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ENVELOPING ALGEBRAS OF LIE SUPERALGEBRAS

ERAZM J. BEHR

We review elementary properties of Lie superalgebras and their representations. These are later used in a discussion of the enveloping algebra $U(L)$ of a Lie superalgebra L from the point of view of non-commutative ring theory. In particular, we show that $U(L)$ has an Artinian ring of quotients, that Harish-Chandra's theorem holds for $U(L)$ and that in several cases $\operatorname{gl.dim}(U(L))$ turns out to be infinite.

Interest in Lie superalgebras, motivated mainly by problems in homotopy theory and particle physics, dates back to the early 60's. As a natural generalization of Lie algebras, Lie superalgebras appear to be promising as an ingredient of unified field theories currently under investigation. They are also interesting from a purely mathematical point of view, their enveloping algebras providing a varied and rich class of associative algebras.

In this paper we will give a brief review of well-known facts about Lie superalgebras and their enveloping algebras; in doing so we will concentrate on those results that seem to be of greatest use for ring-theoretical considerations. We will then present some new results, in the hope that they may eventually have a bearing on representation theory of Lie superalgebras.

Sections 1 and 2 contain notation, definitions and examples of Lie superalgebras and their enveloping algebras. In particular, we give an example of an enveloping algebra whose nilpotent radical is non-zero. In §3 we give some elementary results involving ideals and radicals of an enveloping algebra; we also prove at that point that in several important cases the enveloping algebra has infinite global dimension. In §4 we concentrate on the existence of an Artinian full quotient ring and on the non-commutaitve extension of Hilbert's Nullstellensatz. Section 5 describes the Joseph-Small additivity principle for enveloping algebras, and gives an application of that formula to certain primitive ideals. It also contains Harish-Chandra's theorem generalized to the class of Lie superalgebras. Finally, in §6, we list several unresolved problems which, in our opinion, deserve considerable attention and indicate possible direction of further study of enveloping algebras.

In all of the following, Proposition (Theorem, Example etc.) $X.Y$ means Proposition Y in Section X.

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1. Superalgebras. Throughout this paper, k will denote a field of characteristic 0. A k-algebra A is called a *superalgebra* (or a 2-graded algebra) if A is a direct sum of two vector spaces, $A = A_0 \oplus A_1$, such that for any $a, b \in \{0,1\}$, $A_a A_b \in A_{a \circ b}$, where $a \circ b$ stands for addition modulo 2.

Elements of A_0 are called *even*, while those of A_1 are *odd*. Writing $x \in A$ uniquely as $x_0 + x_1$ $(x_i \in A_i)$ we have a conjugation mapping defined by $\overline{x_0 + x_1} := x_0 - x_1$, which is obviously an automorphism of the algebra A. An ideal I of A is *graded* if $I = \bar{I}$. A superalgebra $L = L_0 \oplus L_1$ is a *Lie superalgebra* if its multiplication, traditionally denoted by $[x, y]$ instead of xy, satisfies two other conditions:

(i) for any $x \in L_a$, $y \in L_b$ $(a, b \in \{0,1\})$

$$[x, y] = -(-1)^{ab}[y, x] \quad \text{and}$$

(ii) for any $x \in L_a$, $y \in L_b$, $z \in L_c$ $(a, b, c \in \{0,1\})$

$$(-1)^{ac}[x,[y,z]] + (-1)^{ab}[y,[z,x]] + (-1)^{bc}[z,[x,y]] = 0.$$

It follows that L_0 is an anticommutative subalgebra of L, in which the standard Jacobi identity holds. L_0 is therefore a Lie algebra.

EXAMPLE 1. Let L_0 be the two-dimensional (abelian) Lie algebra with basis $\{e, f\}$ and let V be a two-dimensional vector space with basis $\{g, h\}$. Define a representation ρ of L_0 on V by

$$\rho(e) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{and} \quad \rho(f) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Finally, let φ be the bilinear symmetric mapping from $V \times V$ into L_0 given by $\varphi(g, h) = e + f$ and $\varphi(g, g) = \varphi(h, h) = 0$. Setting $L = L_0 \oplus V$, we can define multiplication on L as follows:

$$[x, y] := \rho(x)(y) \quad \text{for } x \in L_0,\ y \in V \quad \text{and}$$
$$[x, y] := \varphi(x, y) \quad \text{for } x, y \in V.$$

It is easy to verify that L is a Lie superalgebra, usually denoted by pl(1, 1).

The above example illustrates the fact that in general a Lie superalgebra essentially consists of: a Lie algebra L_0, a finite dimensional representation (ρ, V) of L_0 and a suitably chosen bilinear symmetric mapping $\varphi: V \times V \to L_0$ (the conditions that must be imposed on φ are explicitly stated in **[Sch]**).

