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THE SPACING OF THE MINIMA IN CERTAIN CUBIC LATTICES

H. C. WILLIAMS

Let $\mathscr{K}$ be a cubic field with negative discriminant; let $\mu, \nu \in \mathscr{K}$; and let $\mathscr{R}$ be a lattice with basis $\{1, \mu, \nu\}$ such that 1 is a minimum of $\mathscr{R}$. If

$$1 = \theta_1, \theta_2, \theta_3, \ldots, \theta_n, \ldots$$

is a chain of adjacent minima of $\mathscr{R}$ with $\theta_{i+1} > \theta_i$ $(i = 1, 2, 3, \ldots)$, then

$$\theta_{n+5} \geq \theta_{n+3} + \theta_n.$$

This result can be used to prove that if p is the period of Voronoi's continued fraction algorithm for finding the fundamental unit ε_0 of $\mathscr{K}$, then

$$\varepsilon_0 > \tau^{p/2},$$

where $\tau = (1 + \sqrt{5})/2$. It is also shown that

$$\theta_n > 4^{[(n-1)/7]}.$$

1. Introduction. In order to discuss the problems considered in this paper, it is necessary to give a brief description of the properties of cubic lattices. For a more extensive and more general treatment of these topics we refer the reader to Delone and Faddeev [**1**].

Let $f(x) \in \mathbf{Z}[x]$ be a cubic polynomial, irreducible over the rationals $\mathscr{Q}$ and having a negative discriminant. Let δ be the real zero of $f(x)$ and denote by $\mathscr{K} = \mathscr{Q}(\delta)$ the complex cubic field formed by adjoining δ to $\mathscr{Q}$. If $\mathscr{E}_3$ denotes Euclidean 3-space, we can associate with each $\alpha \in \mathscr{K}$ a point $A \in \mathscr{E}_3$, where

$$A = \left(\alpha, (\alpha' - \alpha'')/2i, (\alpha' + \alpha'')/2\right),$$

$i^2 + 1 = 0$, and α', α'' are the conjugates of α. Since $f(x)$ has a negative discriminant, all three components of A must be real. If $\lambda, \mu, \nu \in \mathscr{K}$ and λ, μ, ν are rationally independent, we define the cubic lattice $\mathscr{L}$ by

$$\mathscr{L} = \left\{ u\Lambda + vM + wN \mid (u, v, w) \in \mathbf{Z}^3 \right\}.$$

We say that $\mathscr{L}$ has a basis $\{\lambda, \mu, \nu\}$ and denote $\mathscr{L}$ by $\langle \lambda, \mu, \nu \rangle$. For the sake of convenience we will often use the expression $\alpha \in \mathscr{L}$ to denote that it is the corresponding point $A \in \mathscr{E}_3$ that is actually in $\mathscr{L}$. Also, if $\mathscr{L} = \langle \lambda, \mu, \nu \rangle$, we define $\alpha\mathscr{L}$ $(\alpha \in \mathscr{K})$ to be the lattice $\langle \alpha\lambda, \alpha\mu, \alpha\nu \rangle$.

If A is any point of $\mathscr{L}$, we define the normed body of A to be

$$\mathscr{N}(A) = \mathscr{N}(\alpha)$$
$$= \{(x, y, z) | (x, y, z) \in \mathscr{E}_3, |x| < |\alpha|, y^2 + z^2 \leq |\alpha'|^2\}.$$

This is a semi-open right circular cylinder, symmetric about the origin O of $\mathscr{E}_3$, with axis the x-axis of $\mathscr{E}_3$. It should be mentioned at this point that if $\alpha, \beta \in \mathscr{K}$ and $|\alpha'| = |\beta'|$, then $\alpha = \pm\beta$ (see [1], p. 274). Thus, if $|\beta'| = |\alpha'|$, then $B \notin \mathscr{N}(\alpha)$.

We say that ϕ $(\neq 0) \in \mathscr{K}$ or the point Φ corresponding to ϕ is a minimum of $\mathscr{L}$ if $\mathscr{N}(\phi) \cap \mathscr{L} = \{0\}$. If ψ and ϕ are minima of $\mathscr{L}$ and $\psi > \phi$, we say that ψ and ϕ are *adjacent* minima when there does not exist a non-zero $\chi \in \mathscr{L}$ such that

$$\phi < \chi < \psi \quad \text{and} \quad |\chi'| < |\phi'|.$$

If

$$\theta_1, \theta_2, \theta_3, \ldots, \theta_n, \cdots \tag{1.1}$$

is a sequence of minima of $\mathscr{L}$ such that $\theta_{i+1} > \theta_i$ and θ_{i+1}, θ_i are adjacent $(i = 1, 2, 3, \ldots, n, \ldots)$, we call (1.1) a *chain* of minima of $\mathscr{L}$. By using Minkowski's theorem (see [1]) we can prove that such chains always exist in $\mathscr{L}$.

If $\mathscr{R} = \langle 1, \mu, \nu \rangle$ and 1 is a minimum of $\mathscr{R}$, we say that $\mathscr{R}$ is a *reduced* lattice. In this paper we shall be concerned with the problem of how closely spaced the minima of $\mathscr{R}$ can be. We will show that if $\theta_1 = 1$ and $\theta_4 < \theta_2 + 1$, then $\theta_2 + \theta_3 = \theta_4 + 1$. We can use this result to prove that if ε_0 is the fundamental unit of $\mathscr{K}$, then

$$\varepsilon_0 > \tau^{p/2},$$

where p is the period of Voronoi's continued fraction algorithm for finding ε_0 and $\tau = (1 + \sqrt{5})/2$. We will also show that $\theta_5 \geq \theta_3 + 1 > 2$ and $\theta_8 > 4$. The methods used to prove these results are completely elementary.

