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THE DIVISOR FUNCTION AT CONSECUTIVE INTEGERS

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Let $d(n)$ be the divisor function. Improving a result of Heath-Brown, we show that for sufficiently large x, $d(n) = d(n+1)$ holds for $\gg x(\log\log x)^{-3}$ integers $n \le x$.

1. Introduction. In a recent paper with the above title, Heath-Brown [**5**] showed that there are infinitely many positive integers n, for which $d(n) = d(n+1)$. In fact, he proved that for sufficiently large x

$$\#\{n \le x\colon d(n) = d(n+1)\} \gg x(\log x)^{-7}. \tag{1.1}$$

This settled a problem of Erdös and Mirsky [**1**]. Heath-Brown's work was motivated by an earlier result of Spiro [**6**], namely that with $a = 5040$, $d(n) = d(n+a)$ holds for infinitely many integers n.

Using a different approach, Erdös, Pomerance and Sarközy [**2**] recently showed that for sufficiently large x

$$\sum_{i=0}^{3} \#\{n \le x\colon d(n) = 2^i d(n+1)\} \gg x(\log\log x)^{-1/2}, \tag{1.2}$$

and in a subsequent paper [**3**] established the upper bound

$$\#\{n \le x\colon d(n) = d(n+1)\} \ll x(\log\log x)^{-1/2}. \tag{1.3}$$

These results strongly suggest that the right order of magnitude for the quantity estimated by (1.1) and (1.3) is $x(\log\log x)^{-1/2}$. A heuristic argument supporting this conjecture has been given P. T. Bateman and C. Spiro.

The main purpose of this paper is to prove the following estimate, which considerably improves on Heath-Brown's bound (1.1) and falls short of the conjectured bound only by a power of $\log\log x$.

THEOREM 1. *For sufficiently large x,*

$$\#\{n \le x\colon d(n) = d(n+1)\} \gg x(\log\log x)^{-3}. \tag{1.4}$$

The idea of the proof is to combine the methods of Heath-Brown and Erdös-Pomerance-Sarközy. We shall give an outline of the proof in §3.

A sieve result plays a crucial role in the proof. We shall use here Lemma 2 below, which constitutes the sharpest known estimate of its type. The bound (1.4) can be improved, if one assumes stronger sieve estimates. For example, if Lemma 2 holds with $g = r = 2$, as has been conjectured, then one gets the bound $\gg x(\log\log x)^{-1/2}$ for the left-hand side of (1.4), which by (1.3) is best-possible.

Our method actually yields the following more general result.

THEOREM 2. *Let $d_1, \ldots, .d_7$ be positive integers. For sufficiently large x*

$$\sum_{1 \le i < j \le 7} \#\left\{ n \le x \colon \frac{d(n+1)}{d(n)} = \frac{d_j}{d_i} \right\} \gg x(\log\log x)^{-3}. \tag{1.5}$$

In the special case $d_1 = \cdots = d_7$, the estimate (1.5) reduces to the estimate (1.4) of Theorem 1. In its general form, Theorem 2 allows to partially settle another conjecture of Erdös, asserting that every positive real number is a limit point of the sequence $\{d(n+1)/d(n)\}$. Equivalently, if E denotes the set of limit points of the sequence $\{\log d(n+1)/d(n)\}$, then Erdös' conjecture asserts that $E = \mathbf{R}$. By Heath-Brown's result, E contains the number 0, but until now this had been the only real number known to belong to E. From Theorem 2 we shall deduce

THEOREM 3. *E contains a positive proportion of all real numbers in the sense that*

$$\liminf_{x \to \infty} \frac{1}{x} |E \cap [0, x]| > 0$$

and

$$\liminf_{x \to \infty} \frac{1}{x} |E \cap [-x, 0]| > 0,$$

where $|\cdot|$ denotes the Lebesgue measure. Moreover, there exists a positive δ, such that E contains the interval $[-\delta, \delta]$.

Similar results can be proved for the function $\Omega(n)$, the number of prime factors of n counted with multiplicity. This is not very surprising, since for squarefree integers n, $d(n) = 2^{\Omega(n)}$. Because of the complete multiplicativity of the function $2^{\Omega(n)}$, the arguments in the case of $\Omega(n)$ are in fact technically simpler. The bound (1.4) of Theorem 1 remains valid with $\Omega(n)$ instead of $d(n)$, without further modifications. Moreover, one can establish the following analogues of Theorems 2 and 3.

