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RANGE OF GATEAUX DIFFERENTIABLE OPERATORS AND LOCAL EXPANSIONS

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Let X and Y be Banach spaces, and P: $X \to Y$ a Gateaux differentiable operator having closed graph. Suppose that there is a continuous function c: $[0, \infty) \to (0, \infty)$ satisfying

$$dP_x(\bar{B}(0;1)) \supseteq \bar{B}(0; c(\|x\|)).$$

Then it is shown that for any $K > 0$ (possibly $K = \infty$), $P(B(0; K))$ contains $B(P(0); \int_0^K c(s)\,ds)$. Similar results are obtained for local expansions and locally strongly ϕ-accretive operators. These results extend a number of known theorems by giving the precise geometric estimations for normal solvability of $Px = y$.

1. Introduction. Let P be a nonlinear operator from a Banach space X into a Banach space Y. Many authors (see [**3**], [**4**], [**6**], [**7**], [**10**], [**12**], [**13**], [**14**] and [**15**]) have studied solvability of the equation $Px = y$, for $y \in Y$, a considerable number of which involve local or infinitesimal assumptions on the operator P, by showing that P is surjective. However, in many cases, in general P need not be surjective, although for some $y \in Y$, the equation $Px = y$ is solvable. For example, let P be a Gateaux differentiable operator having closed graph such that for each $x \in X$,

$$dP_x(\bar{B}(0;1)) \supseteq \bar{B}(0; c(\|x\|))$$

where c: $[0, \infty) \to (0, \infty)$ is a continuous function. In [**13**], Ray and Walker showed that P is surjective, where c is nonincreasing and $\int_0^\infty c(s)\,ds = \infty$. However, although c is not nonincreasing and $\int_0^\infty c(s)\,ds < \infty$, intuitively we may expect that for any $K > 0$ (possibly $K = \infty$), $P(B(0; K))$ contains $B(P(0); \int_0^K c(s)\,ds)$ by considering an elementary integral equation, so that for any $y \in B(P(0); \int_0^K c(s)\,ds) \subseteq Y$, $Px = y$ has a solution x in $B(0; K) \subseteq X$.

In this paper, we show that the fact mentioned above holds, and such an idea can be applied to local expansions and locally strongly ϕ-accretive operators similarly. For this purpose, in §2, we give a fixed point theorem which is a basic tool in proving theorems in §3. And in §3, we apply this result to nonlinear operators.

2. A fixed point theorem. In this section we give a fixed point theorem which is a basic tool in proving theorems in the next section. Actually our theorem is based on the following well-known Caristi-Kirk-Browder fixed point theorem [**5**], which is an equivalent formulation of Ekeland's minimization theorem [**8**, **9**].

THEOREM 2.1. *Let (M, d) be a complete metric space and ϕ be a lower semicontinuous (l.s.c.) function from M to $R \cup \{\infty\}$, $\not\equiv \infty$, bounded from below. Let g be a selfmap of M satisfying,*

$$d(x, g(x)) + \phi(g(x)) \leq \phi(x) \tag{2.1}$$

for all $x \in M$. Then g has a fixed point in M.

THEOREM 2.2. *Let (M, d) be a complete metric space, and ψ be a l.s.c. function from M into $[0, \infty)$. Let c be a continuous nonincreasing function from $[0, \infty)$ into $(0, \infty)$, and let $x_0 \in M$ be fixed. Further suppose that there exist $z \in M$ and $K > 0$ (possibly $K = \infty$) satisfying $\int_{d(x_0, z)}^{K} c(s)\, ds \geq \psi(z)$ (when $K = \infty$, $\int_{d(x_0, z)}^{\infty} c(s)\, ds > \psi(z)$). If g is a selfmap of M satisfying*

$$c(d(x_0, x)) d(x, g(x)) \leq \psi(x) - \psi(g(x)) \tag{2.2}$$

whenever $x \in M$ with $\int_{d(x_0, x)}^{K} c(s)\, ds \geq \psi(x)$, then g has a fixed point in M.