2. Preliminary results. From [1] or Williams and Dueck [3] we see that if $\mathscr{R}_1 = \mathscr{R}$ (a reduced lattice), $\theta_g^{(m)}$ is the minimum of $\mathscr{R}_m$ adjacent to 1 and $\mathscr{R}_{m+1}$ is defined to be $(1/\theta_g^{(m)})\mathscr{R}_m$, then $\theta_n \mathscr{R}_n = \mathscr{R}_1$, where $\mathscr{R}_n$ is a reduced lattice and

$$\theta_n = \prod_{i=1}^{n-1} \theta_g^{(i)}. \tag{2.1}$$

We shall need to make use of these results together with several others established in [3]; however, we first give some simple lemmas concerning points of $\mathscr{R}$. Throughout this work we will use θ to denote the minimum of $\mathscr{R}$ adjacent to 1, ω to denote the minimum of $\mathscr{R}$ adjacent to θ, and χ to denote the minimum of $\mathscr{R}$ adjacent to ω. That is, $\theta = \theta_2$, $\omega = \theta_3$, $\chi = \theta_4$. Note that if $\gamma \in \mathscr{R}$, $|\gamma| < \theta$, and $|\gamma'| \leq 1$, we must have $\gamma = 0$ or $\gamma = \pm 1$. We also have

LEMMA 2.1. *If* $\alpha \in \mathscr{R}$ *and* $0 < \alpha < \theta + 1$, *then either* $\alpha = 1, 2$ *or* $|\alpha' - 1| > 1$. *Further, if* $\alpha, \beta \in \mathscr{R}$, $\alpha \neq \beta$, *and* $\theta < \alpha$, $\beta < \theta + 1$, *then* $|\alpha' - \beta'| > 1$.

Proof. We have $-1 < \alpha - 1 < \theta$; thus, if $|\alpha' - 1| \leq 1$, we get $\alpha - 1 = 0, 1$. Since $\theta < \alpha$, $\beta < \theta + 1$, we have $|\alpha - \beta| < 1$. It follows that if $|\alpha' - \beta'| < 1$, then $\alpha = \beta$. If $|\alpha' - \beta'| = 1$, then $\alpha = \beta \pm 1$, which is also impossible. □

From this result we see that $|\theta' - 1| > 1$ and if $\chi < \theta + 1$, then $|\omega' - 1| > 1$ and $|\chi' - 1| > 1$.

In order to develop further results we define

$$(2.2) \qquad \eta_\alpha = (\alpha' - \alpha'')/2i, \qquad \zeta_\alpha = (\alpha' + \alpha'')/2$$

for any $\alpha \in \mathscr{K}$. Note that

$$(2.3) \qquad |\alpha'|^2 = |\alpha''|^2 = \alpha'\alpha'' = \eta_\alpha^2 + \zeta_\alpha^2.$$

Also, if $\alpha \in \mathscr{R}$ and $\eta_\alpha \in \mathscr{Q}$, then $\alpha \in \mathbf{Z}$ and $\eta_\alpha = 0$ (see [3]). Hence, $\eta_\alpha \neq 0$ if $\alpha = \theta_i$ $(i > 1)$.

LEMMA 2.2. *If* $\alpha, \beta \in \mathscr{R}$, $|\alpha'| < 1$, $|\beta'| < 1$, $|\alpha' - 1| > 1$, $|\beta' - \alpha' + 1| > 1$, *then* $|\beta' - \alpha' + 2| > 1$.

Proof. Since $|\beta'| < 1$, we have $\zeta_\beta > -1$ by (2.3). Further, since $|\alpha'| < 1$ and $|\alpha' - 1| > 1$, we must have $\zeta_\alpha < 1/2$; thus, $\zeta_\beta - \zeta_\alpha + 1 > -1/2$ and

$$|\beta' - \alpha' + 2|^2 = |\beta' - \alpha' + 1|^2 + 2(\zeta_\beta - \zeta_\alpha + 1) + 1 > 1. \qquad \square$$

LEMMA 2.3. *If* $\alpha, \beta \in \mathscr{R}$, $|\alpha' - 1| > 1$, $|\alpha' + 1| > 1$, $|\alpha'| < 1$, $|\beta'| < 1$, $\eta_\beta\eta_\alpha > 0$, *and* $|\beta' - \alpha'| > 1$, *then* $|\eta_\alpha| > |\eta_\beta|$.

Proof. Suppose $|\eta_\alpha| \leq |\eta_\beta|$ and consider Figure 1.

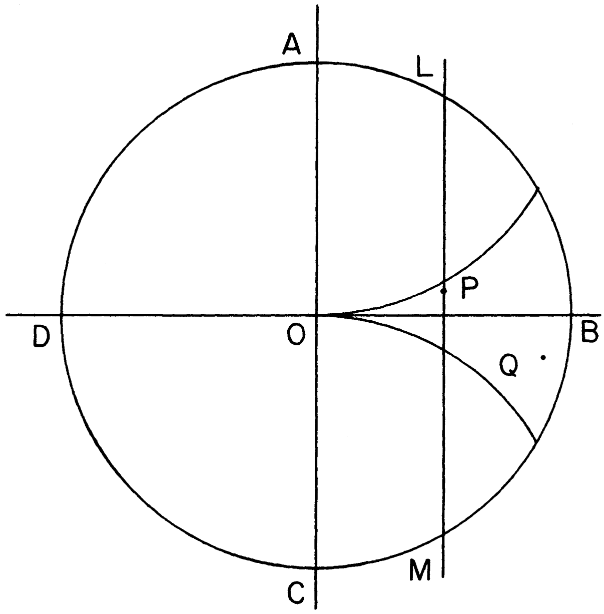

FIGURE 1

Here $P = (|\eta_\alpha|, \zeta_\alpha)$, $Q = (|\eta_\beta|, \zeta_\beta)$. Let the chord through P parallel to AC meet the circle $ABCD$ (radius 1, centre O) at L and M. Since $|\alpha' + 1| > 1$, we have $\overline{PL} < 1$; also, since $|\alpha' - 1| > 1$, we have $\overline{PM} < 1$. Since $\overline{PQ} < \max(\overline{PL}, \overline{PM})$, we get $\overline{PQ} = |\beta' - \alpha'| < 1$, a contradiction. □

In the next sequence of lemmas we prove a number of results concerning points $\alpha \in \mathscr{R}$ such that $|\alpha'| < 1$. We first define $\kappa(\alpha)$ for $\alpha \in \mathscr{R}$ by

$$\kappa(\alpha) = (\zeta_\alpha - 1/2)^2 + \left(\sqrt{3}/2 - |\eta_\alpha|\right)^2 = \zeta_\alpha^2 - \zeta_\alpha + \eta_\alpha^2 - \sqrt{3}|\eta_\alpha| + 1. \tag{2.4}$$

LEMMA 2.4. *If* $\alpha \in \mathscr{R}$, $|\alpha'| < 1$, *and* $\kappa(\alpha) \geq 1$, *then* $\zeta_\alpha \leq 0$, $|\eta_\alpha| \leq \sqrt{3}/2$, *and* $|\zeta_\alpha| \geq |\eta_\alpha|/\sqrt{3}$.