THEOREM 4. *Let $d_1, \ldots, d_7$ be nonnegative integers. For sufficiently large x*

$$\sum_{1 \le i < j \le 7} \#\{n \le x \colon \Omega(n+1) - \Omega(n) = d_j - d_i\} \gg x(\log\log x)^{-3}.$$

THEOREM 5. *Let A denote the set of integers a such that $\Omega(n) - \Omega(n+1) = a$ holds for infinitely many integers n. Then A has positive lower density.*

2. Lemmas. We shall need two lemmas, both of which are crucial in Heath-Brown's method. The first is Heath-Brown's "Key Lemma".

LEMMA 1 [5, p. 142]. *For any positive integer k there exist positive integers $a_1 < \cdots < a_k$ such that if $a_{ij} = a_j - a_i$ then*

$$a_{ij} \mid (a_i, a_j) \qquad (1 \le i < j \le k) \tag{2.1}$$

and

$$d(a_j)d\left(\frac{a_i}{a_{ij}}\right) = d(a_i)d\left(\frac{a_j}{a_{ij}}\right) \qquad (1 \le i < j \le k). \tag{2.2}$$

The second result we shall need is a lower bound sieve estimate for almost-primes represented by products of linear polynomials.

LEMMA 2. *For every integer $g \ge 2$ there exists an integer $r = r(g) > 1$ and positive constants $\delta_i = \delta_i(g)$, $i = 1, 2, 3$, with the following property. Let a_i, b_i, $1 \le i \le g$, be integers satisfying*

$$\prod_{i=1}^{g} a_i \prod_{1 \le t < s \le g} (a_t b_s - a_s b_t) \ne 0, \tag{2.3}$$

and let

$$f(n) = \prod_{i=1}^{g} (a_i n + b_i).$$

Suppose that the polynomial $f(n)$ has no fixed prime divisor. Let

$$S(x) = \#\{n \le x \colon \Omega(f(n)) \le r; \mu^2(f(n)) = 1; p(f(n)) > x^{\delta_2}\},$$

where $p(n)$ denotes the least prime factor of n. Then we have

$$S(x) \ge \delta_1 x(\log x)^{-g}, \tag{2.4}$$

provided x satisfies

$$2\max\{|a_i|, |b_i|: 1 \le i \le g\} \le x^{\delta_3}. \tag{2.5}$$

Moreover, for $g = 7$ *a possible choice for* r *is* $r = r(7) = 27$.

This result is Theorem 10.5 in Halberstam-Richert [**4**], modified as follows:

(i) In the definition of $S(x)$ we have, apart from the restriction on the number of prime factors of $f(n)$, required that $f(n)$ be squarefree and free of prime factors smaller than a certain fixed power of x. These last two properties do not appear in the statement of [**4**, Theorem 10.5]. However, a restriction of the type $p(f(n)) \ge x^{\delta_2}$ is implicit in the proof of that theorem, and using this property, one easily sees that the contribution of the non-squarefree integers is negligible.

(ii) In [**4**, Theorem 10.5] the numbers a_i and b_i are fixed and the bound (2.4) is asserted only for sufficiently large x, and with a constant δ_1 depending on the coefficients a_i and b_i. Here an inspection of the proof shows that if x exceeds a sufficiently large but fixed power of each of the coefficients (as is guaranteed by the condition (2.5)), then the same bound holds with the implied constants depending at most on g.

(iii) The table given in [**4**, p. 285] gives $r = 29$ as admissible value in the case $g = 7$. The value $r = 27$, which is the current record, has been obtained by Xie [**7**] through more careful numerical computations.

We remark that Heath-Brown in his work applies the same result with the above modifications including Xie's improvement, except that he does not use the uniformity in the coefficients.

We shall use the two lemmas only in the cases $k = 7$ resp. $g = 7$. For our purposes therefore the constants δ_i in Lemma 2 can be taken as absolute constants.