If $\int_0^\infty c(s)\, ds = \infty$, then Theorem 2.2 is the same as Theorem 2.1 of [**13**], which is actually equivalent to Ekeland's theorem [**8**, **9**] (see [**11**]). Theorem 2.2 is a slightly extended version of Theorem 2.1, but they are actually equivalent in logic. The advantage of Theorem 2.2 is that we need not examine the inequality (2.2) for all $x \in M$, that is, if for suitable $x \in M$ (2.2) holds, we have the desired conclusion. In fact, in Theorem 2.1, by putting $A = \{x \in M;\ d(x, z) \leq \phi(z) - \phi(x)\}$ for some $z \in M$ with $\phi(z) < \infty$, we have $g(A) \subseteq A$ and g has a fixed point in A. Also this fact gives the basic idea of the proof of Theorem 2.2.

Proof of Theorem 2.2. Now we construct a new function ϕ: $M \to [0, \infty]$, which is $\not\equiv \infty$, l.s.c. and satisfies (2.1), so that by applying Theorem 2.1, g has a fixed point in M. If $K = \infty$ and $\int_0^\infty c(s)\, ds = \infty$, then Park and Bae [**11**] showed that the equality

$$\int_{d(x_0, x)}^{d(x_0, x) + \phi(x)} c(s)\, ds = \psi(x)$$

gives ϕ which is a desired one. Therefore we may assume that $K < \infty$ (if $K = \infty$ and $\int_0^\infty c(s)\, ds < \infty$, then the similar method well do). Now define ϕ as follows: if $\int_{d(x_0,x)}^{K} c(s)\,ds \geq \psi(x)$, then put $\phi(x)$ satisfying $\int_{d(x_0,x)}^{d(x_0,x)+\phi(x)} c(s)\,ds = \psi(x)$, and if $\int_{d(x_0,x)}^{K} c(s)\,ds < \psi(x)$, then put $\phi(x) = \infty$.

To show that ϕ is l.s.c., let $x_n \to x$ and $\underline{\lim}\,\phi(x_n) = t$. If $t = \infty$, then there is nothing to prove, so that we may assume that $t < \infty$. Now we can choose a subsequence $\{x_{n_k}\}$ such that $\lim \phi(x_{n_k}) = t$. Then since $\lim d(x_n, x_0) = d(x_0, x)$, we have

$$\int_{d(x_0,x)}^{d(x_0,x)+t} c(s)\,ds = \lim \int_{d(x_0,x)}^{d(x_0,x)+\phi(x_{n_k})} c(s)\,ds$$
$$= \lim \psi(x_{n_k}) \geq \psi(x),$$

and $d(x_0, x) + t = \lim(d(x_0, x) + \phi(x_{n_k})) \leq K$. Therefore $\phi(x) \leq t$, and consequently ϕ is l.s.c.

To prove that ϕ satisfies (2.1), it suffices to prove that

$$d(x, y) + \phi(y) \leq \phi(x)$$

whenever $\int_{d(x_0,x)}^{K} c(s)\,ds \geq \psi(x)$ and

$$c(d(x_0, x))d(x, y) \leq \psi(x) - \psi(y),$$

since if $\phi(x) = \infty$, then (2.1) is trivially true. Suppose that the latter case holds. Since c is nonincreasing,

$$\int_{d(x_0,x)}^{d(x_0,x)+d(x,y)} c(s)\,ds \leq c(d(x_0, x))d(x, y).$$

Therefore by assuming $\phi(y) < \infty$, we have

$$\int_{d(x_0,x)}^{d(x_0,x)+d(x,y)} c(s)\,ds \leq \int_{d(x_0,x)}^{d(x_0,x)+\phi(x)} c(s)\,ds - \int_{d(x_0,y)}^{d(x_0,y)+\phi(y)} c(s)\,ds.$$

Since $d(x_0, y) \leq d(x_0, x) + d(x, y)$ and c is nonincreasing,

$$\int_{d(x_0,x)}^{d(x_0,x)+d(x,y)} c(s)\,ds + \int_{d(x_0,x)+d(x,y)}^{d(x_0,x)+d(x,y)+\phi(y)} c(s)\,ds \leq \int_{d(x_0,x)}^{d(x_0,x)+\phi(x)} c(s)\,ds,$$

which shows that $d(x, y) + \phi(y) \leq \phi(x)$. In the above case actually $\phi(y) < \infty$. To see this, suppose that $\phi(y) = \infty$. Then we can find $\varepsilon > 0$ such that $\int_{d(x_0,y)}^{K+\varepsilon} c(s)\,ds \leq \psi(y)$, and hence the above inequalities give

$$\int_{d(x_0,x)}^{K+\varepsilon} c(s)\,ds \leq \int_{d(x_0,x)}^{d(x_0,x)+\phi(x)} c(s)\,ds,$$

which is a contradiction to the fact that $\phi(x) < \infty$.