Proof. Since $|\alpha'| < 1$, we have $|\eta_\alpha| < 1$ and $|\zeta_\alpha| < 1$; thus,

$$-\sqrt{3}/2 < \sqrt{3}/2 - 1 < \sqrt{3}/2 - |\eta_\alpha| < \sqrt{3}/2,$$

and $(\zeta_\alpha - 1/2)^2 \geq 1/4$ by (2.4). If $0 < \zeta_\alpha < 1$, this latter result is not possible; hence, $\zeta_\alpha \leq 0$. If $|\eta_\alpha| > \sqrt{3}/2$, then $|\zeta_\alpha| < 1/2$ by (2.3) and the fact that $|\alpha'| < 1$; thus, by (2.4)

$$\kappa(\alpha) < -1/2 + \zeta_\alpha^2 + \eta_\alpha^2 - \zeta_\alpha < 1/2 - \zeta_\alpha < 1,$$

which is also not possible. Since $|\eta_\alpha| < \sqrt{3}/2$, we have $|\eta_\alpha| < 3(\sqrt{3} - 1/\sqrt{3})/4$ and

$$\left(|\eta_\alpha|/\sqrt{3} + 1/2\right)^2 + \left(\sqrt{3}/2 - |\eta_\alpha|\right)^2 \leq 1.$$

It follows that since $\kappa(\alpha) \geq 1$, we must have $|\zeta_\alpha| \geq |\eta_\alpha|/\sqrt{3}$ by (2.4). □

COROLLARY 2.4.1. *If* $\alpha \in \mathscr{R}$, $|\alpha'| < 1$, *and* $\kappa(\alpha) \geq 1$, *then* $|\alpha' + 1| \leq 1$.

Proof. By the lemma, $1 - |\eta_\alpha|/\sqrt{3} > 0$ and $0 < \zeta_\alpha + 1 \leq 1 - |\eta_\alpha|/\sqrt{3}$. Thus,

$$|\alpha' + 1|^2 = (\zeta_\alpha + 1)^2 + \eta_\alpha^2 \leq \left(1 - |\eta_\alpha|/\sqrt{3}\right)^2 + \eta_\alpha^2 \leq 1$$

as $|\eta_\alpha| < \sqrt{3}/2$. □

LEMMA 2.5. *If* $\alpha, \beta \in \mathscr{R}$, $|\alpha'| < 1$, $|\beta'| < 1$, $|\alpha' - 1| > 1$, $\eta_\alpha\eta_\beta > 0$, $\kappa(\alpha) < 1$, *and* $|\alpha' - \beta'| > 1$, *then* $\kappa(\beta) > 1$.

Proof. The point $(\eta_\alpha, \zeta_\alpha)$ must lie in the Reuleaux triangle (see [3]) with vertices O (the origin), $(\sigma\sqrt{3}/2, 1/2)$, $(\sigma\sqrt{3}/2, -1/2)$, where $\sigma = \operatorname{sgn}(\eta_\alpha)$. If $\kappa(\beta) \leq 1$, then $(\eta_\beta, \zeta_\beta)$ is in the same Reuleaux triangle as $(\eta_\alpha, \zeta_\alpha)$; hence, $|\alpha' - \beta'| \leq 1$, which is impossible. □

LEMMA 2.6. *If* $\alpha, \beta \in \mathscr{R}$, $|\alpha'| < 1$, $|\beta'| < 1$, $|\alpha' - 1| > 1$, $|\alpha' + 1| > 1$, *and* $\kappa(\beta) \geq 1$, *then* $|1 - \alpha' - \beta'| > 1$.

Proof. Since $|\alpha'| < 1$, $|\alpha' + 1| > 1$, and $|\alpha' - 1| > 1$, we have $|\zeta_\alpha| < 1/2$ and $1 - 2\zeta_\alpha > 0$. Since $|\alpha' - 1| > 1$ and $\kappa(\beta) \geq 1$, we also have

$$\begin{aligned}(2.5)\quad |1 - \alpha' - \beta'|^2 &\\ &= 1 + \zeta_\beta^2 - 2\zeta_\beta + \eta_\beta^2 + 2\zeta_\alpha\zeta_\beta + 2\eta_\alpha\eta_\beta + \zeta_\alpha^2 - 2\zeta_\alpha + \eta_\alpha^2 \\ &> 1 + \zeta_\beta(-1 + 2\zeta_\alpha) + 2\eta_\alpha\eta_\beta + \sqrt{3}|\eta_\beta|\end{aligned}$$

by (2.4) and the fact that $|\alpha' - 1| > 1$. By Lemma 2.4, we have $\zeta_\alpha \leq 0$; hence, if $\eta_\alpha\eta_\beta \geq 0$, we get $|1 - \alpha' - \beta'| > 1$. If $\eta_\alpha\eta_\beta < 0$, then from (2.5) and Lemma 2.4 we get

$$|1 - \alpha' - \beta'|^2 > 1 + |\eta_\beta|\left((1 - 2\zeta_\alpha)/\sqrt{3} + \sqrt{3} - 2|\eta_\alpha|\right).$$

Since $\zeta_\alpha < \sqrt{1 - \eta_\alpha^2}$, we have

$$(1 - 2\zeta_\alpha)/\sqrt{3} + \sqrt{3} - 2|\eta_\alpha| > \left(1 - 2\sqrt{1 - \eta_\alpha^2}\right)/\sqrt{3} + \sqrt{3} - 2|\eta_\alpha|.$$

But $2/\sqrt{3} > 1 > |\eta_\alpha|$ and $\sqrt{1-\eta_\alpha^2}/\sqrt{3} < 2/\sqrt{3} - |\eta_\alpha|$; hence,

$$\left(1 - 2\sqrt{1-\eta_\alpha^2}\right)/\sqrt{3} + \sqrt{3} - 2|\eta_\alpha| > 0. \qquad \square$$

COROLLARY 2.6.1. *If* $\alpha, \beta \in \mathscr{R}$, $|\alpha'| < 1$, $|\beta'| < 1$, $|\alpha' - 1| > 1$, $|\alpha' + 1| > 1$, $|\beta' - \alpha'| > 1$, *and* $|\beta' - \alpha' + 1| < 1$, *then* $\kappa(\gamma) < 1$, *where* $\gamma = \beta - \alpha + 1$.

