)\\
&\leq \int_0^\infty \liminf_{n\to\infty} \mu_t(T_n)\,dt \quad (\textit{Lemma } 3.4)\\
&\leq \liminf_{n\to\infty} \int_0^\infty \mu_t(T_n)\,dt \quad (\text{usual Fatou's lemma})\\
&= \liminf_{n\to\infty} \tau(T_n).
\end{aligned}$$

(ii) It follows from (i) and

$$\tau(T_n) = \int_0^\infty \mu_t(T_n)\,dt \leq \int_0^\infty \mu_t(T)\,dt = \tau(T).$$ □

3.6. THEOREM. *Let $\{T_n\}$ be a sequence of τ-measurable operators converging to T in the measure topology. Assume that there exist τ-measurable operators S_n, $n = 1, 2, \ldots$, and S in $L^p(\mathcal{M}; \tau)$, $0 < p < \infty$, satisfying the following conditions*:

(i) $\mu_t(T_n) \leq \mu_t(S_n)$ (*it is satisfied if* $|T_n| \leq |S_n|$, *Lemma* 2.5, (ii), (iii)),

(ii) $\|S\|_p = \lim_{n\to\infty}\|S_n\|_p$,

(iii) $\mu_t(S) \le \liminf_{n\to\infty}\mu_t(S_n)$. (*It is satisfied if* $\{S_n\}$ *converges to* S *in the measure topology, Lemma* 3.4, (i).) *Then*; T_n *and* T *are in* $L^p(\mathcal{M};\ \tau)$, *and we have*

$$\lim_{n\to\infty}\|T - T_n\|_p = 0.$$

If $p = 1$, *then we also get*

$$\lim_{n\to\infty}\tau(T_n) = \tau(T).$$

Of course, the dominated convergence theorem described at the beginning of the section is included in this theorem. (Take $p = 1$ and $S_n = S$.)

Proof. At first we note $|\tau(T)| \le \tau(|T|)$ whenever $T \in L^1(\mathcal{M};\ \tau)$ (see V §2, **[24]** for example). The last statement follows from

$$|\tau(T_n) - \tau(T)| = |\tau(T_n - T)| \le \tau\big(|T_n - T|\big) = \|T_n - T\|_1.$$

Obviously, (i) implies $T_n \in L^p(\mathcal{M};\ \tau)$. Also, (any version of) Fatou's lemma implies

$$\|T\|_p \le \liminf_{n\to\infty}\|T_n\|_p \le \liminf_{n\to\infty}\|S_n\|_p = \|S\|_p < \infty$$

and $T \in L^p(\mathcal{M};\ \tau)$. By Lemma 2.5, (iv), we have

$$\mu_t(T - T_n) \le \mu_{t/2}(T) + \mu_{t/2}(T_n) \le \mu_{t/2}(T) + \mu_{t/2}(S_n),$$

and hence,

$$\mu_t(T - T_n)^p \le C_p\big\{\mu_{t/2}(T)^p + \mu_{t/2}(S_n)^p\big\},$$

where $C_p = \mathrm{Max}(1, 2^{p-1})$. Since $\lim_{n\to\infty}\mu_t(T - T_n)^p = 0$ (Lemma 3.1) and $\liminf_{n\to\infty}\mu_{t/2}(S_n)^p \ge \mu_{t/2}(S)^p$ ((iii)), the non-negative function $C_p\{\mu_{t/2}(T)^p + \mu_{t/2}(S_n)^p\} - \mu_t(T - T_n)^p$ satisfies

$$\liminf_{n\to\infty}\Big[C_p\big\{\mu_{t/2}(T)^p + \mu_{t/2}(S_n)^p\big\} - \mu_t(T - T_n)^p\Big] \ge C_p\big\{\mu_{t/2}(T)^p + \mu_{t/2}(S)^p\big\}.$$

Usual Fatou's Lemma thus implies

$$\begin{aligned}\int_0^\infty & C_p\big\{\mu_{t/2}(T)^p + \mu_{t/2}(S)^p\big\}\,dt \\ &\le \int_0^\infty \liminf_{n\to\infty}\Big[C_p\big\{\mu_{t/2}(T)^p + \mu_{t/2}(S_n)^p\big\} - \mu_t(T - T_n)^p\Big]\,dt \\ &\le \liminf_{n\to\infty}\int_0^\infty\Big[C_p\big\{\mu_{t/2}(T)^p + \mu_{t/2}(S_n)^p\big\} - \mu_t(T - T_n)^p\Big]\,dt.\end{aligned}$$

In other words, we have

$$2C_p\left(\|T\|_p^p + \|S\|_p^p\right) \le \liminf_{n\to\infty}\left\{2C_p\left(\|T\|_p^p + \|S_n\|_p^p\right) - \|T - T_n\|_p^p\right\}.$$

By (ii) and the fact that every norm is finite, we get

$$0 \le -\limsup_{n\to\infty}\|T - T_n\|_p^p,$$

that is, $\lim_{n\to\infty}\|T - T_n\|_p = 0$. □

3.7. THEOREM. *Let T_n, $n = 1, 2, \ldots,$ and T be elements in $L^p(\mathscr{M}; \tau)$, $0 < p < \infty$. The following two conditions are equivalent*:

(i) $\lim_{n\to\infty}\|T - T_n\|_p = 0$,

(ii) $\lim_{n\to\infty}\|T_n\|_p = \|T\|_p$ *and* $T_n \to T$ *in the measure topology.*

If, in addition, $1 < p < \infty$, *they are also equivalent to*

(iii) $\lim_{n\to\infty}\|T_n\|_p = \|T\|_p$ *and* $T_n \to T$ *in the* $\sigma(L^p, L^q)$-*topology, where q is the conjugate exponent of p.*

Proof. (i) ⇒ (ii), (i) ⇒ (iii) are obvious. (ii) ⇒ (i) follows from Theorem 3.6 with $S_n = |T_n|$ and $S = |T|$. In fact, (iii) in Theorem 3.6 is checked as follows:

$$\mu_t(S) = \mu_t(T) \le \liminf_{n\to\infty}\mu_t(T_n) = \lim_{n\to\infty}\mu_t(S_n).$$

Here, Lemma 2.5, (ii), and Lemma 3.4,(i) were used. (iii) ⇒ (i) follows from the uniform convexity of $L^p(\mathscr{M}; \tau)$, $1 < p < \infty$. (cf. **[13]**, **[14]**, **[34]**). □

The theorem ((ii) ⇒ (i)) was previously known for special values of p only when $\tau(1) < \infty$. (cf. [14], [21]). Also, here is one subtlety in the above proof worth pointing out. Although we were able to check the condition (iii) in Theorem 3.6, the following problem is still open: Let $\{T_n\}$ be a sequence of τ-measurable operators converging to T in the measure topology. Does the sequence $\{|T_n|\}$ converge to $|T|$ in the measure topology? The answer is affirmative if $\tau(1) < +\infty$. More information can be found in **[19]**, **[20]** (when $\tau(1) < \infty$).

