PACIFIC JOURNAL OF MATHEMATICS
Vol. 126, No. 2, 1987

THE ASYMPTOTIC BEHAVIOR OF A FAMILY OF SEQUENCES

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A class of sequences defined by nonlinear recurrences involving the greatest integer function is studied, a typical member of the class being

$$a(0) = 1, \quad a(n) = a(\lfloor n/2 \rfloor) + a(\lfloor n/3 \rfloor) + a(\lfloor n/6 \rfloor) \quad \text{for } n \geq 1.$$

For this sequence, it is shown that $\lim a(n)/n$ **as** $n \to \infty$ **exists and equals** $12/(\log 432)$**. More generally, for any sequence defined by**

$$a(0) = 1, \qquad a(n) = \sum_{i=1}^{s} r_i a(\lfloor n/m_i \rfloor) \quad \text{for } n \geq 1,$$

where the $r_i > 0$ **and the** m_i **are integers** ≥ 2**, the asymptotic behavior of** $a(n)$ **is determined.**

1. Introduction. Rawsthorne [**R**] recently asked whether the limit $a(n)/n$ exists for the sequence $a(n)$ defined by

$$(1.1) \quad a(0) = 1, \qquad a(n) = a(\lfloor n/2 \rfloor) + a(\lfloor n/3 \rfloor) + a(\lfloor n/6 \rfloor), \quad n \geq 1,$$

where $\lfloor x \rfloor$ denotes the greatest integer $\leq x$. If the limit exists, Rawsthorne also asked for its value. We have answered these questions [**EHOPR**]: the limit exists and equals $12/\log 432$, where, as in the rest of the paper, log denotes the natural logarithm. Our method leads to a more general result about such recursively defined sequences.

Let $a(n)$ be the sequence defined by

$$(1.2) \qquad a(0) = 1, \qquad a(n) = \sum_{i=1}^{s} r_i a(\lfloor n/m_i \rfloor), \quad n \geq 1,$$

where $r_i > 0$ and the m_i's are integers ≥ 2. Let τ be the (unique) solution to

$$(1.3) \qquad \sum_{i=1}^{s} \frac{r_i}{m_i^{\tau}} = 1.$$

We distinguish two cases: if there is an integer d and integers u_i such that $m_i = d^{u_i}$, we are in the *lattice* case, otherwise we are in the *ordinary* case. In the ordinary case, $\lim a(n)/n^{\tau}$ exists; in the lattice case, $\lim a(n)/n^{\tau}$ does not exist, but $\lim_{k \to \infty} a(d^k)/d^{k\tau}$ exists. The limit in either case is readily computable. The proof involves transforming (1.2) into a renewal equation and using the standard limit theorems for that equation. For a precise statement of our results, see Theorem 2.14 below.

We are interested also in the rapidity of convergence. We prove that $(a(n) - a(n-1))/n^{\tau}$ is greater than $\gamma \cdot (\log n)^{-(s-1)/2}$ for some $\gamma > 0$ and infinitely many n. In Rawsthorne's original sequence (1.1), this result can be strengthened (see Theorem 3.46). For $n = 432^t$,

$$\text{(1.4)} \qquad \frac{a(n) - a(n-1)}{n} \sim \left(\frac{6}{5\pi t}\right)^{1/2} \quad \text{as } t \to \infty,$$

and this is, asymptotically, an upper bound. The numbers $J(m, r) := a(2^m 3^r) - a(2^m 3^r - 1)$ satisfy the so-called "square" functional equation; we use the work of Stanton and Cowan and others to help in the asymptotic analysis.

A somewhat different functional equation was studied by Erdös [**E1**], [**E2**]: for $2 \le a_1 \le a_2 \le \cdots$ a sequence of integers, let

$$F(0) = 0, \ F(1) = 1,$$

$$F(n) = \sum_{k=1}^{\infty} F(\lfloor n/a_k \rfloor) + 1 \quad \text{for } n > 1.$$

Both the methods and results are different from ours.

2. An application of renewal theory. We fix the following notation. Let integers m_i, $2 \le m_i \le M$, and positive real numbers r_i be given. Define the sequence $a(n)$ recursively by

$$\text{(2.1)} \qquad a(0) = 1, \qquad a(n) = \sum_{i=1}^{s} r_i a(\lfloor n/m_i \rfloor), \quad n \ge 1.$$

For $x \ge 0$ define

$$\text{(2.2)} \qquad A(x) = a(\lfloor x \rfloor).$$

Since $[x/mn] = [[x/m]/n]$ for positive integers m and n, we may define $A(x)$ directly and, in effect, extend the sequence to a function on the positive reals:

$$\text{(2.3)} \quad A(x) = 1 \quad \text{for } 0 \le x < 1, \qquad A(x) = \sum_{i=1}^{s} r_i A(x/m_i) \quad \text{for } x \ge 1.$$

Note that the function $\phi(u) = \sum r_i/m_i^u$ decreases strictly on the real line from ∞ to 0 so there exists a unique $\tau > 0$ satisfying

$$\phi(\tau) = \sum_{i=1}^{s} \frac{r_i}{m_i^\tau} := \sum_{i=1}^{s} p_i = 1. \tag{2.4}$$

Now let

$$f(x) = A(x)/x^\tau \tag{2.5}$$

so that we may rewrite (2.3) as

$$f(x) = x^{-\tau}, \quad 0 < x < 1; \qquad f(x) = \sum_{i=1}^{s} \frac{r_i}{m_i^\tau} f\left(\frac{x}{m_i}\right) \quad \text{for } x \geq 1. \tag{2.6}$$

Since $p_i > 0$ and $\sum p_i = 1$, $f(x)$ is a convex combination of previous values of f for $x \geq 1$. It is thus unsurprising that f tends to a limit.

