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MAXIMAL FUNCTIONS ON THE UNIT n-SPHERE

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It is shown that the Hardy-Littlewood maximal function on the unit sphere in n-space is weak-type (1, 1) with a weak-type constant cn where c is independent of n.

Introduction. E. M. Stein and J. O. Strömberg [6] have shown that the Hardy-Littlewood maximal function in $\mathbf{R}^n$ is weak-type (1, 1) with a weak-type constant cn with c independent of n. Their approach is to pointwise bound the maximal function by a supremum of averages of members of a certain heat-diffusion semi-group on $\mathbf{R}^n$. They then apply the Hopf abstract maximal ergodic theorem to obtain their result.

We plan to use an analogous version of this approach to show that the maximal function on the unit n-sphere is weak-type (1, 1) with a weak-type constant cn. The best weak-type constant prior to this was $cn\sqrt{n}$, see [4], using an entirely different approach.

Many of the ideas in this paper have already been presented in a paper by C. Herz [3]. In order to obtain the weak-type constant cn, sharper estimates are required than are indicated in Herz's paper. Furthermore, there is an oversight of a primarily technical nature which led this author to perform some contortions to rectify. It should be pointed out that Herz's overall approach applies not only to the unit sphere in $\mathbf{R}^n$ and $\mathbf{R}^n$ itself, but to more general spaces as well.

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NOTATION AND DEFINITIONS. Let S^{n-1} denote the unit sphere in $\mathbf{R}^n$ centered at the origin. Let ω_{n-1} denote its Lebesgue measure (surface area). Let $v(x,t) = T^t f(x)$ be the solution to the initial value problem $\partial v/\partial t = \Delta_S v$ and $v(x,0) = f(x)$ where Δ_S is the spherical Laplacian; that is, the "angular" part of the Laplacian in $\mathbf{R}^n$. If there is any confusion on the reader's part, Δ_S is defined precisely in the proof of Lemma 2 in equation (12).

Define the maximal heat function of f to be

$$M_T f(x) = \sup_{\lambda>0}\left|\frac{1}{\lambda}\int_0^\lambda T^\mu f(x)\,d\mu\right|.$$

The Poisson kernel for the unit ball in $\mathbf{R}^n$ is

$$P(rx, y) = \frac{1 - r^2}{\omega_{n-1}|rx - y|^n}$$

with $x, y \in S^{n-1}$ and $0 \le r \le 1$. If $f \in L^1(S^{n-1})$, then

$$u(rx) = \int_{S^{n-1}} P(rx, y) f(y)\, dy \tag{1}$$

defines an harmonic function in the unit ball where dy is Lebesgue measure on the unit sphere. The Hardy-Littlewood maximal function of f on S^{n-1} is

$$Mf(x) = \sup_{0 \le t \le 2} \int_{S^{n-1}} \frac{1}{|S(x,t)|} \chi_{S(x,t)}(y) |f(y)|\, dy \tag{2}$$

where $S(x,t) = \{ y \in S^{n-1}\colon |x - y| \le t$ and $\chi_{S(x,t)}$ is the characteristic function of the set $S(x,t)$.

The symbol c will stand for a positive constant that may be different at different appearances but will always be less than 10^6.

THEOREM. *If $f \in L^1(S^{n-1})$ and $n \ge 3$, then*

$$|\{x \in S^{n-1}\colon Mf(x) > \lambda\}| \le \frac{cn}{\lambda} \|f\|_1 \tag{3}$$

for all $\lambda > 0$.

It is enough to prove (3) for Schwartz functions g such that $|g^{(\alpha)}(x)| \le N^{|\alpha|}$ for any multi-index α and for some $N = N(g) > 0$. This follows from the well-known fact that for any $f \in L^1(S^{n-1})$ and $\varepsilon > 0$, there exists a Schwartz function g such that $\|f - g\|_1 < \varepsilon$ and $|g^{(\alpha)}(x)| \le N^{|\alpha|}$ for some $N = N(\varepsilon) > 0$.

Before we prove the theorem, we establish four lemmas.