We will now describe another natural way of constructing Lie superalgebras. If $A = A_0 \oplus A_1$ is an associative superalgebra, we will declare that for $x \in A_a$, $y \in A_b$ $(a, b \in \{0,1\})$ $[x, y] := xy - (-1)^{ab}yx$. When this operation is extended by linearity, A becomes a Lie superalgebra. One important special case of this process is the situation in which $A = M_r(k)$, the algebra of all $r \times r$ matrices over the field k. Let $r = m + n$ for some non-negative integers m and n, and let

$$A_0 := \{(a_{ij}) \in M_r(k) \mid a_{ij} = 0 \text{ for } 1 \le i \le m,\ m+1 \le j \le r \text{ and for } m+1 \le i \le r,\ 1 \le j \le m\},$$

$$A_1 := \{(a_{ij}) \in M_r(k) \mid a_{ij} = 0 \text{ for } 1 \le i \le m,\ 1 \le j \le m \text{ and for } m+1 \le i \le r,\ m+1 \le j \le r\}$$

or, symbolically

$$A_0 = \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix} \quad \text{and} \quad A_1 = \begin{pmatrix} 0 & * \\ * & 0 \end{pmatrix}.$$

By applying the above "super commutator" construction to A we obtain the *general linear* Lie superalgebra pl(m, n) (see **[Sch]**.)

Simple Lie superalgebras, which are of particular interest from the standpoint of representation theory, have been completely classified in **[K]**. Their list, along with the most widely used notation to which we will henceforth adhere, can also be found in **[Sch]**.

2. Enveloping superalgebras. Let L be a Lie superalgebra over the field k, and let U be an associative k-algebra with 1. U, together with a linear map $\sigma: L \to U$ is a *universal enveloping algebra* of L if

(i) for any $x \in L_a$, $y \in L_b$ $(a, b \in \{0,1\})$,

$$\sigma([x, y]) = \sigma(x) \circ \sigma(y) - (-1)^{ab}\sigma(y) \circ \sigma(x)$$

and

(ii) for any associative k-algebra U' with 1 and a k-linear map $\sigma': L \to U'$ which satisfies (i), there is a unique algebra homomorphism $r: U \to U'$ with $r(1) = 1$ and $r \circ \sigma = \sigma'$.

It is well known (see eg. **[Ro]**) that a universal enveloping algebra of L can be realized as a quotient of the tensor algebra $T(L)$ modulo the two-sided ideal I generated in $T(L)$ by all elements of the form

$x \otimes y - (-1)^{ab} y \otimes x - [x, y]$, where $x \in L_a$, $y \in L_b$ and $a, b \in \{0, 1\}$. If σ denotes the natural mapping $L \to T(L) \to T(L)/I = U$, then the pair (U, σ) is a universal enveloping algebra of L. We refer to (U, σ) as *the* enveloping algebra of L, denoting it by $U(L)$.

From this point on we will be considering only finite dimensional Lie superalgebras. In order to simplify notation, we will now quote a result which is in fact a corollary to the next theorem.

Fact. The natural mapping $\sigma: L \to U(L)$ is injective.

Therefore we can identify elements of L with their images under σ. We will also use juxtaposition, rather than $\otimes$, to denote multiplication in $U(L)$. The following generalization of the Poincaré-Birkhoff-Witt theorem describes the structure of $U(L)$ as a vector space.

THEOREM 1 ([**Ro**], *Theorem* 2.1). *Given any ordered basis* $\{x_1, \dots, x_s\}$ *of L consisting of homogeneous elements, the set of all products of the form*

$$x_1^{p_1} \cdots x_s^{p_s} \quad \left(\textit{where } x_i^0 = 1,\ p_i \geq 0 \textit{ and } p_i \leq 1 \textit{ whenever } x_i \textit{ is odd}\right)$$

is a basis of $U(L)$.

It is clear that if $x \in L_1$ and $[x, x] = 0$ then $x^2 = [x, x]/2 = 0$ in $U(L)$, which gives an early indication of significant differences between $U(L)$ and $U(L_0)$—the latter never containing any zero divisors.

Another important characteristic of $U(L)$ is the fact that it can be filtered by the following ascending chain of its subspaces: $U(L)^{(0)} = k$, $U(L)^{(1)} = k + L$ and $U(L)^{(n)} =$ the subspace of $U(L)$ spanned by all basis monomials of degree $\leq n$. Then $U(L) = \bigcup U(L)^{(i)}$, and for any $i, j \geq 0$ we have $U(L)^{(i)} U(L)^{(j)} \subseteq U(L)^{(i+j)}$. We can now form the *associated graded algebra* of $U(L)$, which is defined as the k-direct sum

$$\mathrm{Gr}(U(L)) := U(L)^{(0)} \oplus U(L)^{(1)}/U(L)^{(0)} \oplus U(L)^{(2)}/U(L)^{(1)} \oplus \cdots,$$

with multiplication given by

$$\left(a + U(L)^{(i)}\right)\left(b + U(L)^{(j)}\right) := ab + U(L)^{(i+j-1)}$$

and extended to $\mathrm{Gr}(U(L))$ by linearity. It follows immediately from Theorem 1 that

$$\mathrm{Gr}(U(L)) \cong k[X_1, \dots, X_m] \otimes_k \Lambda(Y_1, \dots, Y_n),$$

where $m = \dim_k L_0$, $n = \dim_k L_1$ and $\Lambda(Y_1, \dots, Y_n)$ is the exterior algebra on n symbols. It follows that $\mathrm{Gr}(U(L))$ is a (left- and right-) Noetherian ring, so that $U(L)$ is Noetherian on both sides as well.

The following result, which can be found in [**Sch**], is easy but useful.

