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ANALYTICITY AND SPECTRAL DECOMPOSITIONS OF L^p FOR COMPACT ABELIAN GROUPS

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Let Γ be a dense subgroup of the real line $\mathbf{R}$. Endow Γ with the discrete topology, and let K be the dual group of Γ. Helson's classic theory uses the spectral representation in Stone's Theorem for unitary groups to establish and implement a one-to-one correspondence Φ_2 between the cocycles on K and the normalized simply invariant subspaces of $L^2(K)$. Using our recent extension of Stone's Theorem to UMD spaces, we generalize Helson's theory to $L^p(K)$, $1 < p < \infty$, by producing spectral decompositions of $L^p(K)$ which provide a correspondence analogous to Φ_2. In particular this approach shows that every normalized simply invariant subspace of $L^p(K)$ is the range of a bounded idempotent. However, unlike the situation in the L^2-setting, our spectral decompositions do not stem from a projection-valued measure. Instead they owe their origins to the Hilbert transform of $L^p(\mathbf{R})$. In the context of abstract UMD spaces, we develop the relationships between holomorphic semigroup extensions and the spectral decompositions of bounded one-parameter groups. The results are then applied to describe, in terms of generalized analyticity, the normalized simply invariant subspaces of $L^p(K)$.

More specifically, throughout what follows K will be a compact abelian group other than $\{0\}$ or the unit circle $\mathbf{T}$ such that the dual group of K is archimedean ordered. Equivalently, we shall require that K is the dual group of Γ, where Γ arises as a dense subgroup of the real line $\mathbf{R}$, and Γ is then endowed with the natural order of $\mathbf{R}$ and the discrete topology. For each $\lambda \in \Gamma$ we denote by χ_λ the corresponding character on K (evaluation at λ), and for each $t \in \mathbf{R}$ we let e_t be the element of K defined by $e_t(\lambda) = \exp(it\lambda)$ for all $\lambda \in \Gamma$. As is well-known, $t \to e_t$ is a continuous isomorphism of $\mathbf{R}$ onto a dense subgroup of K. For $1 < p < \infty$ we follow Helson in defining a *simply invariant subspace* of $L^p(K)$ to be a closed subspace M of $L^p(K)$ such that $\chi_\lambda M \subseteq M$ for all $\lambda > 0$, but for some $\alpha < 0$, $\chi_\alpha M$ is not a subset of M. A simply invariant subspace M is said to be *normalized* provided $M = \bigcap\{\chi_\lambda M\colon \lambda \in \Gamma,\ \lambda < 0\}$. The set of all normalized simply invariant subspaces of $L^p(K)$ will be denoted by $\mathscr{S}_p$. A *cocycle* on K is a Borel measurable function $A\colon \mathbf{R} \times K \to \mathbf{T}$ such that

$$A(t+u, x) = A(t, x)A(u, x + e_t), \quad \text{for } t \in \mathbf{R},\ u \in \mathbf{R},\ x \in K.$$

After identifying cocycles which are equal almost everywhere on $\mathbf{R} \times K$ (with respect to the product of Haar measures), we denote by $\mathscr{C}$ the collection of all cocycles on K. Helson's classic generalization of Beurling's Invariant Subspace Theorem establishes a natural one-to-one mapping Φ_2 of $\mathscr{C}$ onto $\mathscr{S}_2$. The construction of Φ_2 relies on the countably additive spectral decompositions of one-parameter unitary groups afforded by Stone's Theorem. Since $\mathscr{S}_p$ has a standard natural one-to-one correspondence with $\mathscr{S}_2$ (see Proposition (2.5) below), an injective mapping Φ_p of $\mathscr{C}$ onto $\mathscr{S}_p$ ensues from Φ_2. The main result shown below is that, in complete analogy with Helson's construction of Φ_2, the mapping Φ_p can be fashioned from spectral decompositions in $L^p(K)$ corresponding to isometric groups induced by cocycles. This new approach establishes that each $M \in \mathscr{S}_p$ is complemented in $L^p(K)$.

Our approach rests on a recent abstract generalization of Stone's Theorem [**3**, Theorems (5.5) and (5.16)] (see Theorem (2.1) below). If X is a Banach space possessing the unconditionality property for martingale differences (in particular, if X is a reflexive L^p-space for an arbitrary measure), this generalization provides that every uniformly bounded one-parameter group of operators on X is the Fourier-Stieltjes transform of a projection-valued function $\mathscr{E}(\cdot)$ defined on $\mathbf{R}$ with values in $\mathscr{B}(X)$, the algebra of bounded operators on X. The spectral decomposition $\mathscr{E}(\cdot)$ arises by transference of the classical Hibert transform (Theorem (2.1)–(iii), (v)) and, in contrast with the development of Φ_2 for $L^2(K)$ ([**11**, §3]), need not stem from a projection-valued measure.

After collecting the necessary background items in §2, we develop the implementation of Φ_p by spectral decompositions in §3 (Theorem (3.3) and Corollary (3.6)). Section 4 is concerned with holomorphic semigroup extensions of uniformly bounded one-parameter groups on UMD spaces. The results of §4 are used in §5 to generalize to $\mathscr{S}_p$ Helson's analytic description of the functions constituting $\Phi_2(A)$, $A \in \mathscr{C}$ (see Theorem (5.7)). In §6 we apply the projections onto elements of $\mathscr{S}_p$ obtained in §3 in order to describe a generalization of the "orthogonal complement" relationship in $L^2(K)$ between the invariant subspaces corresponding to a cocycle and its complex conjugate.

2. Preliminaries. In this section we assemble the tools needed for the sequel.

DEFINITION. Let Y be a Banach space, and let I denote the identity operator of Y. A *spectral family of projections in* Y is a uniformly bounded, projection-valued function $E(\cdot)\colon \mathbf{R} \to \mathscr{B}(Y)$ such that:

(i) $E(s)E(t) = E(t)E(s) = E(s)$ for $s \le t$;

(ii) $E(\cdot)$ is right-continuous at each point of $\mathbf{R}$ in the strong operator topology of $\mathscr{B}(Y)$;

(iii) $E(\cdot)$ has a strong left-hand limit (denoted $E(s^-)$) at each $s \in \mathbf{R}$;

(iv) $E(s) \to I$ (resp., $E(s) \to 0$) in the strong operator topology as $s \to +\infty$ (resp., $s \to -\infty$).