It is now appropriate to review some well-known (to probabilists) results about the renewal equation. We paraphrase Feller [**F**, v. 2, pp. 358–362]. Suppose h is a Riemann integrable function with compact support and $F\{dy\}$ is a probability measure with finite expectation and suppose g satisfies the renewal equation

$$g(u) = h(u) + \int_0^u g(u - \nu) F\{d\nu\}, \qquad u \geq 0. \tag{2.7}$$

If the mass of $F\{d\nu\}$ is concentrated on a set of the form $\{0, \lambda, 2\lambda, \dots\}$, we are in the *lattice* case; otherwise we are in the *ordinary* case. The following limit theorem for g is due to Erdös, Feller, and Pollard in the lattice case and Blackwell in the ordinary case.

Renewal Limit Theorem (*see* [**F**, v. 2, p. 362]).

(i) *In the ordinary case*.

$$\lim_{u \to \infty} g(u) = \frac{\int_0^\infty h(u)\, du}{\int_0^\infty y F\{dy\}}. \tag{2.8}$$

(ii) *In the lattice case, let* λ *be chosen to be maximal*; *then* g *does not converge, but for any fixed* $x \in [0, \lambda)$,

$$\lim_{n \to \infty} g(x + n\lambda) = \frac{\lambda \sum_{k=0}^\infty h(x + k\lambda)}{\int_0^\infty y F\{dy\}}, \tag{2.9}$$

where the limit in (2.9) *is taken over integral* n.

We now return to our problem. Let

$$g(u) = f(e^u). \tag{2.10}$$

Then (2.6) can be rewritten as

$$g(u) = \begin{cases} e^{-\tau u}, & u \le 0; \\ \sum_{i=1}^{s} p_i g(u - \log m_i), & u \ge 0. \end{cases} \tag{2.11}$$

Let $F\{d\nu\}$ be the probability measure with mass p_i at $\log m_i$. Then g satisfies an equation of the form (2.7), where h measures the discrepancy between the full recurrence of (2.11) and that portion provided by the convolution in (2.7). This discrepancy arises from a negative argument of g. Hence,

$$h(u) = \sum_{u < \log m_i} p_i e^{-\tau(u - \log m_i)} = \sum_{u < \log m_i} p_i m_i^\tau e^{-\tau u}, \tag{2.12}$$

and so

$$h(u) = \sum_{i=1}^{s} p_i m_i^\tau e^{-\tau u} \chi_{[0, \log m_i)}(u). \tag{2.13}$$

Having now transformed (1.2) into a renewal equation we must decide which case we are in. The mass of F is concentrated at $\{\log m_i\}$, which is a subset of $\{0, \lambda, 2\lambda, \dots\}$ for some λ (the lattice case) if and only if $m_i = d^{u_i}$ for some integers d and u_i. Alternatively, we are in the ordinary case if and only if $(\log m_i)/(\log m_j)$ is irrational for some (m_i, m_j).

We now combine these discussions into a theorem.

THEOREM 2.14. *Let $a(n)$ be defined by* (2.1) *and let τ be defined as above.*

(i) *If $\tau = 0$ then $a(n) \equiv 1$.*

(ii) *If $\tau \ne 0$ and $(\log m_i)/(\log m_j)$ is irrational for some (m_i, m_j) (the ordinary case), then*

$$\lim_{n \to \infty} \frac{a(n)}{n^\tau} = \frac{\sum_{i=1}^{s} p_i (m_i^\tau - 1)/\tau}{\sum_{i=1}^{s} p_i \log m_i}. \tag{2.15}$$

(iii) *If $\tau \ne 0$ and $m_i = d^{u_i}$, where d and the u_i's are integers and d is maximal (the lattice case), then*

$$\lim_{k \to \infty} \frac{a(d^k)}{d^{k\tau}} = \frac{\sum_{i=1}^{s} p_i (m_i^\tau - 1)}{\sum_{i=1}^{s} p_i \log m_i} \cdot \frac{d^\tau \log d}{d^\tau - 1}. \tag{2.16}$$

Proof. (i) If $\tau = 0$, then $\sum r_i = 1$ and it is easy to see from (2.1) that $a(n) \equiv 1$ by induction. As $u^\tau - 1 \approx \tau \log u$ for τ near 0, this result is consistent with the limiting behavior in (2.15) and (2.16).