PROPOSITION 2. *$U(L_0)$ is a subalgebra of $U(L)$. Moreover $U(L)$, as a $U(L_0)$-module (both left and right), is free with finite basis.*

Proof. Let $\{x_1, \ldots, x_m, y_1, \ldots, y_n\}$ be a homogeneous basis of L such that $x_i \in L_0$ and $y_j \in L_1$ for all $i = 1, \ldots, m$ and $j = 1, \ldots, n$. The vector subspace of $U(L)$ spanned by all monomials $x_1^{p_1} \cdots x_m^{p_m}$ with $p_i \geq 0$ is, by the standard Poincaré-Birkhoff-Witt theorem, isomorphic to $U(L_0)$. The relations $x_i x_j = x_j x_i + [x_i, x_j]$, which hold in both $U(L_0)$ and $U(L)$, make it easy to verify that the vector space isomorphism mentioned above is in fact an algebra isomorphism, which proves the first assertion.

Since there are exactly 2^n monomials $y_1^{q_1} \cdots y_n^{q_n}$ ($q_i = 0$ or 1), they can be labeled $Y_0 = y_1^0 \cdots y_n^0 = 1, Y_1, \ldots, Y_s$–where $s = 2^n - 1$. By Theorem 1 we then have $U(L) \cong U(L_0)Y_0 \oplus U(L_0)Y_1 \oplus \cdots \oplus U(L_0)Y_s$ as left $U(L_0)$-modules. Moreover, this form of elements of $U(L)$ is unique, so that the set $\{Y_0, Y_1, \ldots, Y_s\}$ is a free basis of the left $U(L_0)$-module $U(L)$. The analogous statement for "right" instead of "left" can be proved in the same way by starting with the ordered basis $\{y_1, \ldots, y_n, x_1, \ldots, x_m\}$ of the superalgebra L. □

EXAMPLE 1. We will now use the above remarks to describe (in considerable detail) the enveloping algebra of $L = \mathrm{pl}(1,1)$. If

$$x_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad x_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},$$

$$y_1 = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad y_2 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

Then $\{x_1, x_2\}$ is a basis of L_0 and $\{y_1, y_2\}$ is a basis of L_1. Let $Y_0 = 1$, $Y_1 = y_1$, $Y_2 = y_2$ and $Y_3 = y_1 y_2$. According to Proposition 2 we have $U(L) \cong U(L_0)Y_0 \oplus U(L_0)Y_1 \oplus U(L_0)Y_2 \oplus U(L_0)Y_3$ as left $U(L_0)$-modules. Since L_0 is abelian, $U(L_0)$ is simply the commutative polynomial ring $k[x_1, x_2]$. It follows from the multiplication table for L that we have the commutation relations

$$Y_1 x_1 = (x_1 - 1)Y_1, \quad Y_2 x_1 = (x_1 + 1)Y_2, \quad Y_1 x_2 = (x_2 + 1)Y_1,$$

$$Y_2 x_2 = (x_2 - 1)Y_2, \quad Y_3 x_1 = x_1 Y_3 \quad \text{and} \quad Y_3 x_2 = x_2 Y_3,$$

i.e. $U(L)$ is a normalizing extension of $U(L_0)$. In addition, it is easy to compute that

$$\begin{aligned} &Y_1Y_1 = 0, && Y_1Y_2 = Y_3, && Y_1Y_3 = 0,\\ &Y_2Y_1 = -Y_3 + x_1 + x_2, && Y_2Y_2 = 0, && Y_2Y_3 = (x_1 + x_2)Y_2,\\ &Y_3Y_1 = (x_1 + x_2)Y_1, && Y_3Y_2 = 0 \quad \text{and} && Y_3Y_3 = (x_1 + x_2)Y_3. \end{aligned}$$

All structure constants of $U(L)$ can now be derived from the above data. A laborious but straightforward computation shows that $U(L)$ is a prime ring—even though it has many zero-divisors. However, in general an enveloping algebra of a Lie superalgebra need not even be semiprime, which is the main point of the next example.

EXAMPLE 2. We will consider the 7-dimensional subspace of $M_4(k)$ with basis $x_1 = -e_{11} + e_{22} + e_{33} + e_{44}$, $x_2 = e_{12} - e_{43}$, $x_3 = -e_{21} + e_{34}$, $y_1 = e_{13}$, $y_2 = e_{24}$, $y_3 = e_{14} + e_{23}$ and $y_4 = e_{32} - e_{41}$, where e_{ij} denote the matrix units. It is easy to see that this subspace is actually a subalgebra of the Lie superalgebra pl(2, 2), which is usually denoted by b(2). The even part of b(2) (spanned by x_1, x_2 and x_3) is the special linear Lie algebra sl(2). According to Proposition 2 we can take the following elements of $U = U(\mathrm{b}(2))$ as a free basis over the algebra $U_0 = U(\mathrm{sl}(2))$:

$$\begin{aligned} &Y_1 = 1, && Y_1 = y_1, && Y_2 = y_2,\\ &Y_3 = y_3, && Y_4 = y_4, && \\ &Y_5 = y_1y_2, && Y_6 = y_1y_3, && Y_7 = y_1y_4,\\ &Y_8 = y_2y_3, && Y_9 = y_2y_4, && Y_{10} = y_3y_4,\\ &Y_{11} = y_1y_2y_3, && Y_{12} = y_1y_2y_4, && Y_{13} = y_1y_3y_4,\\ &Y_{14} = y_2y_3y_4 \quad \text{and} && Y_{15} = y_1y_2y_3y_4. && \end{aligned}$$