As a direct consequence of Theorem 2.2, we have the following Corollary by putting c is constant.

COROLLARY 2.3. *Let (M, d) be a complete metric space, ϕ a l.s.c. function from M into $[0, \infty)$ and let $x_0 \in M$ be fixed. Suppose that there exist $z \in M$, $K > 0$ and $c > 0$ such that $c(K - d(x_0, z)) = \phi(z)$. If g is a selfmap of M satisfying*

$$cd(x, g(x)) \leq \phi(x) - \phi(g(x))$$

for all $x \in M$ with $c(K - d(x_0, x)) \geq \phi(x)$, then g has a fixed point in M.

Note that when $z = x_0$, Corollary 2.3 is the same as Theorem 2.1 by considering the set $A = \{x \in M;\ cd(x, z) \leq \phi(z) - \phi(x)\}$, and $g(A) \subseteq A$.

3. Range of operators. In this section we apply Theorem 2.2 to Gateaux differentiable operators, local expansions and locally strongly ϕ-accretive operators. We begin with the Gateaux differentiable operators.

Let X and Y be Banach spaces, and P a mapping from an open subset D of X to Y. We say that P is Gateaux differentiable if, for each $x \in D$, there is a function $dP_x \colon X \to Y$ satisfying

$$\lim_{t \to 0^+} \frac{P(x + ty) - P(x)}{t} = dP_x(y), \qquad y \in X.$$

Easy examples show that Gateaux differentiable operators need not be continuous. Note that we do not require that dP_x is linear. However, it follows from the definition that dP_x is homogeneous, that is, $dP_x(ty) = t\, dP_x(y)$ for all $t \geq 0$.

We say that an operator $P\colon D \to Y$ has closed graph if $\{x_n\} \subseteq D$ with $x_n \to x \in D$ and $Px_n \to y$ as $n \to \infty$, it follows that $Px = y$. We denote by $B(w; r)$ the set $\{y;\ \|y - w\| < r\}$, and $\overline{B}(w; r)$ its closure. Also conveniently we set $B(w; \infty) = X$ (if $w \in X$).

Now we state our first theorem. The techniques used here are analogous to those of Ray and Walker [**13**].

THEOREM 3.1. *Let X and Y be Banach spaces, and P a Gateaux differentiable mapping from $B(0; K) \subseteq X$ to Y having closed graph, where $K > 0$. Let $c\colon [0, K) \to (0, \infty)$ be a continuous nonincreasing function for which, for each $x \in B(0; K)$,*

$$dP_x\big(\overline{B}(0; 1)\big) \supseteq \overline{B}\big(0; c(\|x\|)\big). \tag{3.1}$$

Then $P(B(0; K))$ contains $B(P(0); \int_0^K c(s)\, ds)$.

We remark that Theorem 3.1 shows that actually P is an open mapping, therefore it gives Theorem 3.2 of Cramer and Ray **[6]** and Theorem 2.2 of Ray **[12]**. But in order to prove these they used the maximal principle of Brezis and Browder **[2]**, however our basic tool is Theorem 2.2, which is an equivalent formulation of Ekeland **[8, 9]**. Also Theorem 3.1 can be compared with Theorem 3.1 of Ray and Walker **[13]** and Theorem 2.4 of **[12]**, which treat only the case $K = \infty$ and $\int_0^\infty c(s)\,ds = \infty$; in this case Theorem 3.1 is that of **[13]**, which extends Theorem 4 of **[15]**. Moreover, in Theorem 3.4 we will show that the function c need not be nonincreasing. The advantage of our formulation here is that our results contain the range of operators explicitly and we do not assume that the domain of P is the whole space X.