(ii) From our definitions,

$$\frac{a(n)}{n^{\tau}} = f(n) = g(\log n), \tag{2.17}$$

so that information about the limiting behavior of $g(u)$ from the Renewal Limit Theorem can be translated into information about $a(n)/n^{\tau}$. In either the ordinary or lattice case,

$$\int_0^{\infty} yF\{dy\} = \sum_{i=1}^{s} p_i \log m_i. \tag{2.18}$$

In the ordinary case, as $\tau \neq 0$, we have by (2.13),

$$\begin{aligned} \int_0^{\infty} h(u)\,du &= \sum_{i=1}^{s} p_i m_i^{\tau} \int_0^{\log m_i} e^{-\tau u}\,du \\ &= \sum_{i=1}^{s} p_i m_i^{\tau}(1 - m_i^{-\tau})/\tau. \end{aligned} \tag{2.19}$$

Equation (2.15) follows from the foregoing discussion, (2.8), (2.18), and (2.19).

(iii) The period of the lattice is $\lambda = \log d$, and taking $x = 0$ in (2.9),

$$\begin{aligned} \sum_{k=0}^{\infty} h(k\lambda) &= \sum_{k=0}^{\infty} h(k \log d) \\ &= \sum_{k=0}^{\infty} \left(\sum_{i=1}^{s} p_i m_i^{\tau} e^{-\tau k \log d} \chi_{[0, \log m_i)}(k \log d) \right). \end{aligned} \tag{2.20}$$

Since $m_i = d^{u_i}$,

$$\begin{aligned} &\sum_{k=0}^{\infty} e^{-\tau k \log d} \chi_{[0, \log m_i)}(k \log d) \\ &\quad = \sum_{k=0}^{u_i - 1} e^{-\tau k \log d} = (1 - d^{-u_i \tau})/(1 - d^{-\tau}) \\ &\quad = (1 - m_i^{-\tau}) d^{\tau}/(d^{\tau} - 1). \end{aligned} \tag{2.21}$$

We now exchange the order of summation in (2.20) to obtain

$$\sum_{k=0}^{\infty} h(k\lambda) = \sum_{i=1}^{s} p_i m_i^{\tau}(1 - m_i^{-\tau}) d^{\tau}/(d^{\tau} - 1), \tag{2.22}$$

and (2.16) follows from (2.18), (2.22), and (2.9). □

In the lattice case, it is easy to show by induction that the sequence $a(n)$ is constant on intervals of the form $[d^k, d^{k+1} - 1]$. For any rational

$x = j/d^r$, $d^t < x < d^{t+1}$, $a(xd^k)$ is defined for $k \geq r$, and $a(xd^k) = a(d^{t-r+k})$. Using (2.16), one can compute $\lim a(d^k x)/(d^k x)^\tau$; we omit the details.

As a check of Theorem 2.14, consider the following simple lattice example with $s = 1$:

$$(2.23) \qquad a(0) = 1; \qquad a(n) = d^\alpha a(\lfloor n/d \rfloor), \quad n \geq 1.$$

It is easy to see in this case that $\tau = \alpha$ and $a(d^k) = d^{(k+1)\alpha}$, so that $a(d^k)/d^{k\tau} \equiv d^\alpha$. Substituting $p_1 = 1$, $m_1 = d$, and $\tau = \alpha$ into (2.16) returns d^α, as predicted.

It is perhaps worth mentioning that the existence of $\lim_{n\to\infty} f(n)$ can be proved without recourse to the Renewal Limit Theorem. Here is a sketch of the argument, without proofs. First, from (2.6), $\alpha \geq f(x) \geq \beta$ for $x \in [y, My]$, where $y \geq 1$ and $M \geq \max m_i$ implies that $\alpha \geq f(x) \geq \beta$ for all $x \geq y$. Thus $L = \overline{\lim} f(x)$ and $l = \underline{\lim} f(x)$ are positive and finite. Pick sequences $r_k \to \infty$ and $s_k \to \infty$ with $f(r_k) \to L$, $f(s_k) \to l$ and $r_k < s_k < Mr_k$. The next step in the argument is proved as in §3; $a(n) \neq a(n-1)$ if and only if $n = m_1^{e_1} \cdots m_s^{e_s}$ for some integers e_i and $\tau \neq 0$, $a(n) \geq a(n-1)$ if $\tau > 0$ and $a(n) \leq a(n-1)$ if $\tau < 0$. There is a dichotomy depending on which case arises. In the lattice case, $a(n)$ is constant on intervals $[d^k, d^{k+1} - 1]$ and the substitutions $m = d^{u_i}$, $b(k) = a(d^k)$ show that $\{b(k)\}$ satisfies a linear difference equation for k sufficiently large. By the standard method for solving linear difference equations (see [T, Ch. 4], for example) $a(d^k) = b(k) = c \cdot \beta^k + o(\beta^k) = cd^{k\tau} + o(d^{k\tau})$ for appropriate constants. (See the discussion following Corollary 3.12 for more details.)

