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# WEIGHTED NORM INEQUALITIES FOR THE FOURIER TRANSFORM ON CONNECTED LOCALLY COMPACT GROUPS

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**Let $G$ be a locally compact connected group. If $G$ is also either compact or abelian, sufficient conditions on a non-negative pair of measurable functions $T$ and $V$ are given to imply that there exists a constant $c$ independent of $f$ for which an inequality of the form**

$$\left(\int_\Gamma |\hat{f}(\gamma)|^{p'} T(\gamma)\, dm_\Gamma(\gamma)\right)^{1/p'} \le c\left(\int_G |f(x)|^p V(x)\, dm_G(x)\right)^{1/p}$$

**holds for all integrable $f$ on $G$ ($1 < p \le 2, p' = p/(p-1)$). Here, $\hat{f}$ denotes the Fourier transform of $f$ defined on $\Gamma$ (the dual of $G$) and $m_G$, $m_\Gamma$ are Haar measures on $G, \Gamma$ respectively. Conditions on $T, V$ are also given to esnure that the inequality holds with $p'$ replaced by a more general exponent on a more restricted class of groups.**

**1. Introduction.** In [**6**], sufficient conditions are placed on non-negative pairs of functions $T, V$ to imply

$$\left(\int_{\mathbf{R}^n} |\hat{f}(x)|^q T(x)\, dx\right)^{1/q} \le c\left(\int_{\mathbf{R}^n} |f(x)|^p V(x)\, dx\right)^{1/p}$$

for all integrable functions $f$ on $\mathbf{R}^n$, with $c$ a constant independent of $f$. Our aims here are to first make sense of such inequalities in a more abstract setting (in particular, for functions defined on a fairly broad class of locally compact groups $G$ and their Fourier transforms defined on the suitable dual object $\Gamma$ with $m_G$, $m_\Gamma$ being the appropriate measures on $G$ and $\Gamma$ respectively), and then to produce conditions on $T, V$ analogous to those found in [**6**] so that the inequality will be valid. In the case $1 < p \le 2$ and $q = p' = p/(p-1)$, the main technical difficulty is the proof of the existence of a measurable function $W$: $G \to \mathbf{R}^+$ ($= [0, \infty)$) with the properties

$$\begin{aligned} &V(x) \le 2W(x) < 2V(x) && \text{for all } x \in G \quad \text{and} \\ &m_G\{x \in G; W(x) = \alpha\} = 0 && \text{for all } \alpha > 0. \end{aligned} \tag{1.1}$$

In the case $q \ne p'$ however, it is necessary to have a measurable function $S$: $\Gamma \to \mathbf{R}^+$ satisfying

$$\begin{aligned} &T(\gamma) \le 2S(\gamma) < 2T(\gamma) && \text{for all } \gamma \in \Gamma \quad \text{and} \\ &m_\Gamma\{\gamma \in \Gamma\colon S(\gamma) = \alpha\} = 0 && \text{for all } \alpha > 0. \end{aligned} \tag{1.2}$$

Let $G$ be a locally compact abelian (LCA) group. Let $\Gamma$ denote the dual group of $G$. Equip $G$ and $\Gamma$ with Haar measures $m_G$ and $m_\Gamma$ normalized so that the Plancherel identity is valid. If $\gamma \in \Gamma$ and $f \in L^1(G)$, the Fourier transform of $f$ is defined by

$$\hat{f}(\gamma) = \int_G f(x)\overline{\gamma(x)}\, dm_G(x).$$

The Hausdorff-Young theorem then states that if $1 \le p \le 2$ and $f \in L^p(G)$, then $\hat{f} \in L^{p'}(\Gamma)$ and

$$\left(\int_\Gamma |\hat{f}(\gamma)|^{p'} dm_\Gamma(\gamma)\right)^{1/p'} \le \left(\int_G |f(x)|^p dm_G(x)\right)^{1/p}.$$

Now let $G$ be any group. A representation of $G$ is a homomorphism $U$: $G \to \mathrm{GL}(\mathscr{H}_U)$ where $\mathrm{GL}(\mathscr{H}_U)$ is the group of invertible endomorphisms of some finite-dimensional linear space $\mathscr{H}_U$. $\mathscr{H}_U$ is the representation space of $U$ and the dimension of $\mathscr{H}_U$ is defined to be the dimension of the representation $U$ $(= d(U))$.

A unitary representation of $G$ is a homomorphism $U$: $G \to \mathscr{U}(\mathscr{H}_U)$ where $\mathscr{U}(\mathscr{H}_U)$ is the group of unitary endomorphisms of some finite-dimensional Hilbert space $\mathscr{H}_U$.