4. Convexity and concavity inequalities. Here, we study convexity (and concavity) inequalities involving μ_t and provide "real analysis" proofs to the classical norm inequalities. Not only that, we will carefully compare these classical norm inequalities with inequalities involving μ_t due to A. Grothendieck, J. von Neumann, H. Weyl, and the first named author. We will also show that these "semi-finite techniques" are useful in the theory of Haagerup's L^p-spaces.

The topics here are closely related to **[1]**, **[2]**, **[3]**, **[8]**, **[9]**, **[11]**, **[32]**, and as in **[3]** our philosophy is that, inside a trace, operators behave "like functions."

Until further notice, throughout we will assume that $\mathcal{M}$ is a von Neumann algebra with a faithful semi-finite normal trace τ.

Thanks to Proposition 2.7 and Corollary 2.8, one can derive inequalities involving the trace τ from these involving μ_t. The converse is also possible to some extent thanks to

4.1. LEMMA (*cf. Lemma* 3.3. **[8]**). *Assume that $\mathcal{M}$ has no minimal projection. For any τ-measurable operator T, we have*

$$\int_0^t \mu_s(T)\,ds = \sup\{\tau(E|T|E);\ E \textit{ is a projection in } \mathcal{M} \textit{ with } \tau(E) \leq t\}.$$

If $|T|$ does not have a point spectrum, the supremum may be taken over all projections (of trace at most t) in the von Neumann subalgebra generated by the spectral projections of $|T|$.

Proof. We may assume $T \geq 0$. Let $T = \int_0^\infty \lambda\, dE_\lambda$ be the spectral decomposition. Take an abelian von Neumann subalgebra $\mathfrak{A} = L^\infty(X;\ m)$ of $\mathcal{M}$ containing spectral projections of T and with non-atomic measure m (corresponding to τ). (Such $\mathfrak{A}$ exists by assumption on $\mathcal{M}$.) Therefore, for $T = f$ in $\mathfrak{A}$, we have $\mu_t(T) = f^*(t)$ (Remark 2.3.2).

Since (X, m) is non-atomic, the classical equality

$$\int_0^t f^*(s)\,ds = \sup_{\substack{E \subseteq X \\ m(E) \leq t}} \int_E |f|\,dm$$

is available. (See p. 202, **[26]**.) This means that

$$\begin{aligned}\int_0^t \mu_s(T)\,ds &= \sup\{\tau(ETE);\ E \text{ is a projection in } \mathfrak{A} \text{ with } \tau(E) \leq t\}\\ &\leq \sup\{\tau(ETE);\ E \text{ is a projection in } \mathcal{M} \text{ with } \tau(E) \leq t\}.\end{aligned}$$

Conversely, when a projection E in $\mathcal{M}$ satisfies $\tau(E) \leq t$ we estimate

$$\begin{aligned}\tau(ETE) &= \int_0^\infty \mu_s(ETE)\,ds \qquad \text{(Proposition 2.7)}\\ &= \int_0^t \mu_s(ETE)\,ds \qquad \text{(Lemma 2.6)}\\ &\leq \int_0^t \mu_s(T)\,ds. \qquad \text{(Lemma 2.5, (vi))}. \qquad \square\end{aligned}$$

The reader may think that the assumption on $\mathcal{M}$ (no minimal projection) is quite restrictive. But we can always embed $\mathcal{M}$ into $\mathcal{M} \otimes L^\infty([0,1]; dt)$ without changing the s-number because of the trivial fact

$$\mu_t(T) = \mu_t(T \otimes 1),$$

where the s-number on the right is relative to the tensor product of τ by the trace

$$f \to \int_0^1 f(s)\, ds.$$

Let us discuss the Hölder inequality:

$$\|TS\|_r \le \|T\|_p \|S\|_q \qquad (p, q, r > 0;\ p^{-1} + q^{-1} = r^{-1}). \tag{4}$$

Of course, (4) is well-known when p, q, $r \ge 1$. However, it seems that the only proof of (4) for $0 < r < 1$ is based on the Weyl inequality (see **[8]**)

$$\Lambda_t(TS) \le \Lambda_t(T)\Lambda_t(S), \qquad t > 0.$$

Here, $\Lambda_t(T)$ is defined by

$$\Lambda_t(T) = \exp\int_0^t \log \mu_s(T)\, ds, \qquad t > 0.$$

It is easy to see that if for example T satisfies the "Lorentz space"-type condition

$$T \in \mathcal{M} \quad \text{or} \quad \mu_t(T) \le Ct^{-\alpha} \qquad (C, \alpha > 0),\ t > 0, \tag{5}$$

then $\Lambda_t(T)$ is well-defined (i.e., $\infty - \infty$ does not occur). Whenever $\Lambda_t(T)$ appears in what follows, we will always understand that T satisfies (5). Actually, the assumption (5) is satisfied by "almost all" τ-measurable operators appearing in applications. (Cf. the estimate before Lemma 3.4.)

4.2. THEOREM. *Let T, S be τ-measurable operators.*
(i) $\|TS\|_r \le \|T\|_p \|S\|_q$ $(p, q, r > 0;\ p^{-1} + q^{-1} = r^{-1})$.
(ii) $\Lambda_t(TS) \le \Lambda_t(T)\Lambda_t(S)$, $t > 0$.
(iii) $\int_0^t f(\mu_s(TS))\, ds \le \int_0^t f(\mu_s(T)\mu_s(S))\, ds$
for any increasing function $f\colon \mathbf{R}_+ \to \mathbf{R}$ such that $t \to f(e^t)$ is convex.

This result for τ-compact elements in $\mathcal{M}$ (i.e., elements in $\mathcal{M}$ satisfying the conditions in 3.2) was proved in **[8]**. As mentioned before, not only proofs of (i) ~ (iii) for τ-measurable operators, but we will also show that each of them can be deduced from the others.