In the ordinary case, suppose $(\log m_i)/(\log m_j) \notin Q$ and let $m = m_i$ and $\overline{m} = m_j$. For any $\varepsilon > 0$ there exists E so that every x in $[M^{-1}, M]$ is contained in an interval $[w, w(1+\varepsilon)]$, where $w = m^{e_1}/\overline{m}^{e_2}$ and e_1 and e_2 are positive integers $\leq E$. Let W be any finite set of integers of the form $m_1^{f_1} \cdots m_s^{f_s}$; for k sufficiently large and any $w \in W$, $f(r_k/w)$ is close to L and $f(s_k/w)$ is close to l. (This is proved by induction; basically, if a weighted average like (2.6) is close to the maximum then its components can't be too far off.) Now suppose $\tau > 0$ and $L > l$ and let $x = s_k/r_k$, where k is sufficiently large; let $w = m^{e_1}/\overline{m}^{e_2}$ be chosen so that $x \in [w, w(1+\varepsilon)]$. Then $r' = r_k/\overline{m}^{e_2}$ is a little less than $s' = s_k/m^{e_1}$, but $f(r')$ is close to L and $f(s')$ is close to l. As $\tau > 0$, $a(s') \geq a(r')$, and this gives a contradiction to $L > l$. (More precisely, ε is chosen so that $L - \varepsilon > (1+\varepsilon)^\tau(l+\varepsilon)$.) A similar contradiction can be wrought when $\rho < 0$. In either case, $L = l$ so the limit exists. This method, although self-contained, gives no hint about the actual value of the limit.

3. Rates of convergence. We retain the notation of the last section and continue to assume that $a(n)$ is defined by (2.1). Let

$$J(n) = a(n) - a(n-1) \tag{3.1}$$

denote the jump of the sequence at n. In this section we derive closed forms for $a(n)$ and $J(n)$ and use them to give an indication of the rate of convergence of f. Ideally, one would discuss the behavior of $|f(x) - \lim f(x)|$. As a step in that direction, we consider the "jumps" of f. It is clear from (2.2) and (2.5) that f is everywhere continuous from the right and f is continuous from the left except possibly at certain integers. Let

$$z(n) = f(n) - \lim_{\varepsilon \to 0^+} f(n-\varepsilon) = \frac{a(n) - a(n-1)}{n^\tau} = \frac{J(n)}{n^\tau}. \tag{3.2}$$

We shall show in this section that, in the ordinary case, $|z(n)| > c(\log n)^{-(s-1)/2}$ for some $c > 0$ and infinitely many integers n. In Rawsthorne's original problem, (1.1), the exponent of $\log n$ may be improved from -1 to $-1/2$.

In finding a closed form for $a(n)$, the following notation is useful. Let $\mathbf{i} = (i_1, \ldots, i_l)$, $l \geq 1$, be an l-tuple of integers, $1 \leq i_j \leq s$. Let $I(\mathbf{i})$ be the associated interval:

$$I(\mathbf{i}) = [m_{i_1} \cdots m_{i_{l-1}}, \ldots, m_{i_1} \cdots m_{i_{l-1}} m_{i_l}). \tag{3.3}$$

(If $l = 1$ in (3.3), take the left-hand endpoint to be 1.) As an inverse function to I, for $x \geq 1$, let

$$B(x) = \{\mathbf{i} \colon x \in I(\mathbf{i})\}. \tag{3.4}$$

THEOREM 3.5. *For* $x \geq 1$.

$$A(x) = \sum_{\mathbf{i} \in B(x)} r_{i_1} \cdots r_{i_l}. \tag{3.6}$$

Proof. Recall the basic recurrence (2.3):

$$A(x) = \sum_{i=1}^{s} r_i A(x/m_i).$$

Consider the infinite tree with root "x" and valence s so that each node "y" on the kth level is connected to the nodes "y/m_i", $1 \leq i \leq s$ on the $(k+1)$st level. We use this tree to iterate the recurrence (2.3) until the argument of A goes below 1 for the first time. In this way, the path from x to $x/m_{i_1}, \ldots,$ to $x/(m_{i_1} \cdots m_{i_l})$ acquires the coefficient $r_{i_1} \cdots r_{i_l}$. Since $\mathbf{i} = (i_1, \ldots, i_l)$ is in $B(x)$ by construction and $A(x/\prod m_j) = 1$, (3.6) is established. □

We now derive a recurrence for $J(n) = a(n) - a(n-1)$ and find a closed form for $J(n)$.

THEOREM 3.7.

(i)

$$J(n) = \sum_{m_i|n} r_i J(n/m_i).$$

(ii)

$$J(n) = \left(\sum_{i=1}^{s} r_i - 1\right) \sum_{m_1^{e_1} \cdots m_s^{e_s} = n} \frac{(e_1 + \cdots + e_s)!}{e_1! \cdots e_s!} r_1^{e_1} \cdots r_s^{e_s}.$$

Proof. (i) We have from (2.3)

$$J(n) = \sum_{i=1}^{s} r_i \left(A\left(\frac{n}{m_i}\right) - A\left(\frac{n-1}{m_i}\right)\right). \tag{3.8}$$

If $m_i \nmid n$ then $\lfloor n/m_i \rfloor = \lfloor (n-1)/m_i \rfloor$ so the ith term is zero; if $m_i | n$, then by definition, the ith term is $r_i J(n/m_i)$.