Proof of Theorem 3.1. Let $w \in B(P(0); \int_0^K c(s)\,ds)$, that is, $\|w - P(0)\| < \int_0^K c(s)\,ds$. We can choose $0 < q < 1$ satisfying

$$(1 - q)^{-1}\|w - P(0)\| < \int_0^K c(s)\,ds. \tag{3.2}$$

Also we can take a sufficiently small $\varepsilon > 0$ satisfying

$$(1 - q)^{-1}\|w - P(0)\| \leq \int_0^{K-2\varepsilon} c(s)\,ds.$$

Define a new metric ρ on the set $M = \overline{B}(0; K - \varepsilon)$ by

$$\rho(x, y) = \max\left\{\|x - y\|,\ c(0)^{-1}(1 + q)^{-1}\|P(x) - P(y)\|\right\}.$$

Since P has closed graph, (M, ρ) is a complete metric space. Set $\psi(x) = (1 - q)^{-1}\|Px - w\|$, so that $\psi\colon (M, \rho) \to [0, \infty)$ is continuous and $\psi(0) \leq \int_0^{K-2\varepsilon} c(s)\,ds$.

Now we claim that $w \in P(\overline{B}(0; K - 2\varepsilon))$. We proceed by contradiction and suppose that $w \notin P(\overline{B}(0; K - 2\varepsilon))$. For any $x \in M \setminus \overline{B}(0; K - 2\varepsilon)$, we have $\int_{\rho(0,x)}^{K-2\varepsilon} c(s)\,ds < 0 \leq \psi(x)$, since $\rho(0, x) \geq \|x\| > K - 2\varepsilon$. In this case set $g(x) = 0$ $(\neq x)$. For $x \in \overline{B}(0; K - 2\varepsilon)$, set $v = \|w - Px\|^{-1}c(\|x\|)(w - Px)$. Then by (3.1), there is a $u \in \overline{B}(0; 1) \subseteq X$ such that $dP_x(u) = v$ and so, if $h = c(\|x\|)^{-1}\|w - Px\|u$, then $dP_x(h) = w - Px$. Since P is Gateaux differentiable, we may choose $t \in (0,\ 1]$ so small that $x + th \in \overline{B}(0; K - \varepsilon) = M$ and

$$\|P(x + th) - P(x) - t\,dP_x(h)\| \leq qt\|w - Px\|.$$

By setting $g(x) = x + th$, this implies $g(x) \neq x$ and

$$\|P(g(x)) - P(x) - t(w - Px)\| \leq qt\|w - Px\| \tag{3.3}$$

and

$$c(\|x\|)\|g(x) - x\| \leq t\|w - Px\|. \tag{3.4}$$

From (3.3), we have

$$\|P(g(x)) - P(x)\| \leq (1 + q)t\|w - Px\| \tag{3.5}$$

and

$$\|P(g(x)) - w\| - (1 - t)\|Px - w\| \leq qt\|w - Px\|,$$

which implies

$$(1 - q)t\|Px - w\| \leq \|Px - w\| - \|P(g(x)) - w\|. \tag{3.6}$$

Combining (3.5) and (3.6), we have

$$(1 + q)^{-1}\|P(g(x)) - P(x)\| \leq \psi(x) - \psi(g(x)) \tag{3.7}$$

and combining (3.4) and (3.6), we get

$$c(\|x\|)\|g(x) - x\| \leq \psi(x) - \psi(g(x)). \tag{3.8}$$

Here we may assume that the domain of c is $[0, \infty)$ by putting $c(s) = c(K - 2\varepsilon)$ when $s > K - 2\varepsilon$ without affecting our argument of the proof. Now if $\rho(x, g(x)) = \|x - g(x)\|$, then, since $\|x\| \leq \rho(0, x)$, (3.8) gives (2.2), while if $\rho(x, g(x)) = c(0)^{-1}(1 + q)^{-1}\|P(x) - P(g(x))\|$, then, since c is nonincreasing,

$$\begin{aligned} c(\|x\|)\rho(x, g(x)) &= \frac{c(\|x\|)}{c(0)(1 + q)}\|Px - P(g(x))\| \\ &\leq (1 + q)^{-1}\|P(x) - P(g(x))\| \leq \psi(x) - \psi(g(x)) \end{aligned}$$

by (3.7), so again (2.2) holds. Thus by Theorem 2.2, g has a fixed point in M, a contradiction, and hence consequently $w \in P(\overline{B}(0; K - 2\varepsilon)) \subseteq P(B(0; K))$.