Using the commutation relations in U obtained similarly as in Example 1 one can verify the following facts:

(i) the generators Y_{11} and Y_{15} are normalized by U_0, i.e. $U_0Y_i = Y_iU_0$ for $i = 11, 15$ and

(ii) the vector subspace $I = U_0Y_{11} \oplus U_0Y_{15}$ satisfies $IY_j \subseteq I$ for all $i = 0, \ldots, 15$. This shows that I is a two-sided ideal of U. Moreover, since $IY_{11} = (0)$ and $IY_{15} = (0)$, it follows that $I^2 = (0)$ so that the nilpotent radical of U is non-zero, while an enveloping algebra of a Lie algebra is always not only a prime ring, but even a semiprimitive one.

We will now recall the role that the ring-theoretic structure of $U(L)$ plays in the representation theory of L.

A *representation* of a Lie superalgebra L is a k-linear map ρ of L into the space of endomorphisms of a vector space W, for which $\rho([x, y]) = \rho(x) \circ \rho(y) - (-1)^{ab}\rho(Y) \circ \rho(x)$ whenever $x \in L_a$, $y \in L_b$ and $a, b \in \{0, 1\}$.

It is clear that given any representation $\rho : L \to \operatorname{End}(W)$ the resulting action of L on W makes W into a left $U(L)$-module, and that every left $U(L)$-module can be thought of as being a representation space of L. We will recall that a $U(L)$-module W is *simple* if $U(L)W \neq (0)$ and the only subspaces of W which are invariant under the action of $U(L)$ are (0) and W itself. In this situation the *annihilator* of the $U(L)$-module W, i.e. the ideal $\operatorname{Ann}(W) = \{x \in U(L) \mid xW = (0)\}$ is called *primitive*, while W is then isomorphic, as a $U(L)$-module, to a factor $U(L)/M$ for some maximal left ideal of $U(L)$.

Thus the study of representations of a Lie superalgebra L is equivalent to the investigation of modules over the ring $U(L)$. In particular, irreducible representations of L are equivalent to simple $U(L)$-modules. This in turn motivates our interest in primitive ideals of $U(L)$, which will be evident in the following sections.

3. Elementary properties of the algebra $U(L)$. As in the proof of Proposition 2.2, we will fix an ordered basis $\{x_1, \ldots, x_m, y_1, \ldots, y_n\}$ of L in which $x_i \in L_0$ and $y_j \in L_1$ for $i = 1, \ldots, m$ and $j = 1, \ldots, n$; the monomials $y_1^{q_1} \cdots y_n^{q_n}$ with $q_i \in \{0, 1\}$ will be labeled $Y_0 = 1, Y_1, \ldots, Y_s$. For brevity we will use the symbols U_0 and U to denote the enveloping algebras of L_0 and L respectively.

PROPOSITION 1. (i) *U is a left and right Noetherian ring,*

(ii) *for any right ideal A of U_0, the equality $AU = U$ implies $A = U_0$,*

(iii) *every simple right U_0-module embeds in a simple right U-module,*

(iv) *if $\mathscr{J}(\cdot)$ denotes the Jacobson radical, then*

$$\mathscr{J}(U) \cap U_0 \subseteq \mathscr{J}(U_0) = (0).$$

Proof. (i) See remarks in §2.

(ii) Let $x \in U_0$. Then $x \in U = AU = AY_0 \oplus AY_1 \oplus \cdots \oplus AY_S$, i.e. x is of the form $a_0Y_0 + \cdots + a_sY_s$, with the coefficients a_i being uniquely determined by x. Then $x = a_0$, and hence $U_0 \subseteq A$.

(iii) Let M be a simple right U_0-module. Then $M \cong U_0/A$ for some maximal right ideal A of U_0. Since $M \neq (0)$, (ii) implies that $AU \neq U$. U is right Noetherian, which allows us to pick a maximal right ideal of U for which $AU \subseteq B$. Then U/B is a simple right U-module, and the canonical mapping of U_0-modules $U_0/A \to U/B$ is injective.

(iv) Let $x \in \mathcal{J}(U) \cap U_0$. Then x annihilates every simple right U-module and hence, by (iii), $Mx = (0)$ for every simple right U_0-module M. It follows that $x \in \mathcal{J}(U_0)$. It is well known that U_0 is a semiprimitive ring, so that $\mathcal{J}(U_0) = (0)$.

Part (iv) of the last Proposition can be proved in a more general case:

PROPOSITION 2. *Let I be a two-sided ideal of U, and let $U' = U/I$, $U_0' = U_0/(I \cap U_0)$. Then*
(i) *for any principal right ideal A of U_0', $AU' = U'$ implies $A = U_0'$ and*
(ii) $\mathcal{J}(U') \cap U_0' \subseteq \mathcal{J}(U_0')$.