3. Outline of the method. We first outline Heath-Brown's argument leading to (1.1). It is based on the observation that if $n_1 < \cdots < n_k$ are positive integers satisfying the conditions of Lemma 1, then, for any pair $i < j$, the integers $n_i/(n_i, n_j)$ and $n_j/(n_i, n_j)$ are consecutive, and we have

$$\frac{d(n_j/(n_i, n_j))}{d(n_i/(n_i, n_j))} = \frac{d(n_j)}{d(n_i)}.$$

Hence every such k-tuple $(n_1, \ldots, n_k)$, which satisfies in addition

$$d(n_i) = d(n_j) \quad \text{for some } i < j, \tag{3.1}$$

gives rise to a solution of $d(n) = d(n+1)$. To ensure (3.1), one can take k large and require that the integers n_i have only few prime factors. For example, if

$$\sum_{i=1}^{k} d(n_i) < 1 + 2 + \cdots + k = \frac{1}{2}k(k+1),$$

then (3.1) must hold. k-tuples $(n_1, \ldots, n_k)$ having this additional property can be found among the translates $(a_1 + t, \ldots, a_k + t)$ of a fixed k-tuple $(a_1, \ldots, a_k)$ satisfying the conditions of Lemma 1. In fact, it is easily seen that the translated k-tuple satisfies the same conditions, whenever t is divisible by a_i^2 for $i = 1, \ldots, k$. A sieve result such as Lemma 2 can then be applied to show that among those translates there is a substantial proportion, for which $d(n_1), \ldots, d(n_k)$ are small enough to imply (3.1), provided k was chosen sufficiently large at the outset.

This is, roughly, Heath-Brown's method of producing solutions to $d(n) = d(n+1)$. The drawback of this method is that it detects only solutions which have few prime factors, and thus misses the "typical" solutions, which one would expect to have about $\log\log n$ prime factors. To remedy this, we shall combine Heath-Brown's argument with an idea of Erdös, Pomerance and Sarközy [**2**]. Namely, we shall look for integers $n_1, \ldots, n_k$ satisfying the conditions of Lemma 1 and of the form $n_i = m_i q_i$, where the integers m_i (resp. q_i) are composed of the small (resp. large) prime factors of n_i (in a sense to be made precise), and

$$d(m_i) = d(m_j) \qquad (1 \le i < j \le k). \tag{3.2}$$

The restriction on the size of the prime factors of q_i ensures that q_i has only few prime factors, so that, in view of (3.2), there are only few possible values for the k numbers $d(n_i) = d(m_i)d(q_i)$. For large k, this implies (3.1), and we obtain as before a solution to $d(n) = d(n+1)$. To count these solutions, we shall first fix the numbers m_i and count the k-tuples $(q_1, \ldots, q_k)$, which lead to an "admissible" k-tuple $(n_1, \ldots, n_k) = (m_1 q_1, \ldots, m_k q_k)$, and then sum over all choices of $(m_1, \ldots, m_k)$ subject to (3.2). In this way we pick up substantially more solutions than with Heath-Brown's original argument. The optimal choice of k turns out to be $k = 7$; the argument would go through for larger values of k, but the resulting estimate would then be weaker.

The outlined argument leads to the estimate of Theorem 1. In order to obtain the more general estimate of Theorem 2, it suffices to replace the condition (3.2) by

$$\text{(3.2)}' \qquad \frac{d(m_j)}{d(m_i)} = \frac{d_j}{d_i} \qquad (1 \le i < j \le k).$$

The rest of the argument goes through without change.

4. Proof of Theorem 2. Let $d_1, \dots, d_7$ be given positive integers and fix positive integers $a_1 < \cdots < a_7$ satisfying the conditions (2.1) and (2.2) of Lemma 1. Further let z and δ be positive constants to be specified later. At the moment we shall only suppose

$$\text{(4.1)} \qquad z > a_7, \qquad 0 < \delta < 1.$$

These constants as well as all other constants occurring in the proof are allowed to depend on the numbers a_i and d_i.

Next, let $x > z^{1/\delta}$ be given, and suppose that $m_1, \dots, m_7$ are positive integers satisfying the conditions

$$(*) \qquad \begin{cases} m_i \le x^\delta, p(m_i) > z & (1 \le i \le 7), \\ (m_i, m_j) = 1 & (1 \le i < j \le 7), \\ \dfrac{d(m_j)}{d(m_i)} = \dfrac{d(a_i)d_j}{d(a_j)d_i} & (1 \le i < j \le 7). \end{cases}$$

Consider the system of congruences

$$\begin{aligned} n_0 &\equiv 0 && \operatorname{mod} 7! \prod_{i=1}^{7} a_i^2, \\ n_0 &\equiv -a_i && \operatorname{mod} m_i \qquad (i = 1, \dots, 7). \end{aligned}$$