(ii) Observe that $J(1) = a(1) - a(0) = \sum_{i=1}^{s} r_i - 1$. Then, consider each representation of n as a product $m_1^{e_1} \cdots m_s^{e_s}$. The formula (ii) follows by induction from (i) and the well-known multinomial recurrence:

$$\frac{(e_1 + \cdots + e_s)!}{e_1! \cdots e_s!} = \sum_{e_i \geq 1} \frac{(e_1 + \cdots + e_s - 1)!}{e_1! \cdots (e_i - 1)! \cdots e_s!}. \qquad \square$$

We note that (ii) can also be derived from Theorem 3.5 and a consideration of $B(n) - B(n-1)$ and $B(n-1) - B(n)$. If we consider the representations $n = m_1^{e_1} \cdots m_s^{e_s}$ as formally distinct, we may let $j(f_1, \dots, f_s)$ denote that portion of the jump $J(n)$ contributed by the representation $n = m_1^{f_1} \cdots m_s^{f_s}$. In view of Theorem 3.7 we have the following recurrences and generating function.

$$\begin{cases} j(e_1, \dots, e_s) = \sum_{e_i \geq 1} r_i j(e_1, \dots, e_i - 1, \dots, e_s), \\ j(0, \dots, 0) = \sum_{i=1}^{s} r_i - 1. \end{cases} \tag{3.9}$$

$$j(e_1, \dots, e_s) = \left(\sum_{i=1}^{s} r_i - 1\right) \frac{(e_1 + \cdots + e_s)!}{e_1! \cdots e_s!} r_1^{e_1} \cdots r_s^{e_s}. \tag{3.10}$$

$$\begin{aligned} \mathcal{J}(z_1, \dots, z_s) &= \sum j(e_1, \dots, e_s) z_1^{e_1} \cdots z_s^{e_s} \\ &= \left(\sum_{i=1}^{s} r_i - 1\right)\left(1 - \sum_{i=1}^{s} r_i z_i\right)^{-1}. \end{aligned} \tag{3.11}$$

COROLLARY 3.12. *If $\tau > 0$ then $J(n) > 0$ at all n of the form $m_1^{e_1} \cdots m_s^{e_s}$; if $\tau < 0$ then $J(n) < 0$ at all such n.*

Proof. From Theorem 3.7 (ii), the sign of $J(n)$ is the sign of $\sum_{i=1}^{s} r_i - 1$, which equals $\phi(0) - \phi(\tau)$ in the notation of (2.4). Since ϕ is strictly decreasing, the conclusions follow immediately. □

We now turn our attention to the size of $z(n)$. It is convenient to dispose of the lattice case. As $f(d^k)$ converges to a limit l and $A(x)$ is constant on $[d^{k-1}, d^k)$, $z(d^k) \sim l(1 - d^{-\tau})$. It is more interesting to look at $f(d^{k+1}) - f(d^k)$. Let $m_i = d^{u_i}$ and let $u = \max u_i$. Then from (2.6),

$$f(d^k) = \sum_{i=1}^{s} p_i f(d^{k-u_i}). \tag{3.13}$$

Let $\psi(t) = t^u - \sum p_i t^{u-u_i}$ be the characteristic equation of the linear recurrence satisfied by $f(d^k)$. Clearly $\psi(1) = 0$ and, as $\lim|f(d^k)| < \infty$, it follows that the other roots of ψ have moduli less than one. Hence there exists a polynomial q of degree at most $s - 1$ and λ, $0 \leq \lambda < 1$ so that

$$|f(d^k) - 1| \leq q(k)\lambda^k + o(\lambda^k). \tag{3.14}$$

It follows that $|f(d^{k+1}) - f(d^k)| \leq ck^r\lambda^k$ for sufficiently large k, $r \leq s - 1$ and some $c > 0$.

Henceforth we assume the ordinary case and $\tau \neq 0$. We first need two approximation lemmas. The first follows directly from the Stirling approximation $\Gamma(w + 1) \sim w^w e^{-w}\sqrt{2\pi w}$ and we omit the proof. The second allows us to adjust from real numbers to integers in our asymptotic analysis.