Analogous estimations for range of local expansions can be stated by the following theorem, which gives an extended version of Theorem 3.3 of [**13**] which also generalizes a result of Browder ([**4**], Theorem 4.10).

THEOREM 3.2. *Let X and Y be Banach spaces, P an open mapping from $B(0; K) \subseteq X$ $(K > 0)$ to Y having closed graph, and let c: $[0, K) \to (0, \infty)$ be a continuous nonincreasing function. Suppose for each $x \in B(0; K)$, there is an $\varepsilon > 0$ such that, if $y \in B(x; \varepsilon) \cap B(0; K)$, then*

$$c(\max\{\|x\|, \|y\|\})\|x - y\| \leq \|Px - Py\|. \tag{3.9}$$

Then $P(B(0; K))$ contains $B(P(0); \int_0^K c(s)\, ds)$.

Proof. Let $w \in B(P(0); \int_0^K c(s)\,ds)$, that is, $\|w - P(0)\| < \int_0^K c(s)\,ds$. Then we can choose $\varepsilon_1 > 0$ so small that $\|w - P(0)\| \leq \int_{\varepsilon_1}^{K-\varepsilon_1} c(s)\,ds$ holds.

Introduce a new metric ρ on the set $M = \overline{B}(0; K - \varepsilon_1)$ by setting $\rho(x, y) = \max\{\|x - y\|, c(0)^{-1}\|Px - Py\|\}$, so (M, ρ) is complete, and set $\psi(x) = \|w - Px\|$. Let $\bar{c}(s) = c(s + \varepsilon_1)$. Then $\psi(0) \leq \int_{\varepsilon_1}^{K-\varepsilon_1} c(s)\,ds = \int_0^{K-2\varepsilon_1} \bar{c}(s)\,ds$. Now we claim that $w \in P(\overline{B}(0; K - 2\varepsilon_1))$. As in the proof of Theorem 3.1, we suppose $w \notin P(\overline{B}(0; K - 2\varepsilon_1))$ and obtain a contradiction. Now we define a mapping g: $M \to M$ by setting $g(x) = 0$ $(\neq x)$ when $x \in M \setminus \overline{B}(0; K - 2\varepsilon_1)$; note that in this case $\int_{\rho(0,x)}^{K-2\varepsilon_1} \bar{c}(s)\,ds < 0 \leq \psi(x)$, and if $x \in \overline{B}(0; K - 2\varepsilon_1)$, then choose $\varepsilon > 0$ so small that $\varepsilon \leq \varepsilon_1$ and (3.9) holds. Actually the condition (3.9) can be replaced by the condition that if $\|x - y\| < \varepsilon$, then

$$\bar{c}(\|x\|)\|x - y\| \leq \|Px - Py\|.$$

Since P is an open mapping

$$P(B(x; \varepsilon)) \cap \{tPx + (1 - t)w; 0 \leq t < 1\} \neq \varnothing$$

and hence there is a $g(x) \in B(x; \varepsilon)$ such that $P(g(x)) \in \{tPx + (1 - t)w; 0 \leq t < 1\}$, so that $g(x) \neq x$ and $g(x) \in M$. Since

$$\|P(g(x)) - P(x)\| = \|Px - w\| - \|P(g(x)) - w\| = \psi(x) - \psi(g(x)),$$

it follows that

$$\bar{c}(\|x\|)\|g(x) - x\| \leq \psi(x) - \psi(g(x))$$

and

$$\bar{c}(\|x\|)c(0)^{-1}\|P(g(x)) - Px\| \leq \psi(x) - \psi(g(x)),$$

and hence (2.2) holds by assuming that the domain of $\bar{c}$ is $[0, \infty)$ as in the proof of Theorem 3.1. Thus by Theorem 2.2, g has a fixed point in M, which contradicts to the construction of $g(x)$.