If there is a compact interval $[a, b]$ such that $E(s) = I$ for $s \ge b$ and $E(s) = 0$ for $s < a$, we say that $E(\cdot)$ is concentrated on $[a, b]$.

For a compact interval $[u, v]$ in $\mathbf{R}$, let $AC([u, v])$ be the Banach algebra of all complex-valued, absolutely continuous functions on $[u, v]$ under the norm $\|\cdot\|_{[u,v]}$ defined by

$$\|f\|_{[u,v]} = |f(v)| + \operatorname{var}(f,[u,v]),$$

where "var" denotes total variation. We shall require some aspects of the integration theory for an arbitrary spectral family of projections $E(\cdot)$ in Y [**6**, Chapter 17]. For each $f \in AC([u, v])$, the integral $\int_u^v f(\lambda)\, dE(\lambda)$ exists as a strong limit of Riemann-Stieltjes sums, and we define $\int_{[u,v]}^{\oplus} f(\lambda)\, dE(\lambda)$ by the equation

$$\int_{[u,v]}^{\oplus} f(\lambda)\, dE(\lambda) = f(u)E(u) + \int_u^v f(\lambda)\, dE(\lambda).$$

The mapping $f \to \int_{[u,v]}^{\oplus} f(\lambda)\, dE(\lambda)$ is an algebra homomorphism of $AC([u, v])$ into $\mathscr{B}(Y)$ such that

$$\left\| \int_{[u,v]}^{\oplus} f(\lambda)\, dE(\lambda) \right\| \le \|f\|_{[u,v]} \sup\{\|E(s)\| \colon s \in \mathbf{R}\}.$$

The Banach spaces X possessing the unconditionality property for martingale differences (written $X \in$ UMD) have been extensively studied, and are characterized in [**4**], [**5**] as those spaces for which the Hilbert kernel of $\mathbf{R}$ defines a bounded convolution operator on $L^p(\mathbf{R}, X)$ for some, and hence all, p in the range $1 < p < \infty$. The class UMD contains many of the classical reflexive spaces. In particular, for $1 < p < \infty$, the von Neumann-Schatten p-class and $L^p(\mu)$ (μ an arbitrary measure) are UMD spaces. Moreover, the UMD property is always inherited by subspaces, quotient spaces, and dual spaces. The following recent result [**3**, §5] will play a central role in our considerations (compare [**2**, Theorem (3.6)]).

(2.1) Stone's Theorem for UMD Spaces. *Let* $\{U_t\}$, $t \in \mathbf{R}$, *be a strongly continuous, one-parameter group of operators on a* UMD *space* X

such that $\sup\{\|U_t\|: t \in \mathbf{R}\} < \infty$. *Then*:

(i) *there is a unique spectral family* $\mathscr{E}(\cdot)$ (*called the Stone-type spectral family of* $\{U_t\}$) *such that*

$$U_t x = \lim_{u \to +\infty} \int_{-u}^{u} e^{it\lambda}\, d\mathscr{E}(\lambda) x, \quad \text{for } t \in \mathbf{R},\ x \in X;$$

(ii) *the domain* $\mathscr{D}(\mathscr{G})$ *of the infinitesimal generator* $\mathscr{G}$ *of* $\{U_t\}$ *consists of all* $x \in X$ *such that* $\lim_{a \to +\infty} \int_{-a}^{a} \lambda\, d\mathscr{E}(\lambda) x$ *exists, and*

$$\mathscr{G}x = i \lim_{a \to +\infty} \int_{-a}^{a} \lambda\, d\mathscr{E}(\lambda) x, \quad \text{for } x \in \mathscr{D}(\mathscr{G});$$

(iii) *for each* $s \in \mathbf{R}$,

$$(\pi i)^{-1} \int_{\delta < |t| < \delta^{-1}} t^{-1} e^{ist} U_{-t}\, dt$$

converges in the strong operator topology of $\mathscr{B}(X)$, *as* $\delta \to 0^+$, *to an operator* J_s;

(iv) $J_s = \mathscr{E}(s) + \mathscr{E}(s^-) - I$, *for all* $s \in \mathbf{R}$;

(v) $\mathscr{E}(s) = I + 2^{-1}(J_s - J_s^2)$, *for all* $s \in \mathbf{R}$.

Since its inception in [**11**], Helson's invariant subspace theory has been developed in $L^2(K)$ in terms of a companion notion to that of spectral family. In the Banach space setting this notion takes the following form.

DEFINITION. A *decreasing spectral function in* Y is a uniformly bounded, projection-valued function $Q(\cdot): \mathbf{R} \to \mathscr{B}(Y)$ such that:

(i) $Q(s)Q(t) = Q(t)Q(s) = Q(t)$ for $s \leq t$;

(ii) $Q(\cdot)$ is left-continuous at each point of $\mathbf{R}$ in the strong operator topology;

(iii) $Q(\cdot)$ has a strong right-hand limit $Q(s^+)$ at each $s \in \mathbf{R}$;

(iv) $Q(\lambda) \to 0$ (resp., $Q(\lambda) \to I$) in the strong operator topology as $\lambda \to +\infty$ (resp., $\lambda \to -\infty$).

If $E(\cdot)$ is a spectral family of projections in Y, it is obvious that the equation

$$Q_E(\lambda) = I - E(\lambda^-) \quad \text{for } \lambda \in \mathbf{R} \tag{2.2}$$

defines a decreasing spectral function $Q_E(\cdot)$, and that the pairing of $E(\cdot)$ and $Q_E(\cdot)$ is a one-to-one correspondence between the spectral families and the decreasing spectral functions in Y. Moreover, it is clear from the integration theory of spectral families that $\int_u^v f(\lambda)\, dQ_E(\lambda)$ exists as a strong limit of Riemann-Stieltjes sums for each compact interval $[u, v]$ and each $f \in AC([u, v])$. For later convenience we list here the following easy consequence of (2.1)–(i).