LEMMA 1. *If $f \in L^1(S^{n-1})$ and $f \ge 0$, then we have*

$$Mf(x) \le c \max\left\{ n \sup_{0 \le t \le 1/\sqrt{n}} \left| u\left(\left(1 - \frac{t}{\sqrt{n}}\right)x\right)\right|, \right.$$
$$\left. \sqrt{n} \sup_{(1/\sqrt{n}) \le t \le \sqrt{2(1 - 1/n)}} \frac{1}{t}\left| u\left(\left(1 - \frac{t^2}{2}\right)x\right)\right|, \left| u\left(\frac{1}{n}x\right)\right| \right\}.$$

Proof. By (1) and (2), it is enough to bound

$$\frac{1}{P(rx, y)|S(x,t)|} \chi_{S(x,t)}(y)$$

for every $y \in S^{n-1}$. Since $\chi_{S(x,t)}$ is supported on $S(x,t)$ and equals 1 there and $P(rx,y)$ decreases as $|x-y|$ increases, it is enough to bound

$$I \equiv \frac{1}{P(rx,y)|S(x,t)|}$$

for every y such that $|x-y| = t$ with $0 \le t \le 2$. Using spherical coordinates it is easy to see that

$$\begin{aligned}|S(x,t)| &= \omega_{n-2}\int_0^{2\arcsin(t/2)} \sin^{n-2} u\, du\\ &\ge \omega_{n-2}\int_0^{2\arcsin(t/2)} \sin^{n-2} u \cos u\, du\\ &\ge \frac{c\omega_{n-2}}{n}\left[t^2\left(1-\frac{t^2}{4}\right)\right]^{(n-1)/2}\end{aligned}$$

as long as $0 \le t \le \sqrt{2}$. By the law of cosines it is a straightforward calculation to verify that

$$|rx-y|^2 = (1-r)^2 + rt^2.$$

From this we obtain

$$I \le \frac{cnt\,\omega_{n-1}\sqrt{1-t^2/4}}{\omega_{n-2}(1-r)}\left[\frac{(1-r)^2+rt^2}{t^2(1-t^2/4)}\right]^{(n-1)/2}$$

when $0 \le t \le \sqrt{2}$. If $0 \le t \le 1/\sqrt{n}$, choose $r = 1 - t/\sqrt{n}$. We then have

$$I \le \frac{cn\,\omega_{n-1}\sqrt{n}}{\omega_{n-2}} \le cn. \tag{4}$$

If

$$\frac{1}{\sqrt{n}} \le t \le \sqrt{2\left(1-\frac{1}{n}\right)},$$

choose $r = 1 - t^2/2$. In this case we obtain

$$I \le \frac{cnt\,\omega_{n-1}}{\omega_{n-2}t^2} \le \frac{c\sqrt{n}}{t}. \tag{5}$$

Finally if $\sqrt{2(1-1/n)} \le t \le 2$, pick $r = 1/n$ and observe that $\frac{1}{2}\omega_{n-1} \le |S(x,t)| \le \omega_{n-1}$ whenever $\sqrt{2(1-1/n)} \le t \le 2$ which gives

$$I \le c. \tag{6}$$

Inequalities (4), (5), and (6) imply the conclusion of the lemma.

Define

$$P^\lambda f(x) = u(e^{-\lambda}x) \tag{7}$$

for $0 \le \lambda < \infty$.

LEMMA 2. *$P^\lambda f(x) = \int_0^\infty \phi(\mu, \lambda) T^\mu f(x)\, d\mu$ for $\lambda > 0$ where*

$$\phi(\mu, \lambda) = \frac{1}{2\sqrt{\pi}} \lambda \mu^{-3/2} \exp\left(a\lambda - a^2\mu - \frac{\lambda^2}{4\mu} \right) \tag{8}$$

and $a = (n-1)/2$.

This result is stated and its proof is outlined in Herz's paper ([**3**], p. 231). In order that this paper will be self-contained, we provide a different proof of the lemma.

The solution to the initial-value problem $\partial v/\partial t = \Delta_S v$ for $0 < t < \infty$ and $v(x, 0) = f(x)$ is clearly

$$v(x, t) = e^{t\Delta_S} f(x) \equiv T^t f(x) \tag{9}$$

which is well-defined for any Schwartz function f satisfying $|f^{(\alpha)}(x)| \le N^{|\alpha|}$ for some $N > 0$. We claim that

$$P^\lambda f(x) = \exp\left\{ -\lambda\left[(-\Delta_S + a^2)^{1/2} - a \right] \right\} f(x) \tag{10}$$

as we now demonstrate. Letting $r = e^{-\lambda}$ and recalling (7) we can rewrite (10) as

$$u(rx) = r^{(-\Delta_S + a^2)^{1/2} - a} f(x). \tag{11}$$

It is obvious that $u(x) = f(x)$. If we express the Laplacian in $\mathbf{R}^n$, Δu, in the form:

$$\Delta = r^{-n} \frac{\partial}{\partial r}\left(r^n \frac{\partial}{\partial r} \right) + r^{-2} \Delta_S \tag{12}$$

and then calculate $\Delta u(rx)$ we immediately see that $\Delta u = 0$ which establishes (10). By (9), (10), and the symbolic calculus it is sufficient to show that

$$\exp\left\{ -\lambda\left[(-b + a^2)^{1/2} - a \right] \right\} = \int_0^\infty \phi(\mu, \lambda) e^{\mu b}\, d\mu \tag{13}$$

for any negative number b since the spectrum of Δ_S lies on the negative real axis (see [**5**], p. 70). It is elementary to show that the function on the right hand side of (13) satisfies the differential equation $g'(\lambda) = -[(-b + a^2)^{1/2} - a] g(\lambda)$ from which (13) easily follows.

LEMMA 3. *If* $f \in L^1(S^{n-1})$ *and* $f \geq 0$, *then* $Mf(x) \leq cnM_Tf(x)$ *for all* $x \in S^{n-1}$.

Proof. By Lemma 2, we have

$$P^{\lambda}f(x) = \int_0^{\infty} \phi(\mu, \lambda) T^{\mu}f(x)\, d\mu.$$

Integrating by parts yields

$$P^{\lambda}f(x) = \int_0^{\infty} \frac{1}{\mu}\left[\int_0^{\mu} T^{\gamma}f(x)\, d\gamma\right]\mu\frac{d\phi}{d\mu}(\mu, \lambda)\, d\mu \tag{14}$$

$$\leq \sup_{\mu>0}\left|\frac{1}{\mu}\int_0^{\mu} T^{\gamma}f(x)\, d\gamma\right|\int_0^{\infty}\left|\mu\frac{d\phi}{d\mu}(\mu, \lambda)\right| d\mu$$

$$= M_Tf(x)\int_0^{\infty}\left|\mu\frac{d\phi}{d\mu}(\mu, \lambda)\right| d\mu.$$

One easily calculates from (8) that

$$\mu\frac{d\phi}{d\mu} = \frac{-\mu^{-5/2}\lambda}{8\sqrt{\pi}}(4a^2\mu^2 + 6\mu - \lambda^2)\exp\left(a\lambda - a^2\mu - \frac{\lambda^2}{4\mu}\right).$$

In his paper [3], Herz claims that

$$\int_0^{\infty}\left|\mu\frac{d\phi}{d\mu}(\mu, \lambda)\right| d\mu$$

is bounded independent of λ. In point of fact,

$$II \equiv \int_0^{\infty}\left|\mu\frac{d\phi}{d\mu}(\mu, \lambda)\right| d\mu \leq c(1 + \sqrt{n\lambda}) \tag{15}$$

is best possible. Luckily, this is sufficient for our purposes. Since

$$\mu\frac{d\phi}{d\mu} = \frac{d}{d\mu}(\mu\phi) - \phi,$$

we obtain

$$II \leq \int_0^{\infty} |\phi|\, d\mu + \int_0^{\infty}\left|\frac{d}{d\mu}(\mu\phi)\right| d\mu. \tag{16}$$

In the definition of ϕ, we observe that $\phi \geq 0$ and furthermore $\int\phi\, d\mu \equiv 1$ as can be seen from (13) by letting $b = 0$. It is clear that $\mu\phi \geq 0$ and it vanishes at 0 and ∞. By calculating $d/d\mu\ (\mu\phi)$, it is easy to check that $\mu\phi$ has a single turning point on $(0, \infty)$, say $\mu = \mu_0$, where it attains its maximum. Since $\mu\phi$ increases from 0 on $(0, \mu_0)$ and decreases to 0 on

(μ_0, ∞), we have

$$\int_0^\infty \left| \frac{d}{d\mu}(\mu\phi) \right| d\mu = 2\mu_0\phi(\mu_0, \lambda).$$