Proof. (i) Assume that $A = aU_0'$ and $AU' = U'$. Then $1 \in AU' = aU'$, so that $1 = ay$ for some $y \in U'$. Since the right U_0'-module U' is Noetherian, the chain of U_0'-submodules

$$U_0' \subseteq U_0' + yU_0' \subseteq U_0' + yU_0' + y^2U_0' \subseteq \cdots$$

must stabilize. Hence for some $r \geq 1$ we have

$$y^r = x_0 + yx_1 + y^2x_2 + \cdots + y^{r-1}x_{r-1}$$

with $x_i \in U_0'$ for all $i = 0, \ldots, r-1$. Multiplying this equation on the left by a^r we obtain $1 = a^r x_0 + \cdots + ax_{r-1} \in A$, which shows that $A = U_0'$.

(ii) Let $a \in \mathcal{J}(U') \cap U_0'$. If x is any element of U_0', then $(1 - ax)$ is right invertible in U', or $(1 - ax)U' = U'$. But then by (i) we have $(1 - ax)U_0' = U_0'$, i.e. $(1 - ax)$ is right invertible in U_0'. It follows that $a \in \mathcal{J}(U_0')$. □

We will now note an easy consequence of Proposition 2.2, which deals with dimension GK (Gelfand-Kirillov dimension) and K.dim (classical Krull dimension).

PROPOSITION 3. (i) $\mathrm{GK}(U) = \mathrm{GK}(U_0) = \dim_k L_0 < \infty$,
(ii) $\mathrm{K.dim}(U) \leq \dim_k L_0 < \infty$.

Proof. (i) It follows from §2 that there is a filtration of U such that $\mathrm{Gr}(U) \cong k[X_1, \ldots, X_m] \otimes_k \Lambda(Y_1, \ldots, Y_n)$. By **[KL**, Lemma 3.10] we have $\mathrm{GK}(\mathrm{Gr}(U)) = m$, while **[BK]** implies that $\mathrm{GK}(U) = \mathrm{GK}(\mathrm{Gr}(U))$.

(ii) By **[BK]** again, $\mathrm{K.dim}(U) \leq \mathrm{GK}(U) = \dim_k L_0$. □

It should be remarked here that the frequently considered relations between prime and primitive ideals of U_0 and U (e.g. "lying over", "going up", cutting down") pose serious problems in the situation at hand. To the best of our knowledge, the question whether a prime ideal of U intersected with U_0 must be prime (or even semiprime) in U_0, has not been answered yet. This is also the case with a number of related questions, some of which are listed in §6. Apart from a partial answer in a special case (Proposition 5.2), we have been able to prove only the following fact:

PROPOSITION 4 (*Incomparability*). *Let A be a subring of B such that the right A-module B is finitely generated and let A be right Noetherian. For any prime ideal P of A and any pair of prime ideals Q_1, Q_2 of B satisfying $Q_i \cap A = P$ $(i = 1, 2)$, the inclusion $Q_1 \subseteq Q_2$ implies $Q_1 = Q_2$.*

Proof. Let $A' = A/P$ and $B' = B/Q_1$—both being right Noetherian. Assume that $Q_1 \neq Q_2$ and let $Q = Q_2/Q_1$. Q, as a non-zero ideal of the prime ring B', contains a right regular element. Then [**KL**, Proposition 3.15] implies $\mathrm{GK}(B') < \mathrm{GK}(B'/Q)$. However, both B' and B'/Q are finitely generated as right A'-modules, so that by [**KL**, Proposition 5.5] we have $\mathrm{GK}(B') = \mathrm{GK}(A') = \mathrm{GK}(B'/Q)$. This contradiction proves that $Q_1 = Q_2$. □

COROLLARY. *For any prime ideal P of U_0 the prime ideals Q of U for which $Q \cap U_0 = P$ are incomparable.*

It should be reiterated that it is not known to us whether such prime ideals of U exist at all!

So far we made no mention of the third important dimension function, $\mathrm{gl.dim}(U)$. The reason why it was left out of Proposition 3 is that it behaves in a significantly different way than either Krull or GK dimension, as we shall see shortly. A reasoning fashioned after [**AL**] implies that finiteness of $\mathrm{gl.dim}(U)$ is closely linked with a property considered in §2, namely with the existence of those non-zero elements x of L, for which $[x, x] = 0$. The remainder of this section will be devoted to proving that such elements always exist in the case when L is a classical simple Lie superalgebra. In what follows, we will list all such superalgebras (following [**Sch**]), exhibiting in each of them a non-zero element x with $[x, x] = 0$. We will assume here that k is an algebraically closed field. Without further explanation we will often represent elements of $\mathrm{pl}(m, n)$ as block matrices of the form

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}, \quad \text{where } A \in M_m(k), \quad B \in M_{m,n}(k),$$

$$C \in M_{n,m}(k) \quad \text{and} \quad D \in M_n(k).$$

We note that if x is an odd element of the above form then $A = D = 0$ and

$$[x, x] = 2\begin{pmatrix} BC & 0 \\ 0 & CB \end{pmatrix}.$$

We will denote the $r \times r$ identity matrix by I_r.

Case 1. $L = \mathrm{sp}(m, n)$, $m, n \geq 1$, $m \neq n$. L consists of matrices with supertrace 0, i.e. those block matrices for which $\mathrm{tr}(A) = \mathrm{tr}(D)$. Take x to be a matrix with $A = B = D = 0$, $C \neq 0$. Then $[x, x] = 0$.