(2.3) COROLLARY. *Let $\{U_t\}$ satisfy the hypotheses of Theorem (2.1), and let $\mathscr{E}(\cdot)$ be the Stone-type spectral family of $\{U_t\}$. Then there is a unique decreasing spectral function $Q(\cdot)$ in X such that*

$$(2.4) \qquad U_t x = -\lim_{u \to +\infty} \int_{-u}^{u} e^{it\lambda}\, dQ(\lambda) x, \quad \text{for } t \in \mathbf{R},\ x \in X.$$

Moreover, $Q = Q_{\mathscr{E}}$.

We shall refer to the unique $Q(\cdot)$ in (2.4) as the *decreasing spectral resolution of $\{U_t\}$*.

In conjunction with Theorem (2.1) we shall make use of the following variant of [**9**, Theorem V.6.1]. This variant is readily deduced from [**9**] with the aid of the first lemma in [**12**, §1.6].

(2.5) PROPOSITION. *Suppose $1 < p \le q < \infty$. There is a one-to-one mapping $\theta_{p,q}$ of $\mathscr{S}_p$ onto $\mathscr{S}_q$ given by $\theta_{p,q}(M) = M \cap L^q(K)$ for $M \in \mathscr{S}_p$. For each $N \in \mathscr{S}_q$, the inverse image, $\theta_{p,q}^{-1}(N)$, is the closure of N in $L^p(K)$.*

3. **Invariant subspaces and spectral decompositions in $L^p(K)$.** For $A \in \mathscr{C}$ we put $A_t = A(t, \cdot)$ for each $t \in \mathbf{R}$. By [**9**, Lemma VII.12.1], as t runs through $\mathbf{R}$ the corresponding function A_t moves continuously in $L^r(K)$, $1 \le r < \infty$. For $1 < p < \infty$ and $t \in \mathbf{R}$, the translation operator on $L^p(K)$ corresponding to e_t will be denoted by $R_t^{(p)}$, and we define $U_t^{(A,p)}$ by setting

$$U_t^{(A,p)} f = A_t R_t^{(p)} f, \quad \text{for } f \in L^p(K).$$

Thus $\{U_t^{(A,p)}\}$, $t \in \mathbf{R}$, is a strongly continuous one-parameter group of isometries on $L^p(K)$. We write $\mathscr{E}^{(A,p)}(\cdot)$ (resp., $Q^{(A,p)}(\cdot)$) for the Stone-type spectral family (resp., the decreasing spectral resolution) of the group $\{U_t^{(A,p)}\}$. With this notation Helson's classic one-to-one map of $\mathscr{C}$ onto $\mathscr{S}_2$, here denoted Φ_2, takes the following form

$$\Phi_2(A) = \{Q^{(A,2)}(0)\} L^2(K), \quad \text{for } A \in \mathscr{C}.$$

(3.1) LEMMA. *Suppose $1 < p \le q < \infty$, and $A \in \mathscr{C}$. Then for each $\lambda \in \mathbf{R}$, $\mathscr{E}^{(A,p)}(\lambda) \mid L^q(K)$, the restriction of $\mathscr{E}^{(A,p)}(\lambda)$ to $L^q(K)$, coincides with $\mathscr{E}^{(A,q)}(\lambda)$, and $Q^{(A,p)}(\lambda) \mid L^q(K)$ coincides with $Q^{(A,q)}(\lambda)$.*

Proof. It suffices to obtain the first conclusion, which follows readily from Theorem (2.1)–(iii), (v).

Proposition (2.5) allows us to make the following definition.

(3.2) DEFINITION. For $1 < p < \infty$ we define a one-to-one map Φ_p of $\mathscr{C}$ onto $\mathscr{S}_p$ by putting $\Phi_p = \theta_{2,p} \circ \Phi_2$ (resp., $\Phi_p = \theta_{p,2}^{-1} \circ \Phi_2$) if $2 \leq p$ (resp., $p \leq 2$), where "$\circ$" denotes composition of mappings.

REMARK. It follows from the existence of non-trivial cocycles [**12**, §4.3] that not all elements of $\mathscr{S}_p$ have the obvious form—in contrast to the state of affairs described in Beurling's Theorem for $L^p(\mathbf{T})$ ([**10**, §IV.1]).

We are now in a position to establish the central result of the paper.

(3.3) THEOREM. *If* $A \in \mathscr{C}$ *and* $1 < p < \infty$, *then*

$$\Phi_p(A) = \{Q^{(A,p)}(0)\}L^p(K).$$

Proof. Put $W = \Phi_p(A)$. Suppose first that $p < 2$. Then W is the closure in $L^p(K)$ of $\Phi_2(A) = \{Q^{(A,2)}(0)\}L^2(K)$. Thus by Lemma (3.1),

$$W = p\text{-cl} \cdot \{Q^{(A,p)}(0)\}L^2(K), \tag{3.4}$$

where "p-cl." denotes closure in $L^p(K)$. Hence $W \subseteq \{Q^{(A,p)}(0)\}L^p(K)$. To obtain the reverse inclusion first observe that we can carry over *mutatis mutandis* the argument for $p = 2$ [**12**, pg. 22] to show that $\{Q^{(A,p)}(0)\}L^p(K)$ belongs to $\mathscr{S}_p$. Hence it is the closure in $L^p(K)$ of its intersection with $L^\infty(K)$ [**12**, pg. 12]. Thereafter it suffices to apply equation (3.4). Suppose next that $2 < p$. Then $W = \theta_{2,p}(\Phi_2(A))$, and so

$$W = [\{Q^{(A,2)}(0)\}L^p(K)] \cap L^p(K). \tag{3.5}$$

If $f \in W$, then $f = Q^{(A,2)}(0)g$, for some $g \in L^2(K)$. Hence $Q^{(A,2)}(0)f = f$. By Lemma (3.1), $Q^{(A,p)}(0)f = f$. Thus

$$W \subseteq \{Q^{(A,p)}(0)\}L^p(K).$$

Since

$$\{Q^{(A,p)}(0)\}L^p(K) = \{Q^{(A,2)}(0)\}L^p(K) \subseteq \{Q^{(A,2)}(0)\}L^2(K),$$

reference to (3.5) serves to complete the proof.