LEMMA 3.15. *Fix $\alpha_i > 0$, $\sum_{i=1}^{s} \alpha_i = 1$ and define*

$$F(x_1, \ldots, x_s) = \frac{\Gamma(\sum_{i=1}^{s} x_i + 1)}{\prod_{i=1}^{s} \Gamma(x_i + 1)}. \tag{3.16}$$

Then, as $q \to \infty$

$$F(\alpha_1 q, \ldots, \alpha_s q) \sim (2\pi q)^{-(s-1)/2} \prod_{i=1}^{s} \alpha_i^{-(\alpha_i q + 1/2)}. \tag{3.17}$$

LEMMA 3.18. *Fix $\alpha_i > 0$, $\sum_{i=1}^{s} \alpha_i = 1$ and define*

$$\phi(q; t_1, \ldots, t_s) = \frac{F(\alpha_1 q + t_1, \ldots, \alpha_s q + t_s)}{F(\alpha_1 q, \ldots, \alpha_s q)}. \tag{3.19}$$

Then there exists $c > 0$ so that for all sufficiently large q and all choices of t_i with $|t_i| < 1$ and $\sum_{i=1}^{s} t_i = 0$,

$$c^{-1} < \phi(q; t_1, \ldots, t_s) < c. \tag{3.20}$$

Proof. From (3.16) we have

$$\text{(3.21)}\quad \phi(q; t_1, \ldots, t_s) = \frac{\Gamma(q+1)}{\prod_{i=1}^s \Gamma(\alpha_i q + t_i + 1)} \Big/ \frac{\Gamma(q+1)}{\prod_{i=1}^s \Gamma(\alpha_i q + 1)}$$
$$= \prod_{i=1}^s \frac{\Gamma(\alpha_i q + 1)}{\Gamma(\alpha_i q + t_i + 1)}.$$

Let

$$\text{(3.22)}\qquad H(\alpha, q, t) = \log \Gamma(\alpha q + t + 1) - \log \Gamma(\alpha q + 1).$$

As $\log \Gamma$ is convex, for $|t| < 1$, $t \neq 0$ we have

$$\text{(3.23)}\qquad \log \Gamma(\alpha q + 2) - \log \Gamma(\alpha q + 1)$$
$$\geq \frac{H(\alpha, q, t)}{t} \geq \log \Gamma(\alpha q + 1) - \log \Gamma(\alpha q),$$

hence $H(\alpha, q, t) = t \log(\alpha q + p)$, $0 \leq p \leq 1$, and

$$\text{(3.24)}\quad -\log \phi(q; t_1, \ldots, t_s) = \sum_{i=1}^s H(\alpha_i, q, t_i) = \sum_{i=1}^s t_i \log(\alpha_i q + p_i)$$
$$= \sum_{i=1}^s t_i \log \alpha_i + \sum_{i=1}^s t_i \log q + \sum_{i=1}^s t_i \log\left(1 + \frac{p_i}{\alpha_i q}\right).$$

Since $\sum t_i = 0$, $|t_i| < 1$ and $|p_i| < 1$,

$$\text{(3.25)}\qquad |\log \phi(q; t_1, \ldots, t_s)| \leq \sum_{i=1}^s t_i \log \alpha_i + \sum_{i=1}^s \frac{1}{\alpha_i q},$$

from which (3.20) follows. □

THEOREM 3.26. *In the ordinary case with $\tau \neq 0$ there exists $\gamma > 0$ so that*

$$\text{(3.27)}\qquad |z(n)| > \gamma \cdot (\log n)^{-(s-1)/2}$$

for infinitely many n.

Proof. The main idea is to let $n = (m_1^{p_1} \cdots m_s^{p_s})^q$ for large q. Large in this case means that (3.17) is a good approximation. Since $p_i q$ is not an integer in general, we need the approximation of Lemma 3.18.

To be specific, choose q large and choose integers e_i, $\sum_{i=1}^s e_i = q$ such that $|e_i - p_i q| < 1$ for all i. By Theorem 3.7 (ii), we may ignore other representations of n and

$$\text{(3.28)}\qquad |J(n)| \geq \alpha \cdot \frac{(e_1 + \cdots e_s)!}{e_1! \cdots e_s!} r_1^{e_1} \cdots r_s^{e_s},$$

where $\alpha = |\sum_{i=1}^{s} r_i - 1| = |\phi(\tau) - \phi(0)| > 0$. We now replace e_i by $p_i q$ in (3.28): $\prod r_i^{e_i}$ changes by a bounded factor, and by Lemma 3.18 we have

$$(3.29) \qquad |J(n)| \geq \beta \cdot \frac{\Gamma(q+1)}{\prod_{i=1}^{s}(p_i q + 1)} \cdot \prod_{i=1}^{s} r_i^{p_i q},$$

where β has absorbed all other constants. Finally, by Lemma 3.16,

$$(3.30) \qquad \begin{aligned} |J(n)| &\geq \beta \cdot (2\pi q)^{-(s-1)/2} \prod_{i=1}^{s} p_i^{-(p_i q + 1/2)} r_i^{p_i q}(1-\varepsilon) \\ &= \gamma q^{-(s-1)/2} \prod_{i=1}^{s} (r_i/p_i)^{p_i q} \\ &= \gamma q^{-(s-1)/2} \prod_{i=1}^{s} m_i^{\tau p_i q} = \gamma q^{-(s-1)/2} n^{\tau}. \end{aligned}$$

Since $z(n) = J(n)/n^{\tau}$ and $q = (\log n)/(\sum p_i \log m_i)$, (3.27) follows. □

It is possible to sharpen the constant slightly by noting that for any $\varepsilon > 0$ there are infinitely many q such that $p_i q$ is within ε of an integer for all i. (Standard pigeonhole principle argument.) If s were to equal 1 then (3.27) would violate the convergence of f, except that $s = 1$ is always a lattice case.