Case 2. $L = \mathrm{spl}(m, m)/k \cdot I_{2m}$—same as case 1.

Case 3. $L = \mathrm{osp}(m, n)$, $n = 2r$, $m, r > 0$. Here L_1 consists of those block matrices from $\mathrm{pl}(m, n)$ for which $B^T = HC$, $A = D = 0$, where H stands for the matrix $\left(\begin{smallmatrix} 0 & I_r \\ -I_r & 0 \end{smallmatrix}\right)$. It can be seen that if $C = \left(\begin{smallmatrix} C_1 \\ C_2 \end{smallmatrix}\right)$, C_1 and C_2 being in $M_{r,m}(k)$, then B must be of the form $[C_2^T, -C_1^T]$. Let now e_{ij} denote the (i, j)th elementary matrix and let $\xi \in k$ be any square root of -1. Setting $C_1 = e_{11} + e_{12} \cdot \xi$ and $C_2 = 0$ we obtain matrices B and C such that $BC = 0$ and the only entry of CB which is not trivially 0 has the form $-1 - \xi^2$. Hence the block matrix $x \in L_1$ created in this fashion satisfies $[x, x] = 0$.

Case 4. $L = \mathrm{b}(n)$, $n \geq 3$. L_1 consists of block matrices in which A, B, C, D are all $n \times n$, $A = D = 0$ and $B^T = B$, $C^T = -C$. Take $B = I_n$ and $C = 0$.

Case 5. $L = \mathrm{d}(n)/k \cdot I_{2n}$, $n \geq 3$. L_1 consists of those $2n \times 2n$ block matrices for which $B = C \in \mathrm{sl}(n)$. Take $B = e_{n1}$; then

$$BC = CB = B^2 = 0$$

and so $[x, x] = 0$.

We now continue our list considering those classical simple Lie superalgebras which do not arise as subquotients of $\mathrm{pl}(m, n)$. The last three cases cover the remaining (exceptional) simple superalgebras.

Case 6. $L = \Gamma(\sigma_1, \sigma_2, \sigma_3)$, $\sigma_i \neq 0$, $\sigma_1 + \sigma_2 + \sigma_3 = 0$. For any element $x = x_1 \otimes x_2 \otimes x_3$ of the standard basis of L_1 (see **[Sch]** for a detailed definition of this family of algebras) the bracket $[x, x]$ is given by a linear combination of terms which all contain a factor $\Psi(x, x)$, where Ψ is a certain skew-symmetric bilinear form. It follows that $[x, x] = 0$ for all such $x \in L_1$.

Case 7. $L = \Gamma_2$. Here L_1 is a tensor product of the algebra of trace zero Cayley matrices C_0 and a 2-dimensional vector space V equipped with a skew-symmetric bilinear form $\langle\ ,\ \rangle$. The product of two elements of L is defined by $[c \otimes v, d \otimes w] := \langle v, w\rangle \Phi(c, d) + 4\,\mathrm{tr}(cd)\Psi(v, w)$, with Φ being a certain skew-symmetric form on C_0 and Ψ a symmetric form on V (see e.g. **[Ka]**). Thus $[c \otimes v, c \otimes v] = 4\,\mathrm{tr}(c^2)\psi(v, v)$, and the task of finding a non-zero $x \in L_1$ with $[x, x] = 0$ reduces to the problem of producing a Cayley matrix $c \neq 0$ with $\mathrm{tr}(c) = 0$ and $\mathrm{tr}(c^2) = 0$. This is easily done as follows: let $c_{11} = 1$ and $c_{22} = -1$; the off-diagonal entries of c come from the 3-dimensional vector algebra, whose standard basis we shall label $\{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$. When (for instance) $x = c_{12} = c_{21} = \mathbf{i}$, multiplication in C_0 gives

$$c^2 = \begin{pmatrix} 1 - (x, x) & ** \\ ** & 1 - (x, x) \end{pmatrix},$$

where $(\ ,\ )$ is the standard inner product on V (see e.g. **[J]** for a complete treatment of C_0). It is clear that $\mathrm{tr}(c^2) = 0$, as desired.

Case 8. $L = \Gamma_3$. According to **[SNR]**, L_1 is spanned by vectors x for which $[x, x]$ is a linear combination of the basis vectors of L_0 with coefficients involving, as factors, diagonal elements of two matrices:

$$C = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$ the charge conjugation matrix in $\mathbf{R}^3$, and

$$\tilde{C} = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \end{pmatrix},$$ the charge conjugation matrix in $\mathbf{R}^7$ (see **[P]**.)

It follows that $[x, x] = 0$ for x being any vector in that basis of L_1.

Using terminology of [**AL**] we can now conclude that if L is a classical simple Lie superalgebra over any field k, then L is *not* absolutely torsion free.

PROPOSITION 5. *If U is the enveloping algebra of a classical simple Lie superalgebra then* $\text{gl.dim}(U) = \infty$.