Proof. (i) We may and do assume $\|T\|_p$, $\|S\|_q < \infty$. Then the result follows from Corollary 4.4, (iii), **[8]**, and the trick in the Appendix of **[15]**. But, since this trick will be repeatedly used later, for the convenience of the reader we will recall the arguments. Let $T = U|T|$ and $|T| = \int_0^\infty \lambda\, dE_\lambda$

as usual. For each $n \in \mathbf{N}_+$, we set

$$T_n = U\int_0^n \lambda \, dE_\lambda.$$

As $\|T\|_p < +\infty$, each T_n in $\mathscr{M}$ is τ-compact (cf. p. 315–316, [8]) because of

$$\mu_t(T_n) \le \mu_t(T) \to 0 \quad \text{as } t \to \infty.$$

Define S_n similarly. By Corollary 4.4, [8] (which is essentially (ii), (iii) for τ-compact operators), we get

$$\left\{\int_0^\infty \mu_s(T_nS_n)^r \, ds\right\}^{1/r} \le \left\{\int_0^\infty \mu_s(T_n)^p \, ds\right\}^{1/p} \left\{\int_0^\infty \mu_s(S_n)^q \, ds\right\}^{1/q}$$
$$\le \|T\|_p \|S\|_q.$$

But $T_nS_n \to TS$ in the measure topology so that

$$\begin{aligned}\|TS\|_r &= \left\{\int_0^\infty \mu_s(TS)^r \, ds\right\}^{1/r} \\ &\le \left\{\int_0^\infty \liminf_{n\to\infty} \mu_s(T_nS_n)^r \, ds\right\}^{1/r} \qquad \text{(Lemma 3.4, } (i)) \\ &\le \liminf_{n\to\infty} \left\{\int_0^\infty \mu_s(T_nS_n)^r \, ds\right\}^{1/r} \qquad \text{(usual Fatou's lemma).}\end{aligned}$$

Combining the above two estimates, we get the desired result. (i) $\Rightarrow$ (ii) At first, for τ-measurable T, S, we prove

$$(6) \qquad \left(\int_0^t \mu_s(TS)^r \, ds\right)^{1/r} \le \left(\int_0^t \mu_s(T)^{2r} \, ds\right)^{1/2r} \left(\int_0^t \mu_s(S)^{2r} \, ds\right)^{1/2r}, \quad t > 0,$$

based on (i). By the remark after Lemma 4.1, we may and do assume that $\mathscr{M}$ has no minimal projection. Let $T = U|T|$ and $S = V|S|$ be the polar decompositions. Let E be a projection in $\mathscr{M}$ which commutes with $|TS|$ and satisfies $\tau(E) \le t$. Let F be the support projection of $|ES^*|$ and W be the phase part of the polar decomposition of ES^*. Then

$$\begin{cases} W^*W = F, \\ WW^* = \operatorname{supp}(|SE|) \le E. \end{cases}$$

It is then straightforward to check

$$E|TS|^2E = (ES^*W^*E)(EW|T|^2W^*E)(EWSE).$$

Thus, (i) implies that

$$\begin{aligned}\|E|TS|^2E\|_r &\le \|ES^*W^*E\|_{4r} \|EW|T|^2W^*E\|_{2r} \|EWSE\|_{4r} \\ &\le \left\{\int_0^t \mu_s(S)^{4r} \, ds\right\}^{1/4r} \left\{\int_0^t \mu_s(T)^{4r} \, ds\right\}^{1/2r} \left\{\int_0^t \mu_s(S)^{4r} \, ds\right\}^{1/4r}.\end{aligned}$$

Here, Lemma 2.6 and Lemma 2.5, (ii), (vi) were used. Notice $(E|TS|^2E)^r = E|TS|^{2r}E$. By taking $\mathfrak{A}$ which includes E in the proof of Lemma 4.1 (where T is replaced by $|TS|^{2r}$), we obtain (by Lemma 4.1)

$$\left\{\int_0^t \mu_s(TS)^{2r}\,ds\right\}^{1/r} \le \left\{\int_0^t \mu_s(T)^{4r}\,ds\right\}^{1/2r}\left\{\int_0^t \mu_s(S)^{4r}\,ds\right\}^{1/2r},$$

which is exactly (6) since $r > 0$ is arbitrary. Dividing the both sides of (6) by $t^{1/r}$, we get

$$\left\{\int_0^t \mu_s(TS)^{r}\,\frac{ds}{t}\right\}^{1/r} \le \left\{\int_0^t \mu_s(T)^{2r}\,\frac{ds}{t}\right\}^{1/2r}\left\{\int_0^t \mu_s(S)^{2r}\,\frac{ds}{t}\right\}^{1/2r}.$$

We now assume that T, S (hence TS) satisfy (5). By the well-known equality:

$$\exp\int_0^t \log|f(s)|\,\frac{ds}{t} = \lim_{r\to 0}\left\{\int_0^t |f(s)|^r\,\frac{ds}{t}\right\}^{1/r},$$

$$\text{if } \int_0^t |f(s)|^r\,\frac{ds}{t} < +\infty \text{ for some } r > 0.$$

(see p. 74, [**22**] for example), we get

$$\Lambda_t(TS)^{1/t} \le \Lambda_t(T)^{1/t}\Lambda_t(S)^{1/t}.$$

(ii) $\Rightarrow$ (iii) This follows from Corollary 4.2, [**8**], if T and S are bounded. Using the same trick as in the proof of (i), it is clear that (iii) remains valid for any T, S. (iii) $\Rightarrow$ (i) follows from the classical Hölder inequality (with $f(s) = s^{1/r}$ and $t = +\infty$). □

We now discuss the Minkowsky inequality:

$$\|T + S\|_p \le \|T\|_p + \|S\|_p, \qquad p \ge 1.$$

It was shown in [**8**] that this can be deduced from the von Neumann inequality:

$$\Phi_t(T+S) \le \Phi_t(T) + \Phi_t(S), \qquad t > 0,$$

where

$$\Phi_t(T) = \int_0^t \mu_s(T)\,ds \qquad \text{(recall Lemma 4.1)}.$$

4.3 LEMMA (cf. [**1**], [**15**], [**32**]). *Let T, S be τ-measurable operators. Then there exist partial isometries U, V in $\mathcal{M}$ such that*

$$|T + S| \le U|T|U^* + V|S|V^*.$$

4.4. THEOREM. *Let T, S be τ-measurable operators.*
(i) $\|T + S\|_p \leq \|T\|_p + \|S\|_p$, $p \geq 1$
(ii) $\Phi_t(T + S) \leq \Phi_t(T) + \Phi_t(S)$, $t > 0$,
(iii) $\int_0^t f(\mu_s(T + S))\,ds \leq \int_0^t f(\mu_s(T) + \mu_s(S))\,ds$, $t > 0$,
for any convex continuous increasing function f: $\mathbf{R}_+ \to \mathbf{R}$.