There follows the inequality:

$$II \le 1 + 2 \max_{\mu} \mu\phi(\mu, \lambda) \equiv 1 + 2M.$$

To bound M, express $\mu\phi$ as

$$\mu\phi(\mu, \lambda) = A^{1/2}x^{-1/2}\exp\left[-A\left(x^{1/2} - \tfrac{1}{2}x^{-1/2}\right)^2\right] \tag{17}$$

where $A = a\lambda$ and $x = a\lambda^{-1}\mu$. By taking the derivative of the right hand side of (17), it is easy to check that its maximum is taken on when $x = x_0 \equiv (-1 + \sqrt{1 + A^2})/4A$. It is trivial to verify that $x_0 = O(A)$ when $0 \le A < 1$ and $x_0 = O(1)$ when $A \ge 1$. Substituting these estimates for x_0 into (17) gives $M \le cA^{1/2} \le c\sqrt{n\lambda}$ which in turn implies (15) as we wished to show. Substituting inequality (15) into (14) gives

$$P^\lambda f(x) \le c(1 + \sqrt{n\lambda})M_T f(x). \tag{18}$$

Recall that $u(e^{-\lambda}x) = P^\lambda f(x)$ from which there follows $u(rx) \le c(1 + \sqrt{n\ln(1/r)})M_T f(x)$. Lemma 1 implies that

$$Mf(x) \le c\left\{ n \sup_{0 \le t \le 1/\sqrt{n}} \left(1 + \sqrt{n}\, \ln\left(1/\left(1 - \frac{t}{\sqrt{n}}\right)\right)\right),\right.$$

$$\sqrt{n} \sup_{\frac{1}{\sqrt{n}} \le t \le \sqrt{2(1 - \frac{1}{n})}} \left(\frac{1}{t}\left(1 + \sqrt{n\, \ln\left(1/\left(1 - \frac{t^2}{2}\right)\right)}\right)\right),$$

$$\left.(1 + \sqrt{n}\,\ln n)\right\} M_T f(x)$$

which can be simplified to $Mf(x) \le cnM_T f(x)$ as we wished to show.

LEMMA 4. *If* $f \in L^1(S^{n-1})$ *and* $f \ge 0$, *then* $\|T^\lambda f\|_1 \le \|f\|_1$.

Proof. It is a result in diffusion theory (see Ch. IX, p. 252, [**2**]) that

$$T^\lambda f(x) = \int_{S^{n-1}} K_\lambda(x, y) f(y)\, dy$$

with $K_\lambda \geq 0$. If $f(x) \equiv 1$, then clearly $T^\lambda f(x) \equiv 1$ so that

$$\int_{S^{n-1}} K_\lambda(x, y)\, dy = 1 \tag{19}$$

for any $x \in S^{n-1}$. Since $u(x, \lambda) = K_\lambda(x, y)$ is the solution to the initial value problem $\partial u/\partial\lambda = \Delta_S u$ and $u(x, 0) = \delta_y(x)$ where δ_y is the delta function centered at y, then by the symmetry of the sphere $K_\lambda(x, y) = K_\lambda(y, x)$. Equation (19) shows that

$$\int_{S^{n-1}} K_\lambda(x, y)\, dx = 1 \tag{20}$$

for any $y \in S^{n-1}$. We can now use (20) to conclude the proof of the lemma. We have

$$\begin{aligned} \|T^\lambda f\|_1 &= \int_{S^{n-1}} \int_{S^{n-1}} K_\lambda(x, y) f(y)\, dy\, dx \\ &= \int_{S^{n-1}} \int_{S^{n-1}} K_\lambda(x, y)\, dx f(y)\, dy \\ &= \int_{S^{n-1}} f(y)\, dy = \|f\|_1 \end{aligned}$$

which establishes Lemma 4.

We are now ready to finish the proof of the theorem. It is obvious that $T^\lambda f(x)$ forms a semi-group with respect to λ. Lemma 4 states that $\|T^\lambda f\|_1 \leq \|f\|_1$ for $f \geq 0$. We have $\|T^\lambda f\|_\infty \leq \|f\|_\infty$ for $f \geq 0$ since

$$\begin{aligned} \|T^\lambda f\|_\infty &= \left\| \int_{S^{n-1}} K_\lambda(x, y) f(y)\, dy \right\|_\infty \\ &\leq \|f\|_\infty \int_{S^{n-1}} K_\lambda(x, y)\, dy = \|f\|_\infty . \end{aligned}$$

Finally it is obvious that $T^\lambda(1) \equiv 1$. All the hypotheses of the Hopf abstract maximal ergodic theorem are satisfied (see Lemma 6, p. 690, [**1**]) which yields the result

$$|\{x\colon M_T f(x) > \lambda\}| \leq \frac{1}{\lambda} \|f\|_1 \tag{21}$$

for $\lambda > 0$ and $f \geq 0$. By Lemma 3 and (21), we conclude that

$$|\{x\colon Mf(x) > \lambda\}| \leq \frac{cn}{\lambda} \|f\|_1$$

which proves the theorem.

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