Proof (*cf.* [**AL**], *Proposition* 1.8). Let $x \neq 0$ be an element of L_1 for which $[x, x] = 0$. The one-dimensional subspace $k \cdot x$ of L is an abelian subalgebra of L, whose enveloping algebra $V = U(k \cdot x) \subseteq U$ has infinite global dimension. It follows from Theorem 2.1 that U is free over V, so that any projective U-module is also projective over V. This shows that $\text{gl.dim}(U) = \infty$. □

4. Rings of quotients and Hilbert's Nullstellensatz. It is well known that $U(L_0)$, as a prime Noetherian ring, can be embedded in an Artinian simple full ring of quotients (see e.g. [**Go**]). Even though $U(L)$ is, in general, not even semiprime (cf. Example 2.2), it turns out that it also possesses an Artinian full quotient ring—but of course not always a semisimple one. The technique employed here in proving this result has been used quite frequently and in many variants, e.g. in [**Bo**].

THEOREM 1. *Let A be a Noetherian ring with an Artinian full quotient ring, and let B be a ring containing A in such a way that B is finitely generated as a left and right A-module. If every regular element of A remains regular in B, then B has an Artinian full ring of quotients.*

Proof. Let T be the set of all regular elements of A. We will show that by attaching formally inverses of elements from T to the ring B we obtain an Artinian ring in which all regular elements of B are invertible as well.

Let A_T denote the full quotient ring of A. Define B_T to be the tensor product $B \otimes_A A_T$, which clearly is a finitely generated right A_T-module. It consists of elements of the form $b \otimes t^{-1}$, where $b \in B$ and $t \in T$ (see e.g. [**St**, Proposition II.3.2]). Since A_T is right Artinian, B_T is an Artinian module.

We will now prove that T is an Ore set in the ring B. Given any $t \in T$ let φ_t be the endomorphism of the right A-module B defined by $\varphi_t(b) := tb$. By [**St**, Proposition II.3.5], A_T is flat as a left A-module. This implies that the extension $\gamma_t := \varphi_t \otimes \text{id}$ is an injective endomorphism of

$B \otimes A_T$. But B_T is Artinian, and hence γ_t is also surjective. Then for $b \in B$ the element $b \otimes 1$ is an image of $b_1 \otimes t_1^{-1}$ for some $b_1 \in B$ and $t_1 \in T$. It follows that $bt_1 = tb_1$, which is the required right Ore condition.

Finally, if b is any regular element of B, b is also regular in B_T—which is Artinian. Considering the descending chain of right ideals $\ldots \subseteq b^3B_T \subseteq b^2B_T \subseteq bB_T$ we obtain $b^n = b^{n+1}q$ for some $n \geq 1$ and $q \in B_T$. Then (by regularity of b) $1 = bq$, or $b^{-1} = q \in B_T$. Therefore B_T is a full quotient ring of B. □

COROLLARY. *If L is a finite-dimensional Lie superalgebra then $U(L)$ has a full quotient ring which is Artinian.*

Proof. $U(L)$, as a finite, free module over the prime ring $U(L_0)$, satisfies all hypotheses of the theorem. □

We will now note that $U(L)$ enjoys two more properties which are known for enveloping algebras of Lie algebras.

Several authors have considered the following non-commutative generalization of Hilbert's Nullstellensatz:

DEFINITION. A k-algebra B satisfies the Nullstellensatz if, for every simple B-module M, the division ring $\mathrm{End}_B(M)$ is an algebraic k-algebra.

We also have the following related property:

DEFINITION. A ring B is Jacobson if every prime ideal of S is an intersection of primitive ideals.

Using the results of **[Mc]** we easily obtain

PROPOSITION 2. *If L is a finite-dimensional Lie superalgebra, then $U(L)$ is a Jacobson ring satisfying the Nullstellensatz.*

Proof. $U(L_0)$ is an almost normalizing extension of the field k, which obviously is a Jacobson ring. $U(L)$, as a finite $U(L_0)$-module, is then a poly-(finite/almost normalizing) extension of k. **[Mc**, Theorem 4.6] implies the thesis. □

The Jacobson property of the Noetherian ring $U(L)$ has the following immediate consequence:

COROLLARY. *The Jacobson and prime radicals of $U(L)$ coincide (and hence $\mathcal{J}(U(L))$ is a nilpotent ideal).*

5. Additivity principle for Goldie rank and a theorem of Harish-Chandra. Let A be a Noetherian ring contained as a subring in a ring B. If Q is a prime ideal of B then $P = Q \cap A$ is not, in general, prime in A. However, the set of those prime ideals I of A which are minimal with respect to $P \subseteq I$, is finite. Their intersection is the smallest semiprime ideal of A which contains P. In the remainder of this section we will preserve the notation used above, with $I_1, \dots, I_s$ denoting all the distinct prime ideals of A minimal over P. We will now present an approach, originated in **[JS]**, to the question of relationship between the ideals P and Q.

DEFINITION. The ring extension $A \subseteq B$ (with A Noetherian) satisfies the additivity principle for Goldie rank if, for any prime ideal Q of B, there exist positive integers $z_1, \dots, z_s$ such that

$$\mathrm{rk}(B/Q) = \sum z_j \cdot \mathrm{rk}(A/I_j),$$

where $\mathrm{rk}(\cdot)$ is the Goldie (or uniform) rank of a ring.