(3.6) COROLLARY. *Let $M \in \mathscr{S}_p$, $1 < p < \infty$. Then:*

(i) *if $p \le 2$, the self-adjoint projection of $L^2(K)$ onto $M \cap L^2(K)$ has a unique extension to an idempotent $F_M \in \mathscr{B}(L^p(K))$, and $M = F_M(L^p(K))$;*

(ii) *if $2 \le p$, the self-adjoint projection of $L^2(K)$ onto the closure of M in $L^2(K)$ has for its restriction to $L^p(K)$ an idempotent $G_M \in \mathscr{B}(L^p(K))$ such that $M = G_M(L^p(K))$.*

Proof. By Theorem (3.3) and Lemma (3.1).

4. Holomorphic semigroup extensions. Let $A \in \mathscr{C}$. In [**11**, §6], [**12**, Theorem 17] the invariant subspace $\{Q^{(A,2)}(0)\}L^2(K)$ is shown to consist of all $f \in L^2(K)$ such that, for almost all $x \in K$, $t \to (U_t^{(A,2)}f)(x)$ is the boundary function of a suitable analytic function in the upper half-plane. The proof employed relies crucially on the fact that $Q^{(A,2)}(\cdot)$ induces a projection-valued measure in $L^2(K)$, and thus has not produced an analogue for the classes $\mathscr{S}_p$, $1 < p < \infty$. In order to obtain such an analogous characterization for $\mathscr{S}_p$, a goal accomplished in Theorem (5.7), we develop in this section the abstract machinery of holomorphic semigroup extensions for uniformly bounded one-parameter groups on UMD spaces. The results obtained are of independent interest, and formally resemble the corresponding facts for unitary groups. However, in contrast to the unfettered integration of bounded measurable functions against spectral measures, the general theory of Riemann-Stieltjes integration with respect to a spectral family only guarantees the existence of integrals over an unbounded interval for integrands having bounded variation on the whole interval. This fact separates the treatment in the present section from its counterpart for unitary groups.

Throughout this section we assume that $\{U_t\}$ is a strongly continuous, uniformly bounded, one-parameter group of operators acting on a UMD space X, and we utilize the notation of Theorem (2.1). By [**7**, Theorem VIII.1.11], $\Lambda(\mathscr{G})$, the spectrum of $\mathscr{G}$, is a subset of $i\mathbf{R}$. If $\mathscr{M}$ is a subspace of X invariant under $\{U_t\}$, then (2.1)–(iii), (v) show that $\mathscr{E}(\cdot)|\mathscr{M}$ is the Stone-type spectral family of the group $\{U_t\,|\,\mathscr{M}\}$.

(4.1) LEMMA. *Let $b \in \mathbf{R}$. Then $\mathscr{E}(\lambda) = 0$ for $\lambda < b$ if and only if*

$$\Lambda(\mathscr{G}) \subseteq \{i\lambda\colon \lambda \ge b\}. \tag{4.2}$$

Proof. Firstly, let $\alpha \in \mathbf{R}$ and suppose that $\mathscr{E}(\alpha + \varepsilon) \ne \mathscr{E}(\alpha - \varepsilon)$ for all $\varepsilon > 0$. Let f_ε belong to the range of $\{\mathscr{E}(\alpha + \varepsilon) - \mathscr{E}(\alpha - \varepsilon)\}$ with

$\|f_\varepsilon\| = 1$. A simple calculation shows that

$$(\mathcal{G} - i\alpha)f_\varepsilon = i\int_{\alpha-\varepsilon}^{\alpha+\varepsilon}(\lambda - \alpha)\,d\mathcal{E}(\lambda)f_\varepsilon \to 0$$

as $\varepsilon \to 0^+$, and so $(i\alpha) \in \Lambda(\mathcal{G})$. Thus (4.2) implies that $\mathcal{E}(\lambda)$ is constant, and hence 0, on $(-\infty, b)$.

To prove the converse result, suppose that $\mathcal{E}(\lambda) = 0$ for $\lambda < b$. Let $\mu \in (-\infty, b)$ and choose β so that $\mu < \beta < b$. It is easy to see that the sequence of integrals

$$\int_\beta^n (\mu - \lambda)^{-1}\,d\mathcal{E}(\lambda) \tag{4.3}$$

converges in the uniform operator topology as $n \to \infty$. Denoting the limit of the sequence in (4.3) by $\mathcal{R}_\mu$, we have from standard arguments $\mathcal{R}_\mu(i\mu - \mathcal{G})x = ix$, for $x \in \mathcal{D}(\mathcal{G})$, and $(i\mu - \mathcal{G})\mathcal{R}_\mu x = ix$, for $x \in X$. Hence $(i\mu) \notin \Lambda(\mathcal{G})$, and Lemma (4.1) is established.

We shall denote the upper half-plane $\operatorname{Im} z > 0$ by Π^+.