We conclude this paper by returning to Rawsthorne's original problem:

$$a(0) = 1, \quad a(n) = a(\lfloor n/2 \rfloor) + a(\lfloor n/3 \rfloor) + a(\lfloor n/6 \rfloor).$$

By Theorem 3.7 we know that $a(n)$ jumps only at numbers of the form $2^{e_1}3^{e_2}6^{e_3}$; that is, products of 2 and 3. Let

$$(3.31) \qquad J(m, r) =: J(2^m 3^r).$$

Then $m = e_1 + e_3$, $r = e_2 + e_3$, and by both parts of Theorem 3.7,

$$(3.32) \quad J(m,r) = 2\sum_i \frac{(m+r-i)!}{(m-i)!(r-i)!i!} = 2\sum_i \binom{m+r-i}{m-i}\binom{r}{i},$$

$$(3.33) \quad \begin{cases} J(m,r) = J(m, r-1) + J(m-1, r) + J(m-1, r-1), \\ \hfill m, r \geq 1, \\ J(0,0) = J(m,0) = J(0,r) = 2. \end{cases}$$

Unsurprisingly, such a simply defined recurrence has a large literature; (3.33) is called the "square" functional equation and arises as a natural generalization of Pascal's triangle. (Actually, $\frac{1}{2}J$ is the standard form.) The first problem on the 19th Putnam Competition to show that

$S(n) = \frac{1}{2}\Sigma_i J(i, n-i)$ satisfies the recurrence $S(n+2) = 2S(n+1) + S(n)$ [**GGK**, p. 53]). This recurrence then arose in Golomb's study of sphere packing in the Lee metric [**Go**]. Stanton and Cowan [**SC**] considered (3.32) in its own right and were the first to prove Lemma 3.34 below. A. K. Gupta [**Gu1**] [**Gu2**] gave different proofs and generalized these numbers further, as did Carlitz [**Ca**] and Alladi and Hoggatt [**AH**]. The function $\frac{1}{2}J$ has a natural interpretation as the number of ways to go from $(0,0)$ to (m,r) with steps of size $(1,0)$, $(0,1)$ or $(1,1)$; see Fray and Roselle [**FR**] or Handa and Mohanty [**HM**]. Greene and Knuth [**GK**; pp. 111–113] discuss the asymptotics of $J(m,m)$.

Our analysis of $z(2^m 3^r)$ relies crucially on the following combinatorial lemma.

LEMMA 3.34.

$$J(m,r) = 2\sum_i \binom{m}{i}\binom{r}{i}2^i = 2\sum_i \binom{m+r-i}{m-i}\binom{r}{i}. \tag{3.35}$$

Proof. Consider the coefficient of x^m in

$$2(1+x)^m(1+2x)^r = 2\sum_i \binom{r}{i} x^i (1+x)^{m+r-i}. \qquad \square$$

Stanton and Cowan originally proved this lemma by a sequence of standard combinatorial substitutions. Gupta used a number of methods, including the following hypergeometric representation [**Gu**, Lemma 4]:

$$\tfrac{1}{2}J(m,r) = {}_2F_1(-m,-r;1,2). \tag{3.6}$$

Lemma 3.34 leads to a natural probabilistic interpretation of $J(m,r)$. Let

$$\alpha(m,i) = \frac{\binom{m}{i}}{2^m}, \quad \beta(r,i) = \frac{\binom{r}{i}2^i}{3^r}. \tag{3.37}$$

These denote the probabilities of i successes in m and r Bernoulli trials with $p = \frac{1}{2}$ and $\frac{2}{3}$ respectively. As

$$z(2^m 3^r) = \frac{J(m,r)}{2^m 3^r} = 2\sum_i \alpha(m,i)\beta(r,i), \tag{3.38}$$

one expects $z(2^m 3^r)$ to be largest when the probability distributions peak simultaneously; that is, when $m/2 \approx 2r/3$, cf. Proposition 3.41. As a preliminary bound, note that $\alpha(m,i) \le \alpha(m, m/2)$ and $\beta(r,i) \le \beta(r, 2r/3)$, replacing factorials by Γ as necessary. By Lemma 3.15,

$\alpha(m,i) \leq \gamma_0 m^{-1/2}$ and $\beta(r,i) \leq \gamma_1 r^{-1/2}$ for appropriate $\gamma_i > 0$. Hence

$$z(2^m 3^r) \leq \min\left(\gamma_0 m^{-1/2}, \gamma_1 r^{-1/2}\right). \tag{3.39}$$

Since $\log(2^m 3^r) = m\log 2 + r\log 3$, (3.39) implies that $z(n) \leq \gamma(\log n)^{-1/2}$ for some $\gamma > 0$ and all n.