Theorem 3.2 can be applied to the range of locally strongly ϕ-accretive operators. Let X and Y be Banach spaces with Y^* the dual of Y, and let ϕ: $X \to Y^*$ be a mapping such that

$$\phi(X) \text{ is dense in } Y^*, \text{ for each } x \in X \text{ and each } \xi \geq 0$$
$$\|\phi(x)\| \leq \|x\| \text{ and } \phi(\xi x) = \xi\phi(x).$$

A mapping P from X to Y is said to be strongly ϕ-accretive if there exists a constant $c > 0$ such that, for any $x, y \in X$,

$$(Px - Py, \phi(x - y)) \geq c\|x - y\|^2.$$

The ϕ-accretive mappings were introduced in an effort to unify the theories for monotone mappings (when $Y = X^*$) and for accretive mappings (when $Y = X$). Many authors (see [**3**], [**4**], [**7**], [**10**], [**13**], [**16**] and [**17**]) have studied domain invariance or surjectivity of accretive operators. The following theorem gives an improvement of Theorem 4.11 of [**4**], Corollary 2.2 of [**6**] and Theorem 3.4 of [**13**].

THEOREM 3.3. *Let X and Y be Banach spaces, and P an open mapping from $B(0; K) \subseteq X$ $(K > 0)$ to Y having closed graph. Let c: $[0, K) \to (0, \infty)$ be continuous nonincreasing for which, for any $x \in B(0; K)$, there is an $\varepsilon > 0$ such that for every $y \in B(x; \varepsilon) \cap B(0; K)$,*

$$(3.10) \qquad (Px - Py, \phi(x - y)) \geq c(\max\{\|x\|, \|y\|\})\|x - y\|^2.$$

Then $P(B(0; K))$ contains $B(P(0); \int_0^K c(s)\, ds)$.

Proof. It is easy to show that (3.10) implies (3.9), so that Theorem 3.3 follows from Theorem 3.2.

In Theorem 3.3, if P is locally lipschitzian, and if Y can be renormed so that Y is Frechet differentiable and Y^* is strictly convex, that is, the duality mapping J: $Y \to Y^*$ is single-valued and continuous, then Downing and Ray [**7**] show that P is automatically an open mapping. Also if $Y = X$ and ϕ is the duality mapping, and if P is continuous, then P is an open mapping by [**16**] and [**17**]. Also Theorem 3.3 can be applied to multivalued locally strongly ϕ-accretive mappings as in [**7**].

Note that the continuity of c in Theorem 2.2 and Theorems 3.1–3.3 can be replaced by the piecewise continuity of c without affecting results of those theorems.

Simple geometric intuition and integral equation suggest that c need not be nonincreasing in Theorems 3.1–3.3. Actually by using easy geometric estimation we can prove that such a condition can be removed in the following Theorem 3.4. In fact, Torrejon [**17**] proved that in Theorem 3.2, if $K = \infty$ and $\int_0^\infty c(s)\, ds = \infty$, then the condition that c is nonincreasing is not necessary.

THEOREM 3.4. *The conclusions of Theorems* 3.1–3.3 *hold without the assumption that c is nonincreasing.*

Proof. We may assume that $P(0) = 0$ after parallel transformation. Since c is continuous, for any given $\varepsilon > 0$, there is a partition

$$0 = K_0 < K_1 < \cdots < K_n < K$$

of $[0, K]$ such that by putting

$$m_i = \inf\{c(s); K_{i-1} \le s \le K_i\}, \qquad 1 \le i \le n,$$

the inequality

$$\sum_{i=1}^{n} m_i(K_i - K_{i-1}) \ge \int_0^K c(s)\, ds - \varepsilon$$

holds. Now we will prove that $P(B(0; K_n))$ contains $B(0; \int_0^K c(s)\, ds - \varepsilon)$, and hence we complete the proof since ε is arbitrary. For this purpose, it suffices to prove that for any given $w \in Y$ with $\|w\| = 1$,

(3.11) the segment $\{tw; 0 \le t < M_k\} \subseteq P(B(0; K_k))$ for any k, $1 \le k \le n$, where

$$M_k = \sum_{i=1}^{k} m_i(K_i - K_{i-1}).$$

Then for $k = n$, we have $\{tw; 0 \le t < M_n\} \subseteq P(B(0; K_n))$, and this implies $B(0; \int_0^K c(s)\, ds - \varepsilon) \subseteq B(0; M_n) \subseteq P(B(0; K_n))$.