Proof. Of course (i) is well-known. (For example it follows from the $L^p \times L^q$-duality). Also, as mentioned before Lemma 4.3, it can be proved from the von Neumann inequality (see the proof below). Therefore, as before, our main concern is to clarify relation among (i) ~ (iii).

(ii) As usual, we may assume that $\mathscr{M}$ has no minimal projection. Let E be a projection in $\mathscr{M}$ with $\tau(E) \leq t$. Using U, V in Lemma 4.3, we estimate

$$\begin{aligned}\tau(E|T + S|E) &\leq \tau(EU|T|U^*E) + \tau(EV|S|V^*E)\\ &\leq \int_0^t \mu_s(EU|T|U^*E)\,ds + \int_0^t \mu_s(EV|S|V^*E)\,ds\\ &\leq \int_0^t \mu_s(T)\,ds + \int_0^t \mu_s(S)\,ds\end{aligned}$$

due to Lemma 2.6. hence Lemma 4.1 implies (ii).

(ii) $\Rightarrow$ (iii) By Lemma 4.3, (i), **[8]**, (iii) follows from (ii) when S and T are bounded. For general T, S, (iii) remains valid by the trick in the proof of (i) in Theorem 4.2.

(iii) $\Rightarrow$ (i) follows from the usual Minkowsky inequality (with $f(s) = s^p$, $t = \infty$). □

We point out the formula

$$\Phi_t(T) = \inf\{\|T_1\|_1 + t\|T_2\|_\infty\},$$

where the infimum is taken over all decompositions $T = T_1 + T_2$. This also proves (ii), and it is very important in the theory of real interpolation. Let $T = T_1 + T_2$ be an arbitrary decomposition. For $0 < \alpha < 1$ and $s > 0$, we estimate

$$\begin{aligned}\mu_s(T) &\leq \mu_{\alpha s}(T_1) + \mu_{(1-\alpha)s}(T_2) \quad &&(\text{Lemma 2.5, (v)})\\ &\leq \mu_{\alpha s}(T_1) + \|T_2\|_\infty &&(\text{Lemma 2.5, (i)}),\end{aligned}$$

$$\begin{aligned}\Phi_t(T) &\leq \int_0^t \mu_{\alpha s}(T_1)\,ds + t\|T_2\|_\infty \leq \int_0^\infty \mu_{\alpha s}(T_1)\,ds + t\|T_2\|_\infty\\ &= \alpha^{-1}\int_0^\infty \mu_s(T_1)\,ds + t\|T_2\|_\infty = \alpha^{-1}\|T_1\|_1 + t\|T_2\|_\infty.\end{aligned}$$

Letting $\alpha \uparrow 1$, we get $\Phi_t(T) \leq \inf\{\|T_1\|_1 + t\|T_2\|_\infty\}$. To show the reversed inequality, let $T = u|T|$ be the polar decomposition and $|T| = \int_0^\infty \lambda\, dE_\lambda$ be the spectral decomposition. We set $\alpha = \mu_t(T)$,

$$T_1 = u\int_\alpha^\infty (\lambda - \alpha)\, dE_\lambda, \qquad T_2 = T - T_1.$$

Since $|T_1| = \int_\alpha^\infty(\lambda - \alpha)\, dE_\lambda = f(|T|)$ with

$$f(\lambda) = \begin{cases} 0 & \text{if } 0 \leq \lambda \leq \alpha, \\ \lambda - \alpha & \text{if } \lambda \geq \alpha, \end{cases}$$

we get

$$\begin{aligned} \mu_s(T_1) &= f(\mu_s(T)) \qquad (\text{Lemma 2.5, } (ii), (iv)) \\ &= \begin{cases} \mu_s(T) - \alpha & \text{if } 0 < s < t, \\ 0 & \text{if } s \geq t. \end{cases} \end{aligned}$$

Since $\|T_2\|_\infty \leq \alpha$, we get

$$\begin{aligned} \|T_1\|_1 + t\|T_2\|_\infty &\leq \int_0^\infty \mu_s(T_1)\, ds + t\alpha \\ &= \int_0^t (\mu_s(T) - \alpha)\, ds + t\alpha = \int_0^t \mu_s(T)\, ds. \end{aligned}$$

Thus the above formula was proved.

It is also worth pointing out that when positive τ-measurable operators T, S satisfy $\Phi_t(T) \leq \Phi_t(S)$, $t > 0$, S is said to "submajorize" T. In fact, this ordering is one of the most important orderings in the majorization theory.

Before investigating a counterpart of Theorem 4.4 for $0 < p < 1$, we prove Jensen-type inequalities ([3]) which may be of independent interest.

4.5. LEMMA. *Let T be a positive τ-measurable operator, and U be a contraction in $\mathcal{M}$. For any continuous increasing convex function f on $\mathbf{R}_+$ with $f(0) = 0$, we get*

$$\mu_t(f(UTU^*)) = f(\mu_t(UTU^*)) \leq \mu_t(Uf(T)U^*).$$

If f is concave instead, we get the reversed inequality.