By (4.1) and $(*)$ the moduli involved are pairwise coprime. Hence, by the chinese remainder theorem, this system has a solution, and all its solutions are given by

$$n_0(t) = n_0 + tP \qquad (t \in \mathbf{Z}),$$

where n_0 denotes the least positive solution and P is the product of the moduli, i.e.

$$P = 7! \prod_{i=1}^{7} a_i^2 m_i.$$

For $i = 1, \dots, 7$ let

$$n_i(t) = n_0(t) + a_i = n_0 + a_i + tP.$$

The tuples $(n_1(t), \ldots, n_7(t))$, $t \geq 1$, are all translates of a fixed tuple $(a_1, \ldots, a_7)$ satisfying the conditions of Lemma 1. Among these tuples we shall try to find "good" tuples $(n_1, \ldots, n_7)$, from which a solution to the equation

$$\frac{d(n+1)}{d(n)} = \frac{d_j}{d_i} \tag{4.2}$$

can be constructed, as outlined in the preceding section.

By construction, $n_0 + a_i$ and P are both divisible by a_i and by m_i and hence by $a_i m_i$, since $(a_i, m_i) = 1$ in view of (4.1) and $(*)$. We can therefore write

$$n_i(t) = a_i m_i (P_i t + Q_i) = a_i m_i f_i(t), \tag{4.3}$$

say, where

$$P_i = \frac{P}{a_i m_i}, \qquad Q_i = \frac{n_0 + a_i}{a_i m_i}$$

are positive integers. If now, for some $t \geq 1$ and some $i < j$,

$$(**) \qquad \begin{cases} p\big(f_i(t) f_j(t)\big) > x^{\delta}, \\ d\big(f_i(t)\big) = d\big(f_j(t)\big), \end{cases}$$

then we have, by (2.2) and $(*)$,

$$\frac{d\big(n_j(t)/a_{ij}\big)}{d\big(n_i(t)/a_{ij}\big)} = \frac{d\big(a_j/a_{ij}\big) d(m_j) d\big(f_j(t)\big)}{d\big(a_i/a_{ij}\big) d(m_i) d\big(f_i(t)\big)} = \frac{d_j}{d_i},$$

and since

$$n_j(t) = n_i(t) + a_j - a_i = n_i(t) + a_{ij},$$

we obtain (4.2) for $n = n_i(t)/a_{ij}$.

Thus, for fixed $i < j$, every tuple $(\underline{m}, t) = (m_1, \ldots, m_7, t)$ satisfying $(*)$ and $(**)$ gives rise to a solution of (4.2). This solution $n = n_i(t)/a_{ij}$ satisfies $n \leq x$, if we suppose that $3tP \leq x$, i.e.

$$t m_1 \cdots m_7 \leq \left(3 \cdot 7! \prod_{i=1}^{7} a_i^2\right)^{-1} x = cx,$$

say. On the other hand, if $i < j$ are fixed, then every integer $n \leq x$ satisfying (4.2) arises in this way at most once. For we must have $n_i(t) = n a_{ij}$ and hence

$$n_h(t) = n_i(t) + a_h - a_i = n a_{ij} + a_h - a_i \qquad (1 \leq h \leq 7).$$

Therefore n determines the integers $n_h(t)$, $h = 1, \ldots, 7$, uniquely. But then the integers m_h, $h = 1, \ldots, 7$, and t are also uniquely determined by (4.3), since under the conditions (4.1), $(*)$ and $(**)$ the decomposition (4.3) of $n_h(t)$ is unique.

We therefore obtain for the left-hand side of (1.5) the bound

$$\sum_{1 \le i < j \le 7} \sum_{\substack{n \le x \\ \frac{d(n+1)}{d(n)} = \frac{d_j}{d_i}}} 1 \ge \sum_{(*)} T\left(\underline{m}, \frac{cx}{m_1 \cdots m_7}\right), \tag{4.4}$$

where the summation on the right-hand side is extended over all tuples $\underline{m} = (m_1, \ldots, m_7)$ satisfying $(*)$ and $T(\underline{m}, y)$ denotes the number of positive integers $t \le y$, for which $(**)$ is satisfied for *some* pair $i < j$. Note that the condition $(**)$ depends on the pair (i, j) as well as on the numbers m_i, which are implicit in the coefficients of $f_i(t)$. We shall show that with an appropriate choice of the constants z and δ we have

$$T(\underline{m}, y) \gg y(\log y)^{-7} \qquad (x \ge y \ge x^{1/2}) \tag{4.5}$$

for all sufficiently large x and all tuples $\underline{m}$ satisfying $(*)$, and further

$$\sum_{(*)} \frac{1}{m_1 \cdots m_7} \gg (\log x)^7 (\log\log x)^{-3}. \tag{4.6}$$

The asserted bound (1.5) follows from these two estimates and (4.4).