Proof. We have $|\gamma'| < 1$; thus, if $\kappa(\gamma) \geq 1$, then $|1 - \alpha' - \gamma'| = |\beta'| > 1$, which is not so. $\square$

We will also require some lemmas whose proofs have already appeared in [3]. We will only give the statements of these results here; however, we mention that the proofs of these lemmas are elementary and require, for the most part, only results from simple plane geometry.

LEMMA 2.7 (*Lemma* 6.1 *of* [3]). *If* $\alpha, \beta \in \mathscr{R}$, $|\alpha'| < 1$, $|\beta'| < 1$, *and* $2\alpha = \beta + 1$, *then* $|\alpha' - 1| \leq 1$. $\square$

LEMMA 2.8 (*Lemma* 5.4 *of* [3]). *If* $\alpha, \beta \in \mathscr{R}$, $|\alpha'| < 1$, $|\beta'| < 1$, $|\alpha' - 1| > 1$, $|\beta' - 1| > 1$, $\eta_\alpha\eta_\beta > 0$, $|\alpha' - \beta'| > 1$, *and* $|\alpha' + 1| > 1$ (< 1), *then* $|\beta' + 1| < 1$ (> 1). $\square$

LEMMA 2.9 (*Lemma* 6.2 *of* [3]). *Let* $\alpha, \beta, \gamma \in \mathscr{R}$, *where* α, β, γ *are distinct*, $|\alpha'| < 1$, $|\beta'| < 1$, $|\gamma'| < 1$, *and* $|\alpha' - 1| > 1$, $|\beta' - 1| > 1$, $|\gamma' - 1| > 1$. *If* $\eta_\alpha\eta_\beta > 0$ *and* $\eta_\beta\eta_\alpha > 0$, *there cannot exist any b such that*

$$b \leq \alpha, \beta, \gamma < b + 1. \qquad \square$$

LEMMA 2.10 (*Lemma* 6.3 *of* [3]). *Let* $\alpha, \beta \in \mathscr{R}$ *such that* $|\alpha'| < 1$, $|\beta'| < 1$, $\beta > \alpha > 1$, *and* $|\beta'| < |\alpha'|$. *If* $\eta_\alpha\eta_\beta > 0$ *and* $|\alpha' + 1| \leq 1$, *then* $\beta \geq \alpha + 1$. $\square$

LEMMA 2.11 (*Lemma* 6.5 *of* [3]). *Let* $\alpha, \beta, \gamma \in \mathscr{R}$ *such that* $|\alpha'| < 1$, $|\beta'| < 1$, $|\gamma'| < 1$, $|\alpha' - 1| > 1$, $|\beta' - 1| > 1$, $|\gamma' - 1| > 1$, $|\beta' + 1| \leq 1$, $\eta_\alpha\eta_\beta < 0$, $\eta_\beta\eta_\gamma > 0$. *If* $|\beta' - \alpha'| > 1$ *and* $|\beta' - \gamma' + 1| > 1$, *then either* $|\alpha' - \beta'| \leq 1$ *or* $|\alpha' - \beta' + \gamma' - 1| \leq 1$. $\square$

3. The main results. We are now able to use the lemmas of §2 to prove our main results. We first prove an extension of Lemma 2.11.

THEOREM 3.1. *If* $\alpha, \beta, \gamma \in \mathscr{R}$, $|\alpha'| < 1$, $|\beta'| < 1$, $|\gamma'| < 1$, $|\alpha' - 1| > 1$, $|\beta' - 1| > 1$, $|\gamma' - 1| > 1$, $|\beta' + 1| \leq 1$, $\eta_\alpha\eta_\beta < 0$, $\eta_\beta\eta_\gamma > 0$, *and* $|\beta' - \gamma'| > 1$, *then either* $|\alpha' - \beta'| < 1$ *or* $|\alpha' - \gamma'| \leq 1$, *where* $\lambda = \beta - \gamma + 1$.

Proof. If $|\lambda'| > 1$, the result follows from Lemma 2.11. Note that since $\eta_\alpha \eta_\beta < 0$, we cannot have $|\alpha' - \beta'| = 1$, for this would imply that $\alpha = \beta \pm 1$ and $\eta_\alpha = \eta_\beta$. Similarly $|\alpha' - \gamma'| \neq 1$. If $|\lambda'| = 1$, then $\beta = \gamma$ or $\beta = \gamma - 2$. Since $|\beta' - \gamma'| > 0$ and $|\beta'| < 1$, $|\gamma'| < 1$, neither of these is possible.

If $|\lambda'| < 1$, we will consider two cases; however, we first notice that $|\gamma' + 1| > 1$ by Lemma 2.8 and $\eta_\gamma \eta_\lambda = \eta_\gamma(\eta_\beta - \eta_\gamma) = |\eta_\gamma|(|\eta_\beta| - |\eta_\gamma|) < 0$ by Lemma 2.3. Also $\kappa(\lambda) < 1$ by Corollary 2.6.1.

Case 1 ($\kappa(\alpha) < 1$). In this case we see from Lemma 2.5 that $|\alpha' - \lambda'| \leq 1$.