Proof. We will just consider the convex case. (The concave case can be handled similarly.) Since f is convex, it is of the form

$$f(t) = \sup_{\iota \in I}(a_\iota t + b_\iota), \qquad t \geq 0,$$

with $a_\iota \geq 0$ and $b_\iota \leq 0$. For each unit vector ξ and $\iota \in I$, we have

$$\begin{aligned}(Uf(T)U^*\xi|\xi) &\geq (U(a_\iota T + b_\iota)U^*\xi|\xi) \\ &= a_\iota(UTU^*\xi|\xi) + b_\iota\|U^*\xi\|^2 \geq a_\iota(UTU^*\xi|\xi) + b_\iota\end{aligned}$$

since $b_\iota \leq 0$ and $\|U^*\xi\|^2 \leq 1$. Taking the supremum over ι, we have

$$(Uf(T)U^*\xi|\xi) \geq f((UTU^*\xi|\xi)).$$

By making use of the expression right before Proposition 2.2, we get

$$\begin{aligned}\mu_t(Uf(T)U^*) &= \inf\sup(Uf(T)U^*\xi|\xi) \geq \inf\sup f((UTU^*\xi|\xi)) \\ &= f(\inf\sup(UTU^*\xi|\xi)) \quad (\text{since } f \text{ is continuous and increasing}) \\ &= f(\mu_t(UTU^*)).\end{aligned}$$

□

4.6. Proposition. *Let f be a continuous increasing function on $\mathbf{R}_+$ with $f(0) = 0$. Let a, b be elements in $\mathcal{M}$ with $a^*a + b^*b \leq 1$, and T, S be τ-measurable operators.*

(i) *When f is concave. If T, S are positive, we have*

$$\mu_t(a^*f(T)a + b^*f(S)b) \leq \mu_t(f(a^*Ta + b^*Sb)), \qquad t > 0,$$

hence,

$$\tau(a^*f(T)a) + \tau(b^*f(S)b) \leq \tau(f(a^*Ta + b^*Sb)).$$

Also, for general T, S, we have

$$\tau(f(|T + S|)) \leq \tau(f(|T|)) + \tau(f(|S|)).$$

(ii) *When f is convex. If T, S are positive, we have*

$$\mu_t(a^*f(T)a + b^*f(S)b) \geq \mu_t(f(a^*Ta + b^*Sb)), \qquad t > 0,$$

hence,

$$\tau(a^*f(T)a) + \tau(b^*f(S)b) \geq \tau(f(a^*Ta + b^*Sb)).$$

For positive T, S; we also get

$$\tau(f(T + S)) \geq \tau(f(T)) + \tau(f(S)).$$

Inequalities in (i) are implicit in [3], but for the sake of completeness we will give full details. More detailed information can be found in [3].

Proof. (i) Consider the von Neumann algebra $\mathcal{M} \otimes M_2(\mathbf{C})$ equipped with the trace $\tilde{\tau} = \left[\begin{smallmatrix}\tau & 0\\ 0 & \tau\end{smallmatrix}\right]$. Applying Lemma 4.5 to the contraction $\left[\begin{smallmatrix}a & 0\\ b & 0\end{smallmatrix}\right]^*$ and the positive ($\tilde{\tau}$-measurable) operator $\left[\begin{smallmatrix}T & 0\\ 0 & S\end{smallmatrix}\right]$, we get

$$\mu_t\left(\begin{bmatrix} a^*f(T)a + b^*f(S)b & 0 \\ 0 & 0\end{bmatrix}\right) \leq \mu_t\left(\begin{bmatrix} f(a^*Ta + b^*Sb) & 0 \\ 0 & 0\end{bmatrix}\right).$$

Here, the s-numbers are relative to $\tilde{\tau}$. But this obviously means the first inequality in (i). To show the last inequality in (i), we at first assume T, $S \geq 0$. Since $T, S \leq T + S$, there exist contractions U, V in $\mathscr{M}$ such that

$$T^{1/2} = U(T+S)^{1/2}, \quad S^{1/2} = V(T+S)^{1/2},$$
$$U^*U + V^*V = \mathrm{supp}(T+S).$$

We then estimate

$$\begin{aligned}
\tau(f(T)) + \tau(f(S)) &= \tau(f(U(T+S)U^*)) + \tau(f(V(T+S)V^*)) \\
&\geq \tau(Uf(T+S)U^*) + \tau(Vf(T+S)V^*) \\
&\qquad\qquad \text{(Proposition 2.7 and Lemma 4.5).} \\
&= \tau\left(f(T+S)^{1/2}U^*Uf(T+S)^{1/2}\right) + \tau\left(f(T+S)^{1/2}V^*Vf(T+S)^{1/2}\right) \\
&= \tau\left(f(T+S)^{1/2}\mathrm{supp}(T+S)f(T+S)^{1/2}\right) = \tau(f(T+S)).
\end{aligned}$$

For general T, S, we choose partial isometries U, V in $\mathscr{M}$ such that

$$|T+S| \leq U|T|U^* + V|S|V^* \quad \text{(Lemma 4.3)}.$$

Since $U|T|U^*$, $V|S|V^* \geq 0$, we get

$$\begin{aligned}
\tau(f(|T+S|)) &\leq \tau(f(U|T|U^* + V|S|V^*)) \\
&\qquad\qquad \text{(Lemma 2.5, (iii), and Corollary 2.8)} \\
&\leq \tau(f(U|T|U^*)) + \tau(f(V|S|V^*)) \quad \text{(the previous case)} \\
&\leq \tau(f(|T|)) + \tau(f(|S|)).
\end{aligned}$$

Here, the last inequality follows from $\mu_t(U|T|U^*) \leq \mu_t(|T|)$ and $\mu_t(V|S|V^*) \leq \mu_t(|S|)$. (ii) is proved by the same arguments as (i). (But, notice that the last part based on Lemma 4.3 breaks down.) □

We now discuss the following inequality:

$$\|T+S\|_p^p \leq \|T\|_p^p + \|S\|_p^p, \qquad 0 < p \leq 1, \tag{7}$$

that replaces the Minkowsky inequality, $p \geq 1$. It can be proved that (7) is equivalent to the Grothendieck inequality proved in [**9**] for τ-compact elements in $\mathscr{M}$:

$$\Lambda_t(1 + |T+S|) \leq \Lambda_t(1+|T|)\Lambda_t(1+|S|), \qquad t > 0,$$

and hence to

$$\int_0^t g(\mu_s(T+S))\,ds \leq \int_0^t g(\mu_s(T))\,ds + \int_0^t g(\mu_s(S))\,ds, \quad t > 0, \tag{8}$$

for any increasing function g on $\mathbf{R}_+$ which is operator concave ([**7**]) and $g(0) = 0$ (cf. [**1**], [**9**]). However, operator concavity is a strong condition and (8) was proved in **[3]** (when $t = +\infty$) for any increasing concave function f with $f(0) = 0$. We now state

4.7. THEOREM. *Let T, S be τ-measurable operators.*

(i) $\displaystyle\int_0^t g(\mu_s(T+S))\,ds \le \int_0^t g(\mu_s(T))\,ds + \int_0^t g(\mu_s(S))\,ds, \qquad t > 0,$

for any increasing concave function g on $\mathbf{R}_+$ with $g(0) = 0$.