To prove the first estimate, we shall apply Lemma 2 with the polynomial

$$f(t) = \prod_{i=1}^{7} f_i(t) = \prod_{i=1}^{7} (P_i t + Q_i).$$

To this end we have to check first that the hypotheses of this lemma are satisfied. The coefficients P_i and Q_i are all non-zero. Moreover, we have $P_i Q_j - Q_i P_j \ne 0$ for all $i < j$. For this is equivalent to

$$\frac{P}{a_i m_i} \cdot \frac{n_0 + a_j}{a_j m_j} \ne \frac{P}{a_j m_j} \cdot \frac{n_0 + a_i}{a_i m_i} \qquad (1 \le i < j \le 7),$$

which holds, since we assumed the integers a_i to be pairwise distinct. Hence the analogue of condition (2.3) is satisfied.

It remains to verify that $f(t)$ has no fixed prime divisor, i.e. we have to show that for every prime p there exists an integer t for which

$$f(t) \not\equiv 0 \mod p. \tag{4.7}$$

We proceed here as in Heath-Brown [5, p. 148]. We first establish

$$(P_i, Q_i) = 1 \qquad (1 \le i \le 7). \tag{4.8}$$

To see this, recall that

$$P_i = \frac{7!}{a_i m_i} \prod_{j=1}^{7} a_j^2 m_j$$

and

$$Q_i m_i = \frac{n_0}{a_i} + 1,$$

where

$$n_0 \equiv 0 \mod\left(7! \prod_{j=1}^{7} a_j^2\right).$$

Therefore the only possible common prime factors of P_i and Q_i are the prime factors of m_j, $j = 1, \dots, 7$. By (*) the numbers m_j are pairwise coprime and free of prime factors $\leq z$ and therefore coprime to each a_j. Hence no prime factor of m_i divides P_i. On the other hand, if $j \neq i$, then every prime factor of m_j divides $n_0 + a_j = Q_i a_i m_i + (a_j - a_i)$, and since it is coprime to $(a_i, a_j) = |a_j - a_i|$, it cannot divide Q_i. This proves (4.8).

The condition (4.8) implies that each of the congruences

$$f_i(t) = P_i t + Q_i \equiv 0 \mod p \qquad (i = 1, \dots, 7)$$

has exactly one solution $t_i(p)$ modulo p. If now $p > 7$, then there exists a congruence class modulo p different from each of the classes $t_i(p)$ modulo p, and for t belonging to this class (4.7) holds. If however $p \leq 7$, then for each i, $p | 7! | P_i$ for $p \nmid Q_i$, and (4.7) follows again, this time for all integers t.

The hypotheses of Lemma 2 are therefore satisfied, and we conclude that there are $\gg y(\log y)^{-7}$ positive integers $t \leq y$, for which

$$\text{(4.9)} \qquad \Omega(f(t)) \leq 27, \quad \mu^2(f(t)) = 1, \quad p(f(t)) > y^{\delta_2},$$

provided y is a sufficiently large power of each of the coefficients P_i and Q_i. This last condition is satisfied if we suppose that $y \geq x^{1/2}$ and that the constant δ implicit in (*) is sufficiently small.

The first condition in (4.9) implies

$$\sum_{i=1}^{7} \Omega(f_i(t)) = \Omega(f(t)) \leq 27 < 1 + 2 + \cdots + 7 = 28,$$

and hence $\Omega(f_i(t)) = \Omega(f_j(t))$ for some $i < j$. Taking into account the remaining conditions $\mu^2(f(t)) = 1$ and $p(f(t)) > y^{\delta_2}$, we see that (4.9) implies (**) for some $i < j$, provided that $y \geq x^{1/2}$ and $\delta \leq \delta_2/2$, as we may assume. Therefore the number of positive integers $t \leq y$ satisfying

(4.9) is $\leq T(\underline{m}, y)$, and we obtain for $T(\underline{m}, y)$ the asserted lower bound (4.5).