Case 2 ($\kappa(\alpha) \geq 1$). In this case we have $|\alpha' + 1| < 1$ by Corollary 2.4.1. Suppose $|\alpha' - \gamma'| > 1$ and $|\alpha' - \lambda'| > 1$. Since $|\beta' - \gamma'| > 1$, we have $|\lambda' - 1| > 1$; thus, $|\lambda' + 1| > 1$ by Lemma 2.8. If $\rho = \alpha - \lambda + 1$ and $|\rho'| > 1$, then either $|\gamma' - \rho'| \leq 1$ or $|\gamma' - \alpha'| \leq 1$ by Lemma 2.11. Since $\gamma - \rho = \beta - \alpha$, we get $|\beta' - \alpha'| < 1$. If $|\rho'| = 1$, then $\alpha = \lambda$ or $\alpha = \lambda - 2$ and, as above, neither of these is possible. If $|\rho'| < 1$, then $\kappa(\rho) < 1$ and also $\eta_\rho \eta_\gamma > 0$ (Corollary 2.6.1 and Lemma 2.3). Since $\kappa(\gamma) < 1$ by Corollary 2.4.1, we get $|\gamma' - \rho'| \leq 1$ by Lemma 2.5. □

We are now able to show that if $\theta_4 < \theta_2 + 1$, then $\theta_4 + 1 = \theta_2 + \theta_3$.

THEOREM 3.2. *If* $\chi < \theta + 1$, *then* $\eta_\theta \eta_\omega < 0$, $|\chi' + 1| < 1$, $\kappa(\theta) < 1$, $\kappa(\omega) < 1$, *and* $\chi + 1 = \theta + \omega$.

Proof. We first note that $|\theta'| < 1$, $|\omega'| < 1$, $|\chi'| < 1$, and $|\theta' - 1| > 1$, $|\omega' - 1| > 1$, $|\chi' - 1| > 1$. Also, if $\rho_1, \rho_2 \in \{\theta, \omega, \chi\}$ and $\rho_1 \neq \rho_2$, then $|\rho_1' - \rho_2'| > 1$ by Lemma 2.1.

Case 1 ($\eta_\theta \eta_\omega > 0$). By Lemma 2.4 we must have $\eta_\theta \eta_\chi < 0$. Further, by Lemma 2.10, we must also have $|\theta' + 1| > 1$. By Theorem 3.1, we get $|\rho'| \leq 1$, where $\rho = \chi - \omega + \theta - 1$. Now $0 < \rho < \theta$; thus, $\rho = 1$ and $\chi = \omega - \theta + 2$. Since $\omega - \theta + 1 = \chi - 1$, we have $|\omega' - \theta' + 1| > 1$; consequently, $|\chi'| = |\omega' - \theta' + 2| > 1$ by Lemma 2.2. It follows that we must have

Case 2 ($\eta_\theta \eta_\omega < 0$). Here we have $\eta_\chi \eta_\theta > 0$ or $\eta_\chi \eta_\omega > 0$. In either case, by Lemma 2.10 we get $|\chi' + 1| < 1$. If $\eta_\chi \eta_\omega > 0$, then $|\omega' + 1| > 1$ by Lemma 2.8 and $\kappa(\omega) < 1$ by Corollary 2.4.1. Also, by Theorem 3.1 $|\rho'| \leq 1$, where $\rho = \theta - \chi + \omega - 1$. Since $-1 < \rho < \theta$, we get $\rho = 0$ or 1.

As before, we cannot have $\rho = 1$; hence, $\rho = 0$ and $\chi + 1 = \theta + \omega$. Since $\theta = \chi - \omega + 1$, we get $\kappa(\theta) < 1$ from Corollary 2.6.1. Similarly, if $\eta_\chi \eta_\theta > 0$, then $\kappa(\theta) < 1$, $\kappa(\omega) < 1$, and $\chi + 1 = \theta + \omega$. □

By using the remarks at the beginning of §2, we can extend this result to show that if

$$\theta_{n+3} < \theta_{n+1} + \theta_n$$

in (1.1), then

$$\theta_{n+3} + \theta_n = \theta_{n+1} + \theta_{n+2}.$$

We can also improve two of the results of Theorem 3.2 in

LEMMA 3.3. *If* $\chi < \theta + 1$, *then* $|\theta' + 1| > 1$ *and* $|\omega' + 1| > 1$.

Proof. If $|\theta' + 1| \le 1$, then $\zeta_\theta \le 0$ and

$$-2\zeta_\theta \ge \zeta_\theta^2 + \eta_\theta^2 > \zeta_\omega^2 + \eta_\omega^2 > 2\zeta_\omega \qquad (|\omega' - 1| > 1).$$

It follows that $\zeta_\theta + \zeta_\omega < 0$ and, as a consequence, $|\chi'| = |\theta' + \omega' - 1| > 1$, which is impossible.

If $|\omega' + 1| \le 1$, then $\zeta_\omega \le 0$ and

$$|\eta_\omega| \le \sqrt{1 - (1 - |\zeta_\omega|)^2}. \tag{3.1}$$

Since

$$2|\zeta_\omega| = \zeta_\omega^2 + 1 - (1 - |\zeta_\omega|)^2, \tag{3.2}$$

we get

$$2|\zeta_\omega| \ge 1 - (1 - |\zeta_\omega|)^2 > |\eta_\omega|\left(1 - (1 - |\zeta_\omega|)^2\right). \tag{3.3}$$

Also, since

$$\left(|\eta_\theta|\sqrt{2|\zeta_\omega|} - \sqrt{1 - (1 - |\zeta_\omega|)^2}\right)^2 \ge 0,$$

we see, using (3.2), that

$$(1 - \eta_\theta^2)\zeta_\omega^2 \le \left(\sqrt{2|\zeta_\omega|} - |\eta_\theta|\sqrt{1 - (1 - |\zeta_\omega|)^2}\right)^2$$

and

$$|\zeta_\omega|\sqrt{1 - \eta_\theta^2} + |\eta_\theta|\sqrt{1 - (1 - |\zeta_\omega|)^2} \le \sqrt{2|\zeta_\omega|}$$

by (3.3). Now $\zeta_\theta < \sqrt{1 - \eta_\theta^2}$; hence from (3.1) we get

$$\zeta_\theta|\zeta_\omega| + |\eta_\omega|\,|\eta_\theta| - |\zeta_\omega| < \sqrt{2|\zeta_\omega|} - |\zeta_\omega| \le 1/2.$$

Since $\zeta_\omega \leq 0$ and $\eta_\omega \eta_\theta < 0$, we find that

$$-2\zeta_\omega + 2\zeta_\omega \zeta_\theta + 2\eta_\omega \eta_\theta > -1.$$

But

$$|\chi'|^2 - |\omega'|^2 = |\theta' + \omega' - 1|^2 - |\omega'|^2$$
$$= |\theta' - 1|^2 - 2\zeta_\omega + 2\zeta_\omega \zeta_\theta + 2\eta_\theta \eta_\omega;$$

thus, since $|\theta' - 1| > 1$, we have $|\chi'| > |\omega'|$ when $|\omega' + 1| \leq 1$ and this is impossible. □

We will also need to make use of the following result and its corollaries.