(4.4) THEOREM. *The uniformly bounded one-parameter group* $\{U_t\}$ *on the* UMD *space X can be extended to a strongly continuous semigroup* $\{U_z\}$, $\operatorname{Im} z \geq 0$, *such that* $\{U_z\}$ *is holomorphic on* Π^+ *if and only if there is a real number b such that* $\Lambda(\mathcal{G}) \subseteq \{i\lambda\colon \lambda \geq b\}$. *If this is the case, then the semigroup* $\{U_z\}$, $\operatorname{Im} z \geq 0$, *is uniquely determined, and is given by*:

$$U_z\alpha = \lim_{u\to+\infty}\int_{-u}^{u} e^{i\lambda z}\,d\mathcal{E}(\lambda)\alpha, \quad \text{for } \alpha \in X, \operatorname{Im} z \geq 0.$$

Proof. Suppose first that such a semigroup extension $\{U_z\}$, $\operatorname{Im} z \geq 0$, exists. By [**13**, Theorem 17.9.2], $i\mathcal{G}_0 = -\mathcal{G}$, where $\mathcal{G}_0$ is the infinitesimal generator of the strongly continuous one-parameter semigroup $\{U_{it}\}$, $t \geq 0$. Since $\mathcal{G}_0$ generates a semigroup, there is a real number c such that $\operatorname{Re}\Lambda(\mathcal{G}_0) \leq c$, and so $\operatorname{Im}\Lambda(\mathcal{G}) \geq -c$.

Conversely, suppose $b \in \mathbf{R}$, and $\Lambda(\mathcal{G}) \subseteq \{i\lambda\colon \lambda \geq b\}$. By Lemma (4.1), $\mathcal{E}(\lambda) = 0$ for $\lambda < b$. Applying the last assertion of [**1**, Corollary (4.14)] to this, we see that there is a strongly continuous, one-parameter semigroup $\{S_t\}$, $t \geq 0$, such that

$$S_t\alpha = \lim_{u\to+\infty}\int_{-u}^{u} e^{-\lambda t}\,d\mathcal{E}(\lambda)\alpha, \quad \text{for } t \geq 0,\ \alpha \in X. \tag{4.5}$$

For $z = x + iy$, where $x \in \mathbf{R}$, $y \geq 0$, we define U_z by putting

$$U_z = U_x S_y. \tag{4.6}$$

It is clear that $\{U_z\}$, $\operatorname{Im} z \geq 0$, is a strongly continuous semigroup which extends the given group, $\{U_t\}$, $t \in \mathbf{R}$. We observe that for each $t \geq 0$, $\sup\{\|\int_{-u}^{u} e^{-\lambda t}\, d\mathscr{E}(\lambda)\|: u > 0\} < \infty$. Moreover, the Principle of Uniform Boundedness, together with (2.1)–(i), shows that for each $t \in \mathbf{R}$, $\sup\{\|\int_{-u}^{u} e^{it\lambda}\, d\mathscr{E}(\lambda)\|: u > 0\} < \infty$. Application of these last two observations to (4.6) gives us

$$(4.7) \qquad U_z\alpha = \lim_{u \to +\infty} \int_{-u}^{u} e^{i\lambda z}\, d\mathscr{E}(\lambda)\alpha, \quad \text{for } \alpha \in X, \operatorname{Im} z \geq 0.$$

Suppose now that $\alpha \in X$, and $\varphi \in X^*$, the dual space of X. From (4.7) and an integration by parts, we have for $z \in \Pi^+$

$$(4.8) \qquad \langle U_z\alpha, \varphi\rangle = -iz \int_b^{+\infty} \langle \mathscr{E}(\lambda)\alpha, \varphi\rangle e^{i\lambda z}\, d\lambda,$$

the integral on the right of (4.8) existing as a Lebesgue integral. An application of Morera's Theorem shows that this integral is an analytic function of z on Π^+. To complete the proof of Theorem (4.4), it remains only to establish the uniqueness assertion. Suppose, then, that $\{U_z\}$ and $\{V_z\}$, $\operatorname{Im} z \geq 0$, are two semigroup extensions for $\{U_t\}$, $t \in \mathbf{R}$, as in the statement of the theorem. For $\alpha \in X$, $\varphi \in X^*$, the function g defined by

$$g(z) = \langle U_z\alpha, \varphi\rangle - \langle V_z\alpha, \varphi\rangle$$

is continuous on $\operatorname{Im} z \geq 0$, and analytic on Π^+. Since g vanishes on $\mathbf{R}$, the Schwarz Reflection Principle shows that g vanishes identically.

(4.9) Corollary. *Suppose that the uniformly bounded one-parameter group $\{U_t\}$ on the* UMD *space X satisfies the equivalent conditions of Theorem* (4.4). *Then* $\sup\{\|U_z\|: \operatorname{Im} z \geq 0\} < \infty$ *if and only if* $\mathscr{E}(\lambda) = 0$ *for* $\lambda < 0$.

Proof. Suppose first that $\mathscr{E}(\lambda) = 0$ for $\lambda < 0$. By (4.6) it suffices to show that $\sup\{\|S_t\|: t \geq 0\} < \infty$. Using (4.5) in the present situation, we have

$$S_t = \lim_{u \to +\infty} \int_{[0,u]}^{\oplus} e^{-\lambda t}\, d\mathscr{E}(\lambda), \quad \text{for } t \geq 0,$$

the limit being taken in the strong operator topology. But for $t \geq 0$, $u > 0$, the norm of the function $e^{-(\cdot)t}$ in $AC([0, u])$ is 1.

Conversely suppose $\sup\{\|U_z\|: \operatorname{Im} z \geq 0\} < \infty$. Then $\operatorname{Re}\Lambda(\mathscr{G}_0) \leq 0$, where $\mathscr{G}_0$ is the infinitesimal generator of the semigroup $\{U_{it}\}$, $t \geq 0$. As mentioned at the outset of the proof of Theorem (4.4), $-\mathscr{G} = i\mathscr{G}_0$. Combining these facts, we see that $\Lambda(\mathscr{G}) \subseteq \{i\lambda: \lambda \geq 0\}$. Application of Lemma (4.1) completes the proof.

REMARKS. All the discussion in this section continues to be valid if the uniformly bounded, strongly continuous, one-parameter group $\{U_t\}$ on the UMD space X is replaced by a one-parameter group satisfying the hypotheses of [**1**, Theorem (4.20)] on an arbitrary Banach space. However, we shall not need this additional generality or the technical tools required for it.