In **[JS]** the additivity principle was proved for a certain class of ring extensions, in particular for the case in which $\mathfrak{m}$ is an ideal of a finite-dimensional Lie algebra $\mathfrak{g}$, and $A = U(\mathfrak{m})$, $B = U(\mathfrak{g})$. This result was subsequently generalized in **[Bo]** to the case of so-called restricted ring extensions:

DEFINITION. A ring extension $A \subseteq B$ is *restricted* if, for any $b \in B$, AbA is Noetherian as a left and right A-module.

This easily implies

THEOREM 1. *The ring extension $U(L_0) \subseteq U(L)$ satisfies the additivity principle.*

Proof. If $b \in U(L)$ then $U(L_0)bU(L_0)$ is a submodule of the Noetherian $U(L_0)$-module $U(L)$ and, as such, is Noetherian itself. Therefore the extension in question is restricted. Moreover, by Proposition 3.3(i), $\mathrm{GK}(U(L)) < \infty$ and all hypotheses of **[Bo**, Theorem 7.2] are satisfied.

We will now note that under certain special circumstances the additivity principle can be used to relate simple $U(L)$-modules with simple $U(L_0)$-modules, which in turn may provide a way of classifying irreducible representations of L in terms of the better known irreducible representations of L_0. This is the motivation for the next result.

PROPOSITION 2. *Let Q be a prime ideal of $U(L)$. Suppose that there exists a right $U(L)$-module M such that $Q = \mathrm{Ann}_{U(L)}(M)$ and that M, as a right $U(L_0)$-module, is of finite composition length. Then the prime ideals $I_1, \ldots, I_s$ of $U(L_0)$ are all primitive.*

Proof. Choose a composition series $(0) = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_n = M$ of the $U(L_0)$-module M. If $P_j = \mathrm{Ann}_{U(L_0)}(M_j/M_{j-1})$ then $MP_nP_{n-1} \cdots P_1 = (0)$, so that $P_nP_{n-1} \cdots P_1 \subseteq P \subseteq \cap I_i$. Hence for every $i = 1, \ldots, s$ the product $P_nP_{n-1} \cdots P_1$ is contained in the prime ideal I_i. Therefore there is an integer j $(1 \leq j \leq n)$ for which $P_j \subseteq I_i$. Obviously, $P \subseteq P_j$; moreover, P_j is primitive, and hence prime. It follows from minimality of I_i that $P_j = I_i$, which proves the assertion. □

It should be pointed out here that the Proposition applies, in particular, to the situation when Q is an annihilator of a finite-dimensional simple $U(L)$-module M, which of course will have finite composition length as a $U(L_0)$-module.

Our last result will be a generalization of the well known theorem obtained for Lie algebras by Harish-Chandra.

THEOREM 3. *For every non-zero element $x \in U(L)$ there exists a two-sided ideal I of $U(L)$ such that $x \notin I$ and $\dim_k U(L)/I < \infty$.*

Proof (cf. also [**Bos**, Lemma 1.1]). Let $\mathcal{F}$ be the collection of all right ideals of $U = U(L)$ which are of finite codimension. We will first prove that $\bigcap \mathcal{F} = (0)$. Let x be a non-zero element of U. Invoking the notation of Proposition 2.2 we write x as a sum $x_0Y_0 + \cdots + x_sY_s$ with $x_i \in U_0 = U(L_0)$ for $i = 0, \ldots, s$, so that $x_j \neq 0$ for some j $(0 \leq j \leq s)$. It follows from Harish-Chandra's theorem that there exists an ideal J_0 and U_0 such that $x_j \notin J_0$ and U_0/J_0 is finite-dimensional. Consider $J = \sum J_0Y_i$, which is clearly a right ideal of U. The vector space U/J is isomorphic to a direct sum of $s + 1$ copies of U_0/J_0—hence $\dim_k(U/J)$ is finite. By definition of J and freeness of U, $x \notin J$.

Let us now consider a non-zero $x \in U$ again. By the above we can find a right ideal J of U which has finite codimension and does not contain x. U/J is then a finite-dimensional right U-module. If $I = \mathrm{Ann}_U(U/J)$, $x \notin I$—otherwise we would have $Ux \subseteq J$ and $x \in J$. U/J is also a finite-dimensional faithful right U/I-module, so that U/I can be embedded in the finite-dimensional vector space $\mathrm{End}_k(U/J)$. Therefore I is a two-sided ideal of finite codimension, not containing x. □

6. Some open questions. We conclude with a list of selected problems which the author was unable to solve, and which seem to be important for the understanding of the structure of enveloping superalgebras. In what follows, L is a finite-dimensional Lie superalgebra, $U_0 = U(L_0)$ and $U = U(L)$.

The most obvious type of question regards the relation between ideals of U and those of U_0.

Question 1. Let I be an ideal of U and $I_0 = I \cap U_0$.
(i) If I is prime, is I_0 prime?
(ii) If not, is it always semiprime?
(iii) If I is primitive, is I_0 primitive? prime?
It may be interesting to consider these questions in the special case when L is a classical simple Lie superalgebra.

A related problem, connected with irreducible representations of L, is of considerable interest in view of Proposition 5.2:

Question 2. Does a simple U-module have finite composition length when considered as a U_0-module?

Finally, we can ask how often does the situation described in Example 2.2 occur:

Question 3. What characterizes, in terms of L, those enveloping algebras for which $\mathcal{J}(U) = (0)$?

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UNIVERSITY OF CALIFORNIA
LOS ANGELES, CA 90024