If $G$ is compact, let $\hat{G}$ denote a maximal set of pairwise inequivalent continuous unitary representations of $G$. If $f \in L^1(G)$ and $U \in \hat{G}$, define $\hat{f}(U) \in \mathrm{End}(\mathscr{H}_U)$ by

$$\hat{f}(U) = \int_G f(x)U(x)\, dm_G(x).$$

Let $E(\hat{G}) = \prod_{U \in \hat{G}} \mathrm{End}(\mathscr{H}_U)$. If $\psi \in E(\hat{G})$, define

$$\|\psi\|_p = \left\{\sum_{U \in \hat{G}} d(U)\|\psi(U)\|_{\phi_p}^p\right\}^{1/p} \qquad \text{if } p \in [1, \infty),$$

$$\|\psi\|_\infty = \sup_{U \in \hat{G}} \|\psi(U)\|_{\phi_\infty},$$

where $\|\cdot\|_{\phi_p}$ denotes the $p$th von Neumann norm on $\mathrm{End}(\mathscr{H}_U)$ ([5] Sec. 28.34). Now let $E^p(\hat{G})$ be the linear space of $\psi \in E(\hat{G})$ for which $\|\psi\|_p < \infty$. In this context, the Hausdorff-Young theorem states that if $1 \le p \le 2$ and $f \in L^p(G)$ then $\hat{f} \in E^{p'}(\hat{G})$ and

$$\left(\sum_{U \in \hat{G}} d(U)\|\hat{f}(U)\|_{\phi_{p'}}^{p'}\right)^{1/p'} \le \left(\int_G |f(x)|^p dm_G(x)\right)^{1/p}.$$

The sufficiency conditions for the case $q = p'$ are as follows.

THEOREM 1.1. *Let $G$ be an LCA, non-compact, connected group with dual group $\Gamma$. Let $T$ be a non-nonegative, measurable function on $\Gamma$ and $V$ a non-negative, measurable function on $G$. Suppose $1 < p \leq 2$ and there exist positive constants $A, B$ such that*

$$(1.3)\qquad \left(\int_{T(\gamma)>Br} T(\gamma)\, dm_\Gamma(\gamma)\right)\left(\int_{V(x)<r^{p-1}} V(x)^{-1/(p-1)}\, dm_G(x)\right) \leq A$$

*for all $r > 0$. Then, for every integrable function $f$ on $G$,*

$$(1.4)\qquad \left(\int_\Gamma |\hat{f}(\gamma)|^{p'} T(\gamma)\, dm_\Gamma(\gamma)\right)^{1/p'} \leq c\left(\int_G |f(x)|^p V(x)\, dm_G(x)\right)^{1/p}$$

*with $c$ dependent on $A, B, p$ but not on $f$.*

If $G$ is compact and abelian, $\Gamma$ is discrete and $m_\Gamma$ is just the counting measure.

THEOREM 1.2. *Let $G$ be a compact, connected, abelian group with dual group $\Gamma$. Let $T$ be a non-negative function on $\Gamma$ and $V$ a non-negative, measurable function on $G$. Suppose $1 < p \leq 2$ and there exist positive constants $A, B$ such that*

$$(1.5)\qquad \left(\sum_{T(\gamma)>Br} T(\gamma)\right)\left(\int_{V(x)<r^{p-1}} V(x)^{-1/(p-1)}\, dm_G(x)\right) \leq A$$

*for all $r > 0$. Then for every integrable function $f$ on $G$,*

$$(1.6)\qquad \left(\sum_{\gamma\in\Gamma} |\hat{f}(\gamma)|^{p'} T(\gamma)\right)^{1/p'} \leq c\left(\int_G |f(x)|^p V(x)\, dm_G(x)\right)^{1/p}$$

*with $c$ dependent on $A, B, p$ but not on $f$.*

If $G$ is compact and non-abelian, the statement of the theorem is slightly more complicated.

THEOREM 1.3. *Let $G$ be a compact, connected, non-abelian group with dual object $\hat{G}$. Let $T$ be a non-negative function on $\hat{G}$ and $V$ a non-negative, measurable function on $G$. Suppose $1 < p \leq 2$ and there exist positive constants $A, B$ such that*

$$(1.7)\qquad \left(\sum_{T(U)>Br} T(U) d(U)^2\right)\left(\int_{V(x)<r^{p-1}} V(x)^{-1/(p-1)}\, dm_G(x)\right) \leq A$$

*for all $r > 0$. Then for every integrable function $f$ on $G$,*

$$(1.8)\quad \left(\sum_{U\in\hat{G}} \|\hat{f}(U)\|_{\phi_{p'}}^{p'} T(U)\, d(U)\right)^{1/p'} \le c\left(\int_G |f(x)|^p V(x)\, dm_G(x)\right)^{1/p}$$

*with $c$ dependent on $A$, $B$, $p$ but not on $f$.*

Note that the statement of Theorem 1.3 collapses to that of Theorem 1.2 if $G$ is abelian.

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**2. Non-compact abelian case.** To prove the theorems, several lemmas are required leading to the existence of a function $W$ on $G$ satisfying (1.1).