First note that if c is nonincreasing (in particular, c is a constant function), then the theorem holds by Theorems 3.1–3.3. Therefore if $k = 1$, then (3.11) is trivially true. Suppose that (3.11) is not true for some $k \ge 2$, and k is the smallest integer for which (3.11) does not hold. Then there is a t_0 with $M_{k-1} \le t_0 < M_k$ such that $t_0 w \notin P(B(0; K_k))$, but $\{tw; 0 \le t < M_{k-1}\} \subseteq P(B(0; K_{k-1}))$. Now choose an $\varepsilon_1 > 0$ so small that $\varepsilon_1 < M_k - t_0$. Take $r > 0$ such that $m_k r < \varepsilon_1/4$, and set $m = \min\{m_1, m_2, \ldots, m_k\} > 0$. Then note that, by Theorems 3.1–3.3,

(3.12) if $\|x\| < K_{k-1} + r$, then $P(B(x; r)) \supseteq B(P(x); mr)$

and

(3.13) if $K_{k-1} + r \le \|x\| \le K_k - r$,

then $P(B(x; r)) \supseteq B(P(x); m_k r)$.

(3.12) and (3.13) are possible, since P satisfies the conditions of Theorems 3.1–3.3, respectively, by setting $c(s) = m$ in case (3.12) and $c(s) = m_k$ in case (3.13) on $B(x; r)$.

Also take $\varepsilon_2 > 0$ so small that $\varepsilon_2 < \min\{\varepsilon_1/4, rm\}$. Then since $\{tw; 0 \le t < M_{k-1}\} \subseteq P(B(0; K_{k-1}))$, we can take $x_1 \in B(0; K_{k-1})$ so that $Px_1 = t_1 w$, where $t_1 = M_{k-1} - 2^{-1}\varepsilon_2$. Also by (3.12), we can choose $x_2 \in B(x_1; r)$ so that $Px_2 = t_2 w$, where $t_2 = t_1 + rm - 2^{-2}\varepsilon_2$. Continue this process, we assume that x_j and t_j be chosen for $j \ge 2$ with $\|x_i\| \le K_k - r$ for all $i \le j$. Then if $\|x_j\| < K_{k-1} + r$, then by (3.12) there exists

$x_{j+1} \in B(x_j; r)$ such that $Px_{j+1} = t_{j+1}w$, where $t_{j+1} = t_j + rm - 2^{-j-1}\varepsilon_2$, and if $K_{k-1} + r \leq \|x_j\| \leq K_k - r$, then by (3.13) there exists $x_{j+1} \in B(x_j; r)$ such that $Px_{j+1} = t_{j+1}w$, where $t_{j+1} = t_j + rm_k - 2^{-j-1}\varepsilon_2$. We can continue the above process unless $\|x_j\| > K_k - r$. Now we claim that there is a j such that $t_j \leq t_0 < t_{j+1}$ with $\|x_i\| \leq K_k - r$ for all $i \leq j$, so that $t_0 w \in P(B(x_j; r)) \subseteq P(B(0; K_k))$, which is a contradiction.

To prove our claim, suppose that $\|x_j\| \leq K_k - r$ for all $j = 1, 2, \ldots$. Then for $j \geq 2$, we have

$$\begin{aligned} t_j &\geq t_{j-1} + rm - 2^{-j}\varepsilon_2 \quad (\text{since } m \leq m_k) \\ &\vdots \\ &\geq t_1 + (j-1)rm - (2^{-2} + \cdots + 2^{-j})\varepsilon_2 \\ &\geq M_{k-1} + (j-1)rm - \varepsilon_2 \quad \left(\text{since } t_1 = M_{k-1} - 2^{-1}\varepsilon_2\right). \end{aligned}$$