It remains to establish (4.6). Let $z < p_1 < \cdots < p_7$ be the first seven primes exceeding z, and put

$$q_i = p_i^{\alpha_i}, \qquad \alpha_i = \frac{d_i}{d(a_i)} \prod_{j=1}^{7} d(a_j) - 1,$$

so that

$$\frac{d(q_j)}{d(q_i)} = \frac{\alpha_j + 1}{\alpha_i + 1} = \frac{d(a_i)d_j}{d(a_j)d_i} \qquad (1 \leq i < j \leq 7).$$

Let further

$$x' = x^{\delta} / \max_{i=1}^{7} q_i, \qquad z' = p_7.$$

We now require the integers m_i to be of the form $m_i = q_i r_i$, where

$$(*)' \qquad \begin{cases} r_i \leq x',\ p(r_i) > z' \qquad (1 \leq i \leq 7), \\ \mu^2(r_1 \cdots r_7) = 1, \\ \Omega(r_1) = \cdots = \Omega(r_7). \end{cases}$$

It is easily seen that the conditions $(*)'$ for $(r_1, \ldots, r_7)$ imply the conditions $(*)$ for $(m_1, \ldots, m_7) = (q_1 r_1, \ldots, q_7 r_7)$. Therefore we have

$$\sum_{(*)} \frac{1}{m_1 \cdots m_7} \geq \frac{1}{q_1 \cdots q_7} \sum_{(*)'} \frac{1}{r_1 \cdots r_7},$$

and it suffices to establish the bound (4.6) for the last sum.

For $k \geq 1$ let

$$S_k(x') = \sum_{\substack{r \leq x' \\ p(r) > z',\ \Omega(r) = k}} \frac{1}{r}.$$

We shall presently show that

$$(4.10) \qquad S_k(x') \asymp \frac{\log x'}{\log z' \sqrt{\log\log x'}}$$

holds uniformly in the range

$$(4.11) \qquad |k - \log\log x'| \leq \sqrt{\log\log x'},$$

provided x' is sufficiently large in terms of z'. The lower bound in (4.10) implies

$$\sum_{(*)''} \frac{1}{r_1 \cdots r_7} \gg \frac{(\log x')^7}{(\log z')^7 (\log\log x')^3},$$

where the summation is extended over all tuples $(r_1, \dots, r_7)$ satisfying

$$(*)'' \quad \begin{cases} r_i \le x', \ p(r_i) > z' & (1 \le i \le 7), \\ \Omega(r_1) = \cdots = \Omega(r_7), \ |\Omega(r_1) - \log\log x'| \le \frac{1}{2}\sqrt{\log\log x'}\,. \end{cases}$$

Using the upper bound in (4.10), it is readily seen that in the last sum the contribution of the tuples $(r_1, \dots, r_7)$ with $\mu^2(r_1 \cdots r_7) = 0$ is by a factor $\ll 1/z'$ smaller than the right-hand side. Therefore, if z and hence z' are large enough, then the above bound remains valid with the additional condition $\mu^2(r_1 \cdots r_7) = 1$ in the summation, i.e. with $(*)''$ replaced by $(*)'$, and we obtain the desired estimate.

The proof of (4.10) is routine and we shall therefore merely sketch it. One way to obtain (4.10) would be to first "eliminate" the condition $p(r) > z'$ by means of a sieve argument, and then deduce the estimate (and in fact an asymptotic formula for $S_k(x')$) from the Sathe-Selberg formulae for the number of integers $\le x$ with k prime factors.

A simpler and more direct approach goes as follows. For the upper bound one considers first the modified sum

$$S_k'(x') = \sum_{\substack{n \le x' \\ p(n) > z',\ \Omega(n) = k}} \frac{\mu^2(n)}{n}.$$

Clearly,

$$S_k'(x') \le \frac{1}{k!}\left(\sum_{z' < p \le x'} \frac{1}{p}\right)^k = \frac{1}{k!}\left(\log \frac{\log x'}{\log z'} + O(1)\right)^k,$$

and by means of Stirling's formula this bound is easily seen to be of the required order of magnitude uniformly in k, provided x' is sufficiently large in terms of z'. Since obviously

$$S_k(x') \le \left(\sum \frac{1}{n^2}\right) \max_{k'} S_{k'}'(x'),$$

the same bound holds for $S_k(x')$, as asserted in (4.10).