(ii) $\Lambda_t(1 + g(|T+S|)) \le \Lambda_t(1 + g(|T|))\Lambda_t(1 + g(|S|)), \qquad t > 0,$

for any g as in (i).

Of course (7) follows from (i) (actually from the last inequality in Proposition 4.6, (i)). We will prove (ii) and the bi-implication (i) $\Leftrightarrow$ (ii). But, before its proof, some remarks are in order. For a positive τ-measurable operator T, we have

$$\mu_t(1+T) = \begin{cases} \mu_t(T) + 1 & \text{if } t < \tau(1), \\ 0 & \text{if } t \ge \tau(1), \end{cases}$$

(by comparing the spectral projections of T and those of $1 + T$). Therefore, if $t \le \tau(1)$ ($\tau(1) \in (0, \infty]$), then we have

$$\Lambda_t(1+T) = \exp \int_0^t \log(1 + \mu_s(T))\,ds.$$

On the other hand, if $t > \tau(1)$, then (since we are assuming (5) and $\log(1 + \mu_t(T)) \sim \log(\mu_t(T))$ when $\mu_t(T)$ is large) we get

$$0 \le \int_0^{\tau(1)} \log(\mu_s(1+T))\,ds < +\infty,$$

and consequently

$$\Lambda_t(1+T) = 0.$$

Proof. (i) As usual we may and do assume that $\mathscr{M}$ has no minimal projection. Also, by Lemma 4.3, we may and do assume that T, S are positive. For a projection E in $\mathscr{M}$, $\tau(E) \le t$, commuting with $T + S$, we estimate

$$\begin{aligned} \tau(Eg(T+S)E) &= \tau(g(E(T+S)E)) \\ &\le \tau(g(ETE)) + \tau(g(ESE)) \qquad \text{(Proposition 4.6(i))} \\ &= \int_0^t g(\mu_s(ETE))\,ds + \int_0^t g(\mu_s(ESE))\,ds \qquad \text{(Lemma 2..6)} \\ &\le \int_0^t g(\mu_s(T))\,ds + \int_0^t g(\mu_s(S))\,ds. \end{aligned}$$

Thus, (i) follows from Lemma 4.1.

(i) $\Rightarrow$ (ii) By the comment before the proof, we may and do assume $t \leq \tau(1)$. We apply (i) to the function: $s \to \log(1 + g(s))$ and get

$$\int_0^t \log(1 + g(\mu_s(T + S)))\, ds \leq \int_0^t \log(1 + g(\mu_s(T)))\, ds + \int_0^t \log(1 + g(\mu_s(S)))\, ds.$$

Since $g(\mu_s(T + S)) = \mu_s(g(|T + S|))$, this is exactly (ii) (thanks to the other comment before the proof).

(ii) $\Rightarrow$ (i) To show (i), we may and do assume that T and S are bounded by the usual approximation arguments. Lemma 3.2, **[9]**, asserts that

$$\int_0^t g(\mu_s(T))^p\, ds = \kappa^{-1} p \sin(\pi p) \int_0^\infty \log \Lambda_t(1 + rg(|T|)) r^{-1-p}\, dr$$

for each $0 < p < 1$ (and similar formulas for S and $T + S$). Thus, (ii) implies

$$\int_0^t g(\mu_s(T + S))^p\, ds \leq \int_0^t g(\mu_s(T))^p\, ds + \int_0^t g(\mu_s(S))^p\, ds.$$

Letting $p \nearrow 1$, we get (i). □

We now assume that $\mathscr{M}$ is a general von Neumann algebra and $\mathfrak{A} = \mathscr{M} x_\sigma \mathbf{R}$ is the crossed product explained in 1.6. In the rest of the section, we prove norm inequalities of the Haagerup $L^p(\mathscr{M})$ based on the techniques developed so far.

4.8. LEMMA. *For any T in $L^p(\mathscr{M})$, $0 < p < \infty$, we have*

$$\mu_t(T) = t^{-1/p} \|T\|_p, \qquad t > 0,$$

where μ_t is relative to the canonical trace on $\mathfrak{A}$.

This lemma is proved in **[15]**. Actually, it is an immediate consequence of Lemma 1.7 and Lemma 2.5, (iv). This lemma implies that an element in $L^p(\mathscr{M})$, $0 < p < \infty$, satisfies the condition (5). Also, for T in $L^p(\mathscr{M})$, $0 < p < \infty$, it is elementary to compute

$$\begin{cases} \Phi_t(T) = q t^{1/q} \|T\|_p & \text{if } 1 < p < \infty \text{ and } p^{-1} + q^{-1} = 1, \\ \Lambda_t(T) = \left\{ (et^{-1})^{1/p} \|T\|_p \right\}^t. \end{cases}$$

4.9. THEOREM (cf. **[12]**, **[13]**, **[15]**).
(i) *For T in $L^p(\mathscr{M})$ and S in $L^q(\mathscr{M})$ ($p, q, r > 0$; $p^{-1} + q^{-1} = r^{-1}$),*

we have

$$\|TS\|_r \le \|T\|_p \|S\|_q.$$

(ii) *For T, S in $L^p(\mathcal{M})$, $1 \le p \le \infty$, we have*

$$\|T + S\|_p \le \|T\|_p + \|S\|_p.$$

(iii) *For T, S in $L^p(\mathcal{M})$, $0 < p \le 1$, we have*

$$\|T + S\|_p^p \le \|T\|_p^p + \|S\|_p^p.$$

Proof. (i) Setting $t = 1$ in Theorem 4.2, (ii), we get

$$e^{1/r}\|TS\|_r \le e^{1/p}\|T\|_p e^{1/q}\|S\|_q.$$

(ii) When $p = 1, \infty$, the result is trivial. (The case $p = 1$ is actually proved as (iii).) When $1 < p < \infty$, setting $t = 1$ in Theorem 4.4, (ii), we get

$$q\|T + S\|_p \le q\|T\|_p + q\|T\|_p.$$

(iii) Choose p' with $0 < p' < p(\le 1)$. Setting $g(s) = s^{p'}$ and $t = 1$ in Theorem 4.7, (i), one directly computes

$$(1 - p'/p)^{-1}\|T + S\|_p^{p'} \le (1 - p'/p)^{-1}\left(\|T\|_p^{p'} + \|S\|_p^{p'}\right)$$

based on Lemma 4.8. Clearing $(1 - p'/p)^{-1}$ and letting $p' \nearrow p$, we get (iii). □

As pointed out in 1.6, tr (used to define $\| \|_p$) and the canonical trace τ on $\mathfrak{A} = \mathcal{M} x_\sigma \mathbf{R}$ are different. But, it is possible to relate the L^p-norm on $L^p(\mathcal{M})$ to τ. Namely, the explicit computation of Φ_t before Theorem 4.9 and the formula (with $t = 1$) after Theorem 4.4 imply that

$$\|T\|_p = q^{-1}\inf\left\{\|T_1\|_1 + \|T_2\|_\infty\right\}, \qquad T \in L^p(\mathcal{M}), 1 < p < \infty,$$

$p^{-1} + q^{-1} = 1$, where the infimum is taken over all decompositions $T = T_1 + T_2$ (T_i are τ-measurable operators affiliated with $\mathfrak{A}$) and the L^1-norm in the inf sign is relative to the canonical trace on $\mathfrak{A}$.