Consider now the normal approximation to the binomial distribution, see e.g. [**F**, v. 1, p. 170]. For fixed k,

$$\begin{cases} \alpha\left(m, \dfrac{m}{2} + k\dfrac{\sqrt{m}}{2}\right) \sim \dfrac{2}{\sqrt{m}}\dfrac{1}{\sqrt{2\pi}}e^{-k^2/2}, \\ \beta\left(r, \dfrac{2r}{3} + k\dfrac{\sqrt{2r}}{3}\right) \sim \dfrac{3}{\sqrt{2r}}\dfrac{1}{\sqrt{2\pi}}e^{-k^2/2}. \end{cases} \tag{3.40}$$

Let $\Delta(m,r) = |m/2 - 2r/3|$. We now show that if $\Delta(m,r)$ is comparable to $\sqrt{m}$ and $\sqrt{r}$, then $z(2^m 3^r)$ is quite a bit smaller than $\gamma(\log n)^{-1/2}$.

Proposition 3.41. *Fix $k, \varepsilon > 0$ and suppose*

$$\Delta(m,r) = \left|\frac{m}{2} - \frac{2r}{3}\right| > k\left(\frac{\sqrt{m}}{2} + \frac{\sqrt{2r}}{3}\right). \tag{3.42}$$

Then for sufficiently large m and r,

$$z(2^m 3^r) \leq 2(1+\varepsilon)\left(\frac{2}{\sqrt{m}} + \frac{3}{\sqrt{2r}}\right)\frac{1}{\sqrt{2\pi}}e^{-k^2/2}. \tag{3.43}$$

Proof. If (3.42) holds, then for each i at least one of the inequalities

$$\left|\frac{m}{2} - i\right| > \frac{k\sqrt{m}}{2} \quad \text{or} \quad \left|\frac{2r}{3} - i\right| > \frac{k\sqrt{2r}}{3} \tag{3.44}$$

is valid. Suppose that m and r are large enough that the approximation in (3.40) becomes an inequality after multiplication by $1 + \varepsilon$. Let

$$I = \left\{i : \left|\frac{m}{2} - i\right| > \frac{k\sqrt{m}}{2}\right\};$$

then

$$\begin{aligned} z(2^m 3^r) &\leq 2(1+\varepsilon)\left(\sum_{i\in I}\alpha(m,i)\right)\frac{2}{\sqrt{m}}\frac{1}{\sqrt{2\pi}}e^{-k^2/2} \\ &\quad + 2(1+\varepsilon)\left(\sum_{i\notin I}\beta(r,i)\right)\frac{3}{\sqrt{2r}}\frac{1}{\sqrt{2\pi}}e^{-k^2/2} \\ &\leq 2(1+\varepsilon)\left(\frac{2}{\sqrt{m}} + \frac{3}{\sqrt{2r}}\right)\frac{1}{\sqrt{2\pi}}e^{-k^2/2}. \end{aligned} \tag{3.45}$$

□

We remark that, if $r \sim \alpha m$, where $\alpha \neq 3/4$, then this proposition implies that

$$z(n) = z(2^m 3^r) \leq h_1(\alpha)(\log n)^{-1/2} n^{-h_2(\alpha)},$$

where $h_1(\alpha)$ and $h_2(\alpha)$ are complicated positive algebraic functions of α. We spare the reader the gory details. The asymptotic behavior of

$$\sum \binom{c}{i}\binom{\alpha c}{i} x^i$$

has been studied by Laquer **[La]**; more precise information than Proposition 3.41 can be found there, as can our final estimate, whose proof we sketch.

THEOREM 3.46. *For* $n = 432^t = 2^{4t}3^{3t}$,

$$z(n) \sim \left(\frac{6}{5\pi t}\right)^{1/2} = \left(\frac{6 \log 432}{5\pi \log n}\right)^{1/2}. \tag{3.47}$$

Proof. After a reindexing, (3.38) becomes

$$z(n) = \sum_i \alpha(4t, 2t+i)\beta(3t, 2t+i). \tag{3.48}$$

By Feller [v. 1, p. 170], the estimates (3.40) are valid for $|i| \leq t^{2/3-\varepsilon}$, so the tails can be ignored. These approximations reduce to a Riemann sum:

$$z(n) \sim \sqrt{\frac{3}{2}}\, \frac{1}{\pi} \frac{1}{\sqrt{t}} \sum_i e^{-5i^2/4t} \sim \sqrt{\frac{6}{5\pi t}}\,. \tag{3.49}$$ □

As a measure of the slowness of convergence of f and the accuracy of (3.47), let $n = 432^5 \simeq 1.5 \times 10^{13}$. Then $f(n-1) \simeq 1.8430$, $f(n) \simeq 2.1175$, so $z(n) \simeq .2745$, whereas $(6/(25\pi))^{1/2} \simeq .2764$.

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Received July 19, 1985. The second author was supported by a grant from the Deutsche Forschungsgemeinschaft. The fifth author was supported by the National Science Foundation, by an Alfred P. Sloan Fellowship, and by an Arnold O. Beckman Fellowship at the UIUC Center for Advanced Study.

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