A lower bound for $S_k(x')$ can be derived from the inequality

$$S_k(x') \ge \sum_{\Omega(n) = k}{}' \frac{1}{n^\sigma} - \sum_{\substack{n > x' \\ \Omega(n) = k}}{}' \frac{1}{n^\sigma}$$

$$\ge \frac{1}{k!}\left(\sum_{z' < p \le x'} \frac{1}{p^\sigma}\right)^k - \sum_{\substack{n > x' \\ \Omega(n) = k}}{}' \frac{1}{n^\sigma},$$

where in Σ' the summation runs over integers composed of prime factors in the interval $(z', x']$, and $\sigma > 1$ is arbitrary. Taking $\sigma = 1 + \lambda/\log x'$ with a sufficiently large but fixed λ, the second term can be bounded from above by a quantity smaller than the desired lower bound, while the first term can be estimated as before and shown to be of the desired order of magnitude for the range (4.11).

The proof of Theorem 2 is now complete.

5. Proof of Theorem 3. Let

$$E_1 = \left\{ \log \frac{r}{s} : r, s \in \mathbf{N};\ \frac{d(n+1)}{d(n)} = \frac{r}{s} \text{ for infinitely many } n \in \mathbf{N} \right\},$$

and let $\overline{E}_1$ be the closure of E_1 in the usual topology. Then obviously $\overline{E}_1 \subset E$. Theorem 2 shows that given any positive integers $d_1, \ldots, d_7$ there exist indices $i < j$, such that $\log(d_j/d_i) \in E_1$. It follows that given any positive *real* numbers $u_1, \ldots, u_7$, we have

$$u_j - u_i \in \overline{E}_1 \subset E \quad \text{for some } i < j. \tag{5.1}$$

In fact, this holds if the numbers u_i are of the form $u_i = \log(r_i/s)$ with positive integers r_i and s, and an approximation argument yields the general case. We shall deduce the assertions of Theorem 3 from the property (5.1).

Suppose that the second assertion of Theorem 3 is false, i.e. suppose that for every $\delta > 0$ the interval $[-\delta, \delta]$ contains a number not belonging to E. Then there exists a sequence $x_n \in \mathbf{R} \setminus E$, $n \geq 1$, with $\lim_{n\to\infty} x_n = 0$. Since, by Theorem 1, $0 \in E$, this sequence must contain infinitely many pairwise distinct terms. By taking subsequences, we may suppose that the terms of this sequence are already pairwise distinct and have all the same sign, say $x_n > 0$ for all $n \geq 1$ without loss of generality.

By definition, E is a closed set and hence $\mathbf{R} \setminus E$ is an open set. Therefore there exist positive numbers δ_n such that for all $n \geq 1$

$$[x_n - \delta_n, x_n + \delta_n] \subset \mathbf{R} \setminus E.$$

We may assume that the sequence $\{\delta_n\}$ is non-increasing. Further, by taking a suitable subsequence, we may assume that

$$0 < x_{n+1} < \frac{1}{6}\delta_n \qquad (n \geq 1).$$

Consider now the positive integers

$$u_i = \sum_{n=1}^{i} x_n \qquad (1 \leq i \leq 7).$$

These satisfy

$$u_j - u_i = \sum_{n=i+1}^{j} x_n \in [x_{i+1} - \delta_{i+1}, x_{i+1} + \delta_{i+1}] \subset \mathbf{R} \setminus E$$

$$(1 \le i < j \le 7)$$

contradicting the property (5.1). Hence the second assertion of Theorem 3 is proved.

To obtain the first assertion, we apply (5.1) with $u_i = iu$, $1 \le i \le 7$, getting (in an obvious notation)

$$u \in \bigcup_{n=1}^{6} \frac{E}{n} \qquad (u > 0).$$

Hence we have for $x > 0$

$$x = \left| [0, x] \cap \bigcup_{n=1}^{6} \frac{E}{n} \right| \le \sum_{n=1}^{6} \left| \frac{E}{n} \cap [0, x] \right| \le 6 |E \cap [0, 6x]|$$

and therefore

$$|E \cap [0, x]| \ge \frac{x}{36} \qquad (x > 0).$$

Similarly taking $u_i = (8 - i)u$ yields

$$|E \cap [-x, 0]| \ge \frac{x}{36} \qquad (x > 0).$$

This completes the proof of Theorem 3.

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