5. Analytic characterization of invariant subspaces in $L^p(K)$. For $A \in \mathscr{C}$ and $f \in L^p(K)$, $1 < p < \infty$, let $\mathscr{M}(A, p, f)$ be the closed linear span in $L^p(K)$ of $\{U_t^{(A,p)}f\colon t \in \mathbf{R}\}$. (We shall abbreviate $\mathscr{M}(A, p, f)$ by writing $\mathscr{M}_f$.) Obviously $\mathscr{M}_f$ is the closed linear manifold in $L^p(K)$ spanned by $\{\mathscr{E}^{(A,p)}(s)f\colon s \in \mathbf{R}\}$. We shall denote by $\{U_t^{(A,p,f)}\}$ the restriction of the group $\{U_t^{(A,p)}\}$, $t \in \mathbf{R}$, to $\mathscr{M}_f$.

As noted earlier, the main result of this section, Theorem (5.7), generalizes to $\mathscr{S}_p$ Helson's result for $\mathscr{S}_2$. It will be convenient to observe at the outset that, by virtue of the relation $Q^{(A,p)}(0) = I - \mathscr{E}^{(A,p)}(0^-)$, we have:

$$\text{for all } f \in L^p(K),\ f \in \Phi_p(A) \text{ if and only if} \quad \mathscr{E}^{(A,p)}(\lambda)f = 0 \text{ for all } \lambda < 0. \tag{5.1}$$

As a preliminary step we use (5.1) to establish the following characterization of $\mathscr{S}_p$ by vector-valued analyticity.

(5.2) THEOREM. *Suppose $A \in \mathscr{C}$ and $f \in L^p(K)$, $1 < p < \infty$. Then $f \in \Phi_p(A)$ if and only if the one-parameter group $\{U_t^{(A,p,f)}\}$ has an extension to a semigroup of operators belonging to $\mathscr{B}(\mathscr{M}_f)$, $\{U_z^{(A,p,f)}\}$,* $\operatorname{Im} z \geq 0$, *such that $\{U_z^{(A,p,f)}\}$ is holomorphic on Π^+, uniformly bounded on* $\operatorname{Im} z \geq 0$, *and continuous with respect to the strong operator topology of $\mathscr{B}(\mathscr{M}_f)$ on* $\operatorname{Im} z \geq 0$.

Proof. Since the Stone-type spectral family of the group $\{U_t^{(A,p,f)}\}$ is $\mathscr{E}^{(A,p)}(\cdot)\,|\,\mathscr{M}_f$, the desired logical equivalence follows from (5.1) together with Lemma (4.1) and Corollary (4.9).

We shall also require some classical technical machinery (in analogy with [**12**, §3.1]) which is described in (5.3)–(5.6). Let L be the linear fractional transformation given by $L(z) \equiv i(1 + z)(1 - z)^{-1}$. Thus L is the standard conformal mapping of the unit disc $\mathbf{D}$ onto Π^+. We shall follow [**8**], [**14**] for basic facts concerning the usual Hardy spaces $H^p(\mathbf{D})$, $H^p(\Pi^+)$. $H^p(\mathbf{R})$ will denote the non-tangential boundary functions (identified modulo equality a.e.) corresponding to the class $H^p(\Pi^+)$. Let $\mathscr{T}_p$ be the linear space of all complex-valued functions ψ on Π^+ such that $\psi \circ L \in H^p(\mathbf{D})$.

(5.3) PROPOSITION. *Suppose* $1 < p < \infty$, *and* ψ *is a complex-valued function on* Π^+. *The following are equivalent.*

(i) $\psi \in \mathcal{T}_p$;

(ii) $\psi(z)(1 - iz)^{-2/p}$ *belongs to* $H^p(\Pi^+)$;

(iii) $\psi(z)$ *is analytic on* Π^+, $\psi(z)$ *has a non-tangential limit* $\psi(t)$ *for almost all* $t \in \mathbf{R}$, *and* $\psi(t)(1 - it)^{-2/p}$ *belongs to* $H^p(\mathbf{R})$.

If the conditions (i)–(iii) *hold, then the function* $\psi(z)(1 - iz)^{-2/p}$ *on* Π^+ *is the Poisson integral of the function* $\psi(t)(1 - it)^{-2/p}$ *belonging to* $H^p(\mathbf{R})$.

Proof. The equivalence of (i) and (ii), as well as the implication (ii) $\Rightarrow$ (iii) are standard facts about Hardy spaces. The remaining assertions of the proposition follow from Privalov's Uniqueness Theorem [**15**, pg. 212].

DEFINITION. For $1 < p < \infty$, let $B_p(\mathbf{R})$ be the linear space of all complex-valued functions $G(t)$ on $\mathbf{R}$ (identified modulo equality almost everywhere) such that $G(t)(1 - it)^{-2/p}$ belongs to $H^p(\mathbf{R})$.

We restate Proposition (5.3) in the following convenient form.

(5.4) PROPOSITION. *Suppose* $1 < p < \infty$. *The correspondence which sends each* $\varphi \in \mathcal{T}_p$ *to its non-tangential boundary function is a one-to-one linear mapping of* $\mathcal{T}_p$ *onto* $B_p(\mathbf{R})$. *The inverse mapping is given by*

$$G(t) \in B_p(\mathbf{R}) \to F(z)(1 - iz)^{2/p},$$

where $F(z)$ *is the Poisson integral in* Π^+ *of* $G(t)(1 - it)^{-2/p}$.

With the aid of the M. Riesz projection for $L^p(\mathbf{R})$ and a version of the Paley-Wiener Theorem for $H^1(\mathbf{R})$ [**8**, Theorem 11.10], it is not difficult to obtain the following generalization of [**12**, Lemma, pg. 29].