THEOREM 3.4. *If* $\chi < \theta + 1$ *and there exists some* $\rho \in \mathscr{R}$ *such that* $\rho \notin \{\theta, \omega, \chi\}$, $|\rho'| < 1$, $|\rho' - 1| > 1$, *then* $|\rho' - \psi'| < 1$ *for some* $\psi \in \{\theta, \omega, \chi\}$.

Proof. Suppose that there exists some $\rho \in \mathscr{R}$ such that $\rho \notin \{\theta, \omega, \chi\}$, $|\rho'| < 1$, $|\rho' - 1| > 1$, and $|\rho' - \psi'| \geq 1$ for each $\psi \in \{\theta, \omega, \chi\}$. We first note that if $|\rho' - \psi'| = 1$, then $\rho = \psi + 1$. If $\rho = \psi - 1$, then $0 < \rho < \theta$, which contradicts the definition of θ. If $\rho = \psi + 1$, then $|\rho' - 1| = |\psi'| < 1$, which is also impossible. Thus, $|\rho' - \psi'| > 1$ for all $\psi \in \{\theta, \omega, \chi\}$. Since $\eta_\theta \eta_\omega < 0$, $|\theta' + 1| > 1$, $|\omega' + 1| > 1$, we must have $|\rho' + 1| < 1$ (Lemma 2.8). Put α equal to that one of θ or ω such that $\eta_\alpha \eta_\rho < 0$ and let β be the other one. We have $\alpha + \beta = \theta + \omega = \chi + 1$. Further, $|\rho' - \alpha'| > 1$ and $|\alpha' + 1| > 1$; thus, by Theorem 3.1, we get $|\beta' - \lambda'| \leq 1$, where $\lambda = \rho - \alpha + 1$. Since $\beta - \lambda = \beta - \rho + \alpha - 1 = \chi - \rho$, this is impossible. □

COROLLARY 3.4.1. *If* $\chi < \theta + 1$ *and there exists* $\rho \in \mathscr{R}$ *such that* $\rho \in \{\theta, \omega, \chi\}$, $|\rho'| < 1$, *and* $|\rho| < \theta + 1$, *then* $\rho = 0$.

Proof. Since $|-\rho'| = |\rho'|$, we may assume with no loss of generality that if $\rho \neq 0$, then $\rho > 0$. Since $|\rho'| < 1$, we must have $\theta < \rho < \theta + 1$. Thus, by Lemma 2.1, $|\rho' - \psi'| \geq 1$ for all $\psi \in \{\theta, \omega, \chi\}$, which is impossible by the theorem. □

COROLLARY 3.4.2. *If* $\chi < \theta + 1$, *there does not exist any* $\rho \in \mathscr{R}$ *such that* $|\rho'| < 1$ *and* $\chi < \rho < \chi + 1$.

Proof. Suppose such a ρ does exist. If $|\rho' - 1| < 1$, then, since $|\rho - 1| < \theta + 1$, we can only have $\rho - 1 \in \{\theta, \omega, \chi\}$ by the previous result. Since $\rho \neq \chi + 1$, $|\theta' + 1| > 1$, $|\omega' + 1| > 1$, we must have $|\rho' - 1| > 1$ and, as a consequence, $|\rho' - \psi'| < 1$ for some $\psi \in \{\theta, \omega, \chi\}$. Since $0 < \rho - \chi < \chi + 1 - \chi \leq \omega$, we find by the previous corollary that $\rho - \psi = \theta$. If $\psi = \omega$ or χ, then $\rho \geq \chi + 1$; thus, $\psi = \theta$ and $\rho = 2\theta$. If $\rho = 2\theta$, then $|\omega'| < |\theta'| < 1/2$ and $|\omega' - \theta'| < 1$, which is impossible. □

Let $\rho = \theta_5$, the minimum adjacent to $\chi = \theta_4$. We can now show the following unconditional result concerning ρ.

THEOREM 3.5. *$\rho \geq 1 + \omega$ or $\theta_{n+5} \geq \theta_{n+3} + \theta_n$ in* (1.1).

Proof. Suppose $\rho < 1 + \omega$ and let $\mathscr{R}^* = (1/\theta)\mathscr{R}$. If $\theta^* = \omega/\theta$, $\omega^* = \chi/\theta$, $\chi^* = \rho/\theta$, then θ^* is the minimum adjacent to 1 in $\mathscr{R}^*$, ω^* is the minimum adjacent to θ^*, and χ^* is the minimum adjacent to ω^*. Since $\rho < 1 + \omega$, we have $\chi^* < (1 + \omega)/\theta < \omega/\theta + 1 = \theta^* + 1$. By Theorem 3.2, we have $\theta^* + \omega^* = \chi^* + 1$ and

$$\omega + \chi = \rho + \theta.$$

If $\chi \geq \theta + 1$, then $\rho \geq \omega + 1$. If $\chi < \theta + 1$, then $\rho \geq \chi + 1 > \omega + 1$ by Corollary 3.4.2. □

In fact, we actually get cases in which $\rho = 1 + \omega$. For example, consider $D = 239$, $\delta^3 = D$, $\mathscr{R}_1 = \langle 1, \delta, \delta^2 \rangle$. In $\mathscr{R} = \mathscr{R}_{312}$, we get

$$\begin{aligned} \theta &= (6 + 17\delta + 7\delta^2)/247, \\ \omega &= (74 + 45\delta + 4\delta^2)/247, \\ \chi &= (253 + 17\delta + 7\delta^2)/247 = \theta + 1, \\ \rho &= (321 + 45\delta + 4\delta^2)/247 = \omega + 1. \end{aligned}$$

Note also that if $\mathscr{R} = \mathscr{R}_{313}$ here, we have $\theta = (191 - 3\delta + 7\delta^2)/332$, $\omega = (217 + 47\delta + \delta^2)/332$, $\chi = (76 + 44\delta + 8\delta^2)/332$. In this case $\chi < \theta + 1$ and $\chi = \theta + \omega - 1$. Also, $\rho = (408 + 44\delta + 8\delta^2)/332 = \chi + 1$.

If we let $\mathscr{R}_1 = \langle 1, \mu, \nu \rangle$, where $\{1, \mu, \nu\}$ is a basis of the algebraic integers of $\mathscr{K}$, then $\mathscr{R}_1$ is a reduced lattice and there exists an integer p such that $\mathscr{R}_{p+1} = \mathscr{R}_1$. In this case ε_0 (> 1), the fundamental unit of $\mathscr{K}$, is given by the formula

$$\varepsilon_0 = \theta_{p+1} = \prod_{i=1}^{p} \theta_g^{(i)}. \tag{3.4}$$

The value p is called the period of Voronoi's continued fraction algorithm for finding ε_0. By using the reasoning similar to that of Pen and Skubenko [2], we can prove

THEOREM 3.6. *If p is the period of Voronoi's continued fraction algorithm for finding ε_0, then $\varepsilon_0 > \tau^{p/2}$, where $\tau = (1 + \sqrt{5})/2$.*

Proof. If $\mathscr{R} = \mathscr{R}_i$, then $\rho \geq \omega + 1$ and

$$\theta_g^{(i)}\theta_g^{(i+1)}\theta_g^{(i+2)}\theta_g^{(i+3)} \geq 1 + \theta_g^{(i)}\theta_g^{(i+1)}.$$

Since $\mathscr{R}_{p+1} = \mathscr{R}_1$, $\mathscr{R}_{p+2} = \mathscr{R}_2$, $\mathscr{R}_{p+3} = \mathscr{R}_3$, we get $\theta_g^{(p+1)} = \theta_g^{(1)}$, $\theta_g^{(p+2)} = \theta_g^{(2)}$, $\theta_g^{(p+3)} = \theta_g^{(3)}$; thus, we get

$$\begin{aligned}\varepsilon_0^4 = \left(\prod_{i=1}^{p} \theta_g^{(i)}\right)^4 &= \prod_{i=1}^{p} \theta_g^{(i)}\theta_g^{(i+1)}\theta_g^{(i+2)}\theta_g^{(i+3)} \\ &\geq \prod_{i=1}^{p}\left(1 + \theta_g^{(i)}\theta_g^{(i+1)}\right) \\ &\geq \prod_{i=1}^{p}\left(1 + \left(\prod_{i=1}^{p} \theta_g^{(i)}\theta_g^{(i+1)}\right)^{1/p}\right)^p \\ &= \left(1 + \varepsilon_0^{2/p}\right)^p.\end{aligned}$$

If we put $\eta = \varepsilon_0^{2/p} > 1$, then $\eta^2 \geq \eta + 1$. It follows that $\varepsilon_0^{2/p} > \tau$. □

Thus, if R is the regulator of $\mathscr{K}$, we have $R > p(\log \tau)/2$.

4. Further results. In this section we will obtain some results on the spacing of the first few minima of $\mathscr{R}$. We first require the following technical lemma.

LEMMA 4.1. *If $\chi < \theta + 1$, then*
(i) $|\theta'|, |\omega'| > 1/2$;
(ii) $|2\omega' + \chi'| > |\omega'|$, $|2\theta' + \chi'| > |\theta'|$, $|2\theta' + \omega'| > |\theta'|$;
(iii) $|\theta' + \chi'| > |\chi'|$;
(iv) $|2\chi' + \theta'| > |\chi'|$.

Proof. (i) The method of proof of (i) is given in the proof of Corollary 3.4.2.

(ii) Since $|\omega'| > |\chi'|$, we have

$$|2\omega' + \chi'| \geq 2|\omega'| - |\chi'| > |\omega'|.$$

Similarly, $|2\theta' + \chi'| > |\theta'|$ and $|2\theta' + \omega'| > |\theta'|$.

(iii) We note that

$$(4.1)\quad 2\zeta_\chi\zeta_\theta + 2\eta_\chi\eta_\theta = |\chi' + 1|^2 - |\chi' + 1 - \theta'|^2 + |\theta' - 1|^2 - 1.$$

Since $\omega = \chi + 1 - \theta$, we get

$$|\theta' + \chi'|^2 = |\theta'|^2 + |\chi'|^2 + |\chi' + 1|^2 - |\omega'|^2 + |\theta' - 1|^2 - 1.$$

Since $|\theta'| > |\omega'|$ and $|\theta' - 1| > 1$, we have

$$|\theta' + \chi'| > |\chi'|.$$

(iv) From (4.1) we get

$$|2\chi' + \theta'|^2 - |\chi'|^2 = |\chi'|^2 + 2|\chi'|^2 + 2|\chi' + 1|^2 - |\omega'|^2$$
$$+|\theta'|^2 - |\omega'|^2 + 2|\theta' - 1|^2 - 2.$$

Since

$$|\chi'|^2 + |\chi' + 1|^2 \geq \zeta_\chi^2 + (\zeta_\chi + 1)^2$$
$$= \frac{1}{2}(4\zeta_\chi^2 + 4\zeta_\chi + 1) + \frac{1}{2} \geq \frac{1}{2}$$

we get

$$|2\chi' + \theta'| - |\chi'| > 0. \qquad \square$$

We are now able to find possible candidates for further minima when $\chi < \theta + 1$.

LEMMA 4.2. *If* $\chi < \theta + 1$, $\chi + 1 < \rho < \chi + 2$, *and* $|\rho'| < 1$, *then* $\rho \in \{\chi + \theta, \chi + \omega, 2\chi\}$.

Proof. Since $\chi < \rho - 1 < \chi + 1$, we cannot have $|\rho' - 1| \leq 1$, by Corollary 3.4.2. Since $|\rho' - 1| > 1$, by Theorem 3.4, we must have some $\psi \in \{\theta, \omega, \chi\}$ such that $|\rho' - \chi'| < 1$. If $\psi = \theta$, then

$$\omega = \chi + 1 - \theta < \rho - \psi < \chi + 2 - \theta = \omega + 1 < \chi + 1;$$

hence, $\rho - \theta = \chi$ by Corollary 3.4.1 and 3.4.2. If $\psi = \omega$, then $\theta < \rho - \chi < \theta + 1$. By Corollary 3.4.1, we can only have $\rho = 2\omega$, which is impossible by Lemma 4.1, or $\rho = \omega + \chi$. If $\psi = \chi$, then $1 < \rho - \psi < 1 + \theta$ and $\rho - \chi \in \{\theta, \omega, \chi\}$. $\square$

COROLLARY 4.2.1. *If ρ satisfies the conditions of the lemma and ρ is also a minimum of $\mathscr{R}$, then $\rho = \chi + \omega$.*

Proof. If $\rho = 2\chi$ or $\rho = \theta + \chi$, then $|\rho'| > |\chi'|$, which is not possible. $\square$

LEMMA 4.3. *If* $\chi < \theta + 1$, $\chi + 2 < \rho < \chi + 3$, *and* $|\rho'| < 1$, *then*

$$\rho \in \{\theta + \chi, \omega + \chi, 2\chi, \chi + \theta + 1, \chi + \omega + 1, \chi + 2\theta, \chi + 2\omega, 2\chi + 1, 2\chi + \theta, 2\chi + \omega, 3\chi\}.$$

Proof. Since $\chi + 1 < \rho - 1 < \chi + 2$, we see by Lemma 4.2 that if $|\rho' - 1| < 1$, then $\rho = \chi + \theta + 1$, $\chi + \omega + 1$, $2\chi + 1$. If $|\rho' - 1| \geq 1$, then $|\rho' - \psi'| < 1$ for some $\psi \in \{\theta, \omega, \chi\}$. If $\psi = \theta$, then

$$\chi < \omega + 1 < \chi + 2 - \theta < \rho - \psi < \chi + 3 - \theta = \omega + 2 < \chi + 2.$$

Thus, $\rho - \theta \in \{\chi + 1, \chi + \theta, \chi + \omega, 2\chi\}$. (Note that $\theta + \omega + \chi = 2\chi + 1$.) If $\psi = \omega$, then $\chi < \rho - \chi < \chi + 2$ and $\rho - \omega \in \{\chi + 1, \chi + \theta, \chi + \omega, 2\chi\}$. If $\psi = \chi$, then $2 < \rho < \chi + 2$ and $\rho - \chi \in \{\theta, \chi, \omega, \chi + 1, \chi + \theta, \chi + \omega, 2\chi\}$. □

COROLLARY 4.3.1. *If* ρ *satisfies the conditions of the lemma and* ρ *is a minimum of* $\mathscr{R}$, *then*

$$\rho \in \{\omega + \chi, \omega + \chi + 1, 2\chi + 1, 2\chi + \omega\}.$$

Proof. We have $2|\chi'|, 3|\chi'| > |\chi'|$; the other possibilities are ruled out by Lemma 4.1. □

THEOREM 4.4. *If* $\chi < \theta + 1$, *there does not exist a set of minima* $\{\mu_1, \mu_2, \mu_3, \mu_4\}$ *of* $\mathscr{R}$ *such that*

$$\chi + 1 \leq \mu_1 < \mu_2 < \mu_3 < \mu_4 < \chi + 3.$$

Proof. Put $\mathscr{R}^* = (1/\mu_1)\mathscr{R}$, $\theta^* = \mu_2/\mu_1$, $\omega^* = \mu_3/\mu_1$, $\chi^* = \mu_4/\mu_1$. Since $\chi^* < (\chi + 3)/(\chi + 1) < 1 + \theta^*$, we must have

$$\mu_4 + \mu_1 = \mu_2 + \mu_4 \qquad \text{(Theorem 3.2)}, \tag{4.2}$$

and $\mu_1, \mu_2, \mu_3, \mu_4 \in \{\chi + 1, \chi + \omega, \chi + \omega + 1, 2\chi + 1, 2\chi + \omega\}$ by Corollaries 4.2.1 and 4.3.1. If $\mu_1 \neq \chi + 1$, then (4.2) cannot hold. If $\mu_1 = \chi + 1$ and $\mu_2 \neq \chi + \omega + 1$, then (4.2) again cannot hold. Thus, we must have $\omega_1 = \chi + 1$ and $\mu_2 = \chi + \omega + 1$. It follows that $\mu_2 - \mu_1 = \omega - 1$ and we can only have $\mu_3 = 2\chi + 1$, $\mu_4 = 2\chi + \omega$.

Since $\chi + 1$ is a minimum, we have $|\chi' + 1| < |\chi'|$, and therefore $\zeta_\chi < -1/2$. Since $\zeta_\omega < 1/2$, we get $2\zeta_\chi + \zeta_\omega < -1/2$ and $|2\chi' + \omega' + 1| < |\omega' + \chi'|$. Thus, if μ_5 is the minimum adjacent to $\mu_4 = 2\chi + \omega$, then $\mu_5 \leq 2\chi + \omega + 1$. Since $\rho^* = \mu_5/\mu_1$, the minimum adjacent to χ^* in $\mathscr{R}^*$, must satisfy $\rho^* \geq \chi^* + 1$, we get $\mu_5 \geq \mu_4 + \mu_1 = 3\chi + \omega + 1 > 2\chi + \omega + 1$, a contradiction. □

COROLLARY 4.4.1. *If* $\theta_1 = 1$ *in* (2.1), *then* $\theta_8 > 4$.

Proof. If $\theta_4 \geq \theta_1 + 1$, put $\mathscr{R}^* = (1/\theta_4)\mathscr{R}$, $\theta^* = \theta_5/\theta_4$, $\omega^* = \theta_6/\theta_4$, $\chi^* = \theta_7/\theta_4$, $\rho^* = \theta_8/\theta_4$. By Theorem 3.5, we have $\rho^* \geq \omega^* + 1$; hence, $\theta_8 = \theta_4\rho^* \geq (\theta_1 + 1)(\omega^* + 1) > 4$. If $\theta_4 < \theta_1 + 1$, then $\theta_8 > \theta_5 + 3 > 4$ by the theorem. □

It follows from Corollary 4.4.1 that in $\mathscr{R}_i$ we have

$$\prod_{j=0}^{6} \theta_g^{(i+j)} > 4;$$

hence, from (2.1), we get

$$\theta_n > 4^{[(n-1)/7]}.$$

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UNIVERSITY OF MANITOBA
WINNIPEG, MANITOBA
CANADA R3T 2N2