We now specialize ourselves to the commutative case. The following fact does not seem to have been noticed in classical analysis. Let (X, m) be a measure space, and f be a measurable function on X. For $1 < p < \infty$ ($p^{-1} + q^{-1} = 1$), define the function $\tilde{f}$ on $X \times \mathbf{R}$ by

$$\tilde{f}(x, t) = f(x)e^{t/p}, \qquad x \in X,\ t \in \mathbf{R}.$$

Then, $\|f\|_p = (\int_X |f|^p\, dm)^{1/p}$ is given by

$$\|f\|_p = q^{-1}\inf\left\{\|f_1\|_1 + \|f_2\|_\infty\right\}.$$

Here, the infimum is taken over all decompositions $\tilde{f} = f_1 + f_2$ (f_i are functions on $X \times \mathbf{R}$) and $\| \|_1$ in the inf sign is relative to the measure

$dm \otimes e^{-t}\,dt$ on $X \times \mathbf{R}$. Full details are left to the reader. (It is indeed an amusing exercise.)

Finally let us point out the reason why "semi-finite techniques" were useful in Theorem 4.9. Actually, the Haagerup L^p-space is sitting inside the "non-commutative weak L^p-space" (i.e., the Lorentz space $L^{p\infty}(\mathfrak{A};\ \tau)$ associated with $\mathfrak{A} = \mathscr{M} \times_\sigma \mathbf{R}$ and the canonical trace τ) on $\mathfrak{A}$ (although $L^p(\mathscr{M}) \subseteq L^p(\mathfrak{A};\ \tau)$ is false). Furthermore, $\|T\|_p$, $T \in L^p(\mathscr{M})$, is exactly the weak L^p-space norm of T. Details on this theory will be published elsewhere.

5. The Clarkson-McCarthy inequalities. This section is devoted to a study of the Clarkson-McCarthy inequalities:

$$\|T + S\|_p^p + \|T - S\|_p^p \le 2^{p-1}\left(\|T\|_p^p + \|S\|_p^p\right), \qquad 2 \le p < \infty, \tag{9}$$

$$\|T + S\|_p^q + \|T - S\|_p^q \le 2\left(\|T\|_p^p + \|S\|_p^p\right)^{q/p}, \qquad 1 < p \le 2, \tag{10}$$

$p^{-1} + q^{-1} = 1$, where T, S are in the Haagerup $L^p(\mathscr{M})$. Of course, these inequalities, which imply uniform convexity for the L^p-spaces, $1 < p < \infty$, are known. (cf. **[4]**, **[5]**, **[30]**, **[34]** for the semi-finite case and **[12]**, **[13]** for the general case.) The inequality (9) is relatively easy to prove.

LEMMA 5.1. *Assume* $1 \le p < \infty$, *and let* T, S *be in* $L^p(\mathscr{M})_+$. *Then we have*

$$2^{1-p}\|T + S\|_p^p \le \|T\|_p^p + \|S\|_p^p \le \|T + S\|_p^p.$$

Proof. The first inequality is valid for any T, S in $L^p(\mathscr{M})$, and it is just Theorem 4.9, (ii) together with the convexity of t^p, $t \ge 0$. The second inequality is basically the last statement of Proposition 4.6, (ii) (in the semi-finite case). Since tr and τ are different, the arguments in Proposition 4.6 should be modified as follows:

$$\begin{aligned}
\|T\|_p^p + \|S\|_p^p &= \mathrm{tr}\big((U(T+S)U^*)^p\big) + \mathrm{tr}\big((V(T+S)V^*)^p\big)\\
&\qquad \big(\text{Here, } T^{1/2} = U(T+S)^{1/2},\ S^{1/2} = V(T+S)^{1/2},\\
&\qquad\qquad U^*U + V^*V = \mathrm{supp}(T+S),\ U, V \in \mathscr{M}\big)\\
&= \mu_1\big((U(T+S)U^*)^p\big) + \mu_1\big((U(T+S)V^*)^p\big) \qquad \text{(Lemma 4.8)}\\
&\le \mu_1\big(U(T+S)^pU^*\big) + \mu_1\big(V(T+S)^pV^*\big) \qquad \text{(Lemma 4.5)}\\
&= \mathrm{tr}\big(U(T+S)^pU^*\big) + \mathrm{tr}\big(V(T+S)^pV^*\big)\\
&= \mathrm{tr}\big((T+S)^{p/2}\mathrm{supp}(T+S)(T+S)^{p/2}\big)\\
&= \mathrm{tr}\big((T+S)^p\big) = \|T+S\|_p^p.
\end{aligned}$$

□

When $0 < p \leq 1$, all the inequalities in the above Lemma are reversed. In fact, the first inequality follows from the operator concavity of t^p, $t \geq 0$ ($0 < p \leq 1$) (see [7]) while the second is just Theorem 4.9, (iii).