(5.5) LEMMA. *Suppose* $1 < p < \infty$, *and* $p^{-1} + q^{-1} = 1$. *Let*

$$F \in L^p\big(\mathbf{R}, (1 + t^2)^{-1}\, dt\big).$$

Then $F \in B_p(\mathbf{R})$ *if and only if*

$$\int_{\mathbf{R}} F(t) G(t)(1 - it)^{-2}\, dt = 0, \quad \textit{for all } G \in B_q(\mathbf{R}).$$

Let F be a Borel function on $\mathbf{R} \times K$ such that

$$F \in L^p\big((1 + t^2)^{-1}\, dt \times d\sigma\big),$$

where σ denotes the normalized Haar measure of K. In particular, for σ-almost all x, $F(\cdot, x) \in L^p(\mathbf{R}, (1 + t^2)^{-1}\, dt)$, and, for almost all $t \in \mathbf{R}$,

$F_t \equiv F(t, \cdot)$ belongs to $L^p(\sigma)$. For $g \in L^q(\sigma)$, where $p^{-1} + q^{-1} = 1$, put

$$\langle F_t, g\rangle = \int_K F(t,x)g(x)\,d\sigma(x).$$

It is easy to see that $\langle F_t, g\rangle \in L^p(\mathbf{R}, (1+t^2)^{-1}\,dt)$.

By using Lemma (5.5) in analogy with the reasoning of [**12**, pg. 30], we obtain the following result.

(5.6) LEMMA. *Suppose* $1 < p < \infty$, *and* $p^{-1} + q^{-1} = 1$. *Let* F *be a Borel function on* $\mathbf{R} \times K$ *such that* $F \in L^p((1+t^2)^{-1}\,dt \times d\sigma)$. *Then the following are equivalent*:

(i) *for each* $g \in L^q(\sigma)$, $\langle F_t, g\rangle \in B_p(\mathbf{R})$;

(ii) *for* σ-*almost all* x, $F(\cdot, x) \in B_p(\mathbf{R})$.

The stage is now set for the main result of this section, the generalization to $L^p(K)$ of Helson's analyticity criterion in $L^2(K)$ [**11**, §6].

(5.7) THEOREM. *Suppose* $1 < p < \infty$, f *is a Borel function in* $L^p(K)$, *and* $A \in \mathscr{C}$. *Then* $f \in \Phi_p(A)$ *if and only if for* σ-*almost all* x *in* K *the function of* $t \in \mathbf{R}$, $A(t,x)f(x+e_t)$ *belongs to* $B_p(\mathbf{R})$.

Proof. Put $F(t,x) = A(t,x)f(x+e_t)$, for $t \in \mathbf{R}$, $x \in K$. Obviously F is a Borel function in $L^p((1+t^2)^{-1}\,dt \times d\sigma)$. Suppose first that $f \in \Phi_p(A)$. Then the group $\{U_t^{(A,p,f)}\}$, $t \in \mathbf{R}$, has the holomorphic extension $\{U_z^{(A,p,f)}\}$ described in Theorem (5.2). Let $g \in L^q(K)$, where $p^{-1} + q^{-1} = 1$, and let γ be the linear functional on $\mathscr{M}_f$ given by integration against $g\,d\sigma$. Thus

$$\langle F_t, g\rangle = \langle U_t^{(A,p,f)}f, \gamma\rangle, \quad \text{for } t \in \mathbf{R}.$$

Hence $\langle F_t, g\rangle$ is the boundary function of the bounded holomorphic function $\psi \equiv \langle U_z^{(A,p,f)}f, \gamma\rangle$ on Π^+. In particular, $\psi \in \mathscr{T}_p$. By Proposition (5.4), $\langle F_t, g\rangle \in B_p(\mathbf{R})$. Application of Lemma (5.6) gives the desired conclusion.

Conversely, suppose that for σ-almost all x, $F(\cdot, x) \in B_p(\mathbf{R})$. Then for each $g \in L^q(K)$, $\langle U_t^{(A,p)}f, g\rangle$ belongs to $B_p(\mathbf{R})$. If $\lambda \in \mathbf{R}$, then

$$\langle U_t^{(A,p)}\mathscr{E}^{(A,p)}(\lambda)f, g\rangle \equiv \langle U_t^{(A,p)}f, \{\mathscr{E}^{(A,p)}(\lambda)\}^* g\rangle$$

belongs to $B_p(\mathbf{R})$. It is now easy to see that $\langle U_t^{(A,p,f)}\alpha, \varphi\rangle$ belongs to $B_p(\mathbf{R})$ for each $\alpha \in \mathscr{M}_f$, $\varphi \in \mathscr{M}_f^*$. Hence $\langle U_t^{(A,p,f)}\alpha, \varphi\rangle$ is the boundary function of a unique $\psi_{\alpha,\varphi} \in \mathscr{T}_p$. By [**8**, Theorem 2.11],

$$\sup\left\{|\psi_{\alpha,\varphi}(z)| : z \in \Pi^+\right\} \le \|\alpha\|\,\|\varphi\|.$$

Clearly $\psi_{\alpha,\varphi}$ is linear in each of α and φ separately. It follows from these considerations, since $\mathcal{M}_f$ is a reflexive space, that for each $z \in \Pi^+$ there is a unique $V_z \in \mathcal{B}(\mathcal{M}_f)$ such that $\psi_{\alpha,\varphi}(z) = \langle V_z\alpha, \varphi\rangle$ for all $\alpha \in \mathcal{M}_f$, $\varphi \in \mathcal{M}_f^*$. Moreover, $\|V_z\| \leq 1$ for $z \in \Pi^+$. We define V_t for $t \in \mathbf{R}$ by setting $V_t = U_t^{(A,p,f)}$. Thus, for fixed $\alpha \in \mathcal{M}_f$, $\varphi \in \mathcal{M}_f^*$, the function (of $t \in \mathbf{R}$) $\langle V_t\alpha, \varphi\rangle$ is the non-tangential boundary function of the holomorphic function $\langle V_z\alpha, \varphi\rangle$ on Π^+. By Proposition (5.4),