LEMMA 2.1 (*Price, Hogan*). *Let $(X, m_X)$ be a $\sigma$-finite measure space. Let $V$ be a non-negative, measurable function on $X$ which is greater than 0 $m_X$-a.e. For each $r > 0$, let $B_r = \{x \in X;\ V(x) = r\}$ and $R = \{r > 0;\ m_X(B_r) > 0\}$. Then $R$ is countable.*

*Proof.* $X = \bigcup_{n\in\mathbf{Z}} K_n$, $m_X(K_n) < \infty$ for each $n \in \mathbf{Z}$. Let $R_n = \{r \in R;\ m_X(B_r \cap K_n) > 0\}$ and $R_n^m = \{r \in R;\ m_X(B_r \cap K_n) > 1/m\}$ for $m \in \mathbf{Z}^+$ $(= \{n \in \mathbf{Z};\ n > 0\})$. Then $m_X(B_r \cap K_n) \le m_X(K_n) < \infty$ for $n \in \mathbf{Z}$. So $\#R_n^m$ ($=$ cardinality of $R_n^m$) $< \infty$. But $R_n = \bigcup_{m=1}^{\infty} R_n^m$, hence $R_n$ is countable. Let $r \in R$, $0 < m_X(B_r) \le \sum_{n\in\mathbf{Z}} m_X(B_r \cap K_n)$. Therefore, $r \in R_n$ for some $n \in \mathbf{Z}$, i.e. $R \subseteq \bigcup_{n\in\mathbf{Z}} R_n$ and hence, $R$ is countable. □

The utility of this lemma lies in the observation that if $G$ is a connected LCA group then by Theorem 2.4.1 of [**8**],

$$(2.1)\qquad G \simeq H \times \mathbf{R}^n \qquad (n \ge 0)$$

with $H$ compact and abelian. (Here $\simeq$ denotes a topological isomorphism.) Of course, if $G$ is non-compact then $n > 0$ in (2.1). $G$ is then $\sigma$-compact. The regularity of the measure $m_G$ implies the $\sigma$-finiteness of the measure space $(G, m_G)$, so Lemma 2.1 applies to $G$.

LEMMA 2.2. *If $G$ is a connected, non-compact, LCA group and $V$ is a non-negative, measurable function on $G$ which is greater than 0 $m_G$-a.e., then there exists a measurable function $W$ on $G$ satisfying* (1.1).

*Proof.* $G \simeq H \times \mathbf{R}^n$ with $H$ compact and $n \ge 1$. Let $m_H$ denote the unique Haar measure on $H$ satisfying $m_H(H) = 1$. Let $m_n$ denote ordinary Lebesgue measure on $\mathbf{R}^n$. Then $m_G = m_H \times m_n$ is a Haar measure on $G$. (Any other Haar measure on $G$ will be just a constant multiple of this one, so sets of measure zero are independent of the choice involved.) Denote the Euclidean norm on $\mathbf{R}^n$ by $\|\cdot\|$. Let $B_r = \{(h, x) \in G; V(h, x) = r\}$ $(r > 0)$, a Borel subset of $G$. Let $E = \bigcup_{m_G(B_r)>0} B_r$. By Lemma 2.1, $E$ is also a Borel set (being the countable union of Borel sets). Let $S_r(y)$ be the sphere in $\mathbf{R}^n$ with centre $y$, radius $r$. Let $G(h, x) = \|x\|$ and $g(h, x) = \sin\|x\|$.

$$G(h_1, x_1) = G(h_2, x_2) \Leftrightarrow \|x_1\| = \|x_2\| \ (= r \text{ say})$$
$$\Leftrightarrow (h_1, x_1), (h_2, x_2) \in H \times S_r(0).$$

Therefore,

$$m_G\{(h, x) \in G; G(h, x) = r\} = m_G(H \times S_r(0))$$
$$= m_H(H) m_n(S_r(0)) = 0 \quad \text{for all } r > 0.$$

Now, $\{(h, x) \in G; g(h, x) = \alpha\}$ is either empty (if $|\alpha| > 1$), or a countable union of sets of the form $H \times S_r(0)$, so $m_G\{(h, x) \in G; g(h, x) = \alpha\} = 0$ for all $\alpha > 0$. Let

$$W(h, x) = \begin{cases} 3V(h, x)/4 & \text{on } G \setminus E \\ 3r/4 + rg(h, x)/8 & \text{on } B_r. \end{cases}$$

$W$ satisfies (1.1) and the measurability of $V$ and continuity of $g$ imply the measurability of $W$. □

In what follows, $\chi_E$ denotes the characteristic function of a measurable set $E$ and we adopt the convention $0 \cdot \infty = 0$.

*Proof of Theorem* 1.1. If $V(x) = 0$ on a set of positive measure then (1.3) implies $T(\gamma) = 0$ $m_\Gamma$-a.e., so that (1.4) is true trivially. If $m_G\{x \in G; V(x) = \infty \text{ and } f(x) \neq 0\} > 0$, then the right side of (1.4) is infinite so that (1.2