5.2. THEOREM. *Let $\mathscr{M}$ be a general von Neumann algebra, and T, S be in $L^p(\mathscr{M})$ with $2 \leq p < \infty$. Then we have*

$$\|T + S\|_p^p + \|T - S\|_p^p \leq 2^{p-1}\left(\|T\|_p^p + \|S\|_p^p\right).$$

Proof. Set $p' = p/2 \in [1, \infty)$. We have

$$\begin{aligned}
\|T + S\|_p^p + \|T - S\|_p^p &= \left\| \, |T + S|^2 \right\|_{p'}^{p'} + \left\| \, |T - S|^2 \right\|_{p'}^{p'} \\
&\leq \left\| \, |T + S|^2 + |T - S|^2 \right\|_{p'}^{p'} = 2^{p'} \left\| \, |T|^2 + |S|^2 \right\|_{p'}^{p'} \\
&\leq 2^{p'} 2^{p'-1}\left(\left\| \, |T|^2 \right\|_{p'}^{p'} + \left\| \, |S|^2 \right\|_{p'}^{p'} \right) \\
&= 2^{p-1}\left(\|T\|_p^p + \|S\|_p^p\right).
\end{aligned}$$

Here, Lemma 5.1 was used twice. □

From the above proof and the remark after Lemma 5.1, it is clear that, when $0 < p \leq 2$, the inequality (9) is reversed.

The inequality (10) is more difficult to prove, and some false proofs exist in the literature. Basically, all correct proofs have been done in the framework of the "complex interpolation method" (cf. [**13**], [**34**]), and the first named author would like to point out that the "real analysis" proof of (10) presented in [**9**] is incorrect. We present a direct proof by adopting a method due to Cleaver, [**4**].

5.3. THEOREM. *Let $\mathscr{M}$ be a general von Neumann algebra, and T, S be in $L^p(\mathscr{M})$ with $1 < p \leq 2$. Then we have*

$$\|T + S\|_p^q + \|T - S\|_p^q \leq 2\left(\|T\|_p^p + \|S\|_p^p\right)^{q/p},$$

where $p^{-1} + q^{-1} = 1$.

Proof. Let $T + S = U|T + S|$ and $T - S = V|T - S|$ be the polar decompositions, and

$$A = \|T + S\|_p^{q-p}|T + S|^{p-1}U^*, \quad B = \|T - S\|_p^{q-p}|T - S|^{p-1}V^*.$$

Then, A, B are in $L^q(\mathscr{M})$; and we have

$$\operatorname{tr}(A(T + S)) = \|T + S\|_p^q = \|A\|_q^p, \quad \operatorname{tr}(B(T - S)) = \|T - S\|_p^q = \|B\|_q^p.$$

It is easy to see that the theorem follows from

$$\begin{aligned} &|\operatorname{tr}(A(T+S)+B(T-S))| \\ &\quad \le 2^{1/q}\left(\|T\|_p^p+\|S\|_p^p\right)^{1/p}\left(\|A\|_q^p+\|B\|_q^p\right)^{1/p} \end{aligned} \tag{11}$$

for any A, B in $L^q(\mathcal{M})$ and any T, S in $L^p(\mathcal{M})$. Let $A = U|A|$ and $B = V|B|$ be the polar decompositions, and let $T = HX$ and $S = KY$ be the right polar decompositions ($H, K \ge 0$). For $1/2 \le \operatorname{Re} z \le 1$, we set

$$T(z) = H^{pz}X, \quad S(z) = K^{pz}Y,$$

$$A(z) = \|A\|_q^{pz-q(1-z)}U|A|^{q(1-z)},$$

$$B(z) = \|B\|_q^{pz-q(1-z)}V|B|^{q(1-z)}.$$

Then the function

$$f(z) = \operatorname{tr}(A(z)(T(z)+S(z)) + B(z)(T(z)-S(z)))$$

is bounded continuous on $1/2 \le \operatorname{Re} z \le 1$ and holomorphic in the interior. For $\operatorname{Re} z = 1$, we have

$$|\operatorname{tr}(A(z)T(z))| = \|A\|_q^p\left|\operatorname{tr}\left(U|A|^{-iq\operatorname{Im} z}H^pH^{ip\operatorname{Im} z}X\right)\right| \le \|A\|_q^p\|T\|_p^p.$$

Using similar bounds for the other terms, we have

$$|f(z)| \le \left(\|A\|_q^p+\|B\|_q^p\right)\left(\|T\|_p^p+\|S\|_p^p\right), \qquad \operatorname{Re} z = 1.$$

For $\operatorname{Re} z = 1/2$, all of $T(z)$, $S(z)$, $A(z)$, and $B(z)$ are in $L^2(\mathcal{M})$ and we have

$$\begin{aligned} |f(z)| &\le \|A(z)\|_2\|T(z)+S(z)\|_2 + \|B(z)\|_2\|T(z)-S(z)\|_2 \\ &\le \left(\|A(z)\|_2^2+\|B(z)\|_2^2\right)^{1/2}\left(\|T(z)+S(z)\|_2^2+\|T(z)-S(z)\|_2^2\right)^{1/2} \\ &= \sqrt{2}\left(\|A(z)\|_2^2+\|B(z)\|_2^2\right)^{1/2}\left(\|T(z)\|_2^2+\|S(z)\|_2^2\right)^{1/2}, \end{aligned}$$

where the parallelogram law was used. But we estimate

$$\|T(z)\|_2^2 \le \|H^{p/2}\|_2^2 = \|T\|_p^p,$$

$$\|S(z)\|_2^2 \le \|K^{p/2}\|_2^2 = \|S\|_p^p.$$

We similarly get

$$\|A(z)\|_2^2 \le \|A\|_q^{p-q}\left\| |A|^{q/2}\right\|_2^2 = \|A\|_q^p, \quad \|B(z)\|_2^2 \le \|B\|_q^p.$$

Therefore we have proved

$$|f(z)| \le \sqrt{2}\left(\|A\|_q^p+\|B\|_q^p\right)^{1/2}\left(\|T\|_p^p+\|S\|_p^p\right)^{1/2}, \qquad \operatorname{Re} z = 1/2.$$

By the three line theorem, we get

$$|f(1/p)| \leq (\alpha\beta)^{(1/p-1/2)2}\left(\sqrt{2}\sqrt{\alpha}\sqrt{\beta}\right)^{(1-1/p)2} = 2^{1/q}\alpha^{1/p}\beta^{1/p},$$

where α, β are $\|A\|_q^p + \|B\|_q^p$, $\|T\|_p^p + \|S\|_p^p$ respectively. Since $|F(1/p)|$ is exactly the left side of (11), the proof is complete. □

The main emphasis of the present article has been real analysis methods. Yet, the authors' effort to obtain a real analysis proof of (10) has not been successful. Even in the $B(\mathfrak{H})$ case, the authors are very interested in a real analysis proof of (10). Let us point out that a "real analysis proof" of (10) presented in **[17]** is incorrect. (Actually, the argument in **[17]** is based on false inequalities for real numbers.)

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