$$\langle V_z\alpha, \varphi\rangle = W(z)(1 - iz)^{2/p} \quad \text{for } z \in \Pi^+,$$

where $W(z)$ is the Poisson integral of $\langle V_t\alpha, \varphi\rangle(1 - it)^{-2/p}$. Since the latter function is continuous on $\mathbf{R}$, we infer that $\langle V_z\alpha, \varphi\rangle$ is continuous on $\operatorname{Im} z \geq 0$. For fixed $s \in \mathbf{R}$, the functions $\langle V_z\alpha, V_s^*\varphi\rangle$ and $\langle V_{z+s}\alpha, \varphi\rangle$ are bounded and continuous on $\operatorname{Im} z \geq 0$, holomorphic on Π^+, and equal on $\mathbf{R}$. Hence

$$V_{z+s} = V_sV_z \quad \text{for } s \in \mathbf{R}, \operatorname{Im} z \geq 0.$$

For fixed z_0 such that $\operatorname{Im} z_0 \geq 0$, similar reasoning applied to the functions $\langle V_{z+z_0}\alpha, \varphi\rangle$ and $\langle V_zV_{z_0}\alpha, \varphi\rangle$ now shows that $\{V_z\}$, $\operatorname{Im} z \geq 0$, is a semigroup (which we have already seen to be continuous in the weak operator topology). It follows that $\{V_{iy}\}$, $y \geq 0$, is a strongly continuous semigroup. Thus $V_z \equiv V_xV_{iy}$ is strongly continuous on $\operatorname{Im} z \geq 0$. By Theorem (5.2), $f \in \Phi_p(A)$.

6. The invariant subspace corresponding to the complex conjugate cocycle. Let $A \in \mathscr{C}$ and put $M = \Phi_2(A)$. $\bar{A}$, the complex conjugate of A, is also a cocycle on K. In [**12**, Theorem 18] it is shown that $\Phi_2(\bar{A})$ is the "normalization" of $\overline{(M^\perp)}$. Since there is no orthogonal complementation operation which maps the set of all subspaces of $L^p(K)$ into itself, this result for L^2 has hitherto lacked an analogue in $\mathscr{S}_p$. Since each element of $\mathscr{S}_p$ is the range of a canonical projection (Theorem (3.3)), an analogue can now be obtained (Corollary (6.3) below). We fix $A \in \mathscr{C}$ and $p \in (1, +\infty)$. Straightforward calculations using the formulas of Theorem (2.1), (iii)–(v) show that

$$\text{(6.1)} \qquad \overline{\mathscr{E}^{(\bar{A},p)}(s)f} = Q^{(A,p)}(-s)\bar{f}, \quad \text{for } f \in L^p(K),\ s \in \mathbf{R}.$$

Subtracting both sides of (6.1) from $\bar{f}$, and letting $s \to 0^-$, we obtain the following theorem.

(6.2) THEOREM. *Let $A \in \mathscr{C}$, and suppose $1 < p < \infty$. Then*

$$Q^{(\bar{A},p)}(0)f = \overline{\{I - Q^{(A,p)}(0^+)\}\bar{f}}, \quad \text{for } f \in L^p(K).$$

Hence by Theorem (3.3) we obtain the desired description of $\Phi_p(\bar{A})$.

(6.3) COROLLARY. *Under the hypotheses of Theorem* (6.2),

$$\Phi_p(\bar{A}) = \overline{\{I - Q^{(A,p)}(0^+)\} L^p(K)}.$$

REFERENCES

[1] H. Benzinger, E. Berkson, and T. A. Gillespie, *Spectral families of projections, semigroups, and differential operators*, Trans. Amer. Math. Soc., **275** (1983), 431–475.

[2] E. Berkson and T. A. Gillespie, *The existence of spectral decompositions in L^p-subspaces*, Proc. Japan Acad., **61A** (1985), 172–175.

[3] E. Berkson, T. A. Gillespie, and P. S. Muhly, *Abstract spectral decompositions guaranteed by the Hilbert transform*, Proc. London Math. Soc., to appear.

[4] J. Bourgain, *Some remarks on Banach spaces in which martingale difference sequences are unconditional*, Arkiv for Matematik, **21** (1983), 163–168.

[5] D. L. Burkholder, *A geometric condition that implies the existence of certain singular integrals of Banach-space-valued functions*. Proceedings of Conference on Harmonic Analysis in Honor of Antoni Zygmund (Chicago, 1981), ed. by W. Beckner et al., Wadsworth, 1983.

[6] H. R. Dowson, *Spectral theory of linear operators*, London Math. Soc. Monographs, No. 12, Academic Press, New York, 1978.

[7] N. Dunford and J. T. Schwartz, *Linear operators*, I. *General theory*, Pure and Applied Math., vol. 7, Interscience, New York, 1958.

[8] P. L. Duren, *Theory of H^p Spaces*, Pure and Applied Math., **38**, Academic Press, New York, 1970.

[9] T. W. Gamelin, *Uniform Algebras*, Prentice-Hall, Englewood Cliffs, N. J., 1969.

[10] H. Helson. *Lectures on Invariant Subspaces*, Academic Press, New York, 1964.

[11] ______, *Compact groups with ordered duals*, Proc. London Math. Soc., (3) **14A** (1965), 144–156.

[12] ______, *Analyticity on compact abelian groups*. Algebras in Analysis, Proceedings of 1973 Birmingham Conference—NATO Advanced Study Institute, Academic Press, London, 1975, 1–62.

[13] E. Hille and R. S. Phillips, *Functional analysis and semi-groups*. Amer. Math. Soc. Colloq. Publ., vol. **31**, Amer. Math. Soc., Providence, R. I., 1957.

[14] K. Hoffman, *Banach Spaces of Analytic Functions*, Prentice-Hall, Englewood Cliffs, N. J., 1962.

[15] I. I. Privalov, *Randeigenschaften analytischer Funktionen*, Deutscher Verlag der Wissenschaften, Berlin, 1956.

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