Since $rm > 0$, for some sufficiently large j, we can have $t_0 < t_{j+1}$. Also since the sequence $\{t_i\}$ is increasing, our claim is proved. Now suppose that for some j, $\|x_{j+1}\| > K_k - r$ and $\|x_i\| \leq K_k - r$ for all $i \leq j$. Since $\|x_i - x_{i+1}\| < r$ $(1 \leq i \leq j)$ and $rm_k < \varepsilon_1/4 \leq (M_k - M_{k-1})/4$, we have $4r < K_k - K_{k-1}$, so that there is a j_0 with $1 \leq j_0 < j - 1$ satisfying $\|x_{j_0}\| < K_{k-1} + r$ and $K_{k-1} + r \leq \|x_i\| \leq K_k - r$ for all $j_0 + 1 \leq i \leq j$. Then we have $\|x_{j+1} - x_{j_0+1}\| > K_k - K_{k-1} - 3r$, and hence $(j - j_0)r > K_k - K_{k-1} - 3r$. Note that $t_i < t_{i+1}$ for all $1 \leq i \leq j$, and $t_{i+1} = t_i + rm_k - 2^{-i-1}\varepsilon_2$ for all $j_0 + 1 \leq i \leq j$. Therefore we have

$$\begin{aligned} t_{j+1} &= t_j + rm_k - 2^{-j-1}\varepsilon_2 \\ &\vdots \\ &> t_{j_0+1} + (j - j_0)rm_k - 2^{-j_0}\varepsilon_2 \\ &> t_1 + (K_k - K_{k-1} - 3r)m_k - 2^{-j_0}\varepsilon_2 \\ &> M_{k-1} + M_k - M_{k-1} - 3rm_k - \varepsilon_2 \\ &> M_k - \varepsilon_1 > t_0. \end{aligned}$$

so that we complete the proof.

We list here one final conclusion as the following

THEOREM 3.5. *Let X and Y be Banach spaces, and P an operator from X to Y having closed graph. Let c: $[0, \infty) \to (0, \infty)$ be a continuous function for which one of the following conditions holds.*

(a) *P is Gateaux differentiable and for each $x \in X$,* (3.1) *holds.*

(b) *P is an open mapping and for each $x \in X$, there is an $\varepsilon > 0$ such that for every $\|x - y\| < \varepsilon$, (3.9) holds.*

(c) *P is an open mapping and for each $x \in X$, there is an $\varepsilon > 0$ such that for every $\|x - y\| < \varepsilon$, (3.10) holds.*

Then for any $K > 0$ (possibly $K = \infty$), $P(B(0; K))$ contains $B(P(0); \int_0^K c(s)\, ds)$, in particular, if $\int_0^\infty c(s)\, ds = \infty$, then P is surjective.

As a final remark, the condition (3.2) can be applied to the following extended version of Theorem 2 of **[14]**. The proof of the following theorem follows from (3.2) and Lemma of **[14]**. There is no significant variation in the proof, and so we omit it.

THEOREM 3.6. *Let X and Y be Banach spaces, and let P and Q be Gateaux differentiable mappings from $B(0; K) \subseteq X$ ($K > 0$, possibly $K = \infty$) to Y. Let c: $[0, K) \to (0, \infty)$ be a continuous function. Suppose for each $x \in B(0; K)$, that*

(a) *dQ_x is a bounded linear operator from X to Y, and*

(b) *$dP_x(\overline{B}(0; 1)) \supseteq \overline{B}(0; c(\|x\|))$.*

Suppose in addition, for some $\mu \in (0,1)$ and each $x \in X$ that

(c) *$c(\|x\|)^{-1}\|dQ_x\| < \mu$.*

If the mapping $R = P + Q$ has closed graph, then $R(B(0; K))$ contains $B(R(0); (1 - \mu)\int_0^K c(s)\, ds)$, in particular R is an open mapping. And if $\int_0^K c(s)\, ds = \infty$, then $R(B(0; K)) = Y$.

In the same situation of Theorem 3.6 Ray and Walker **[14]** showed that if P and Q have closed graphs, then so does $P + Q$. In **[14]**, in order to prove that R is an open mapping, they used actually the Brezis and Browder principle **[2]**, which was recently generalized in **[1]** and **[18]**. However, our Theorem 3.6 can be proved by using only Theorem 2.2 (actually Theorem 2.1 by assuming that c is a constant function) and combining Theorem 3.4, and it gives a precise estimation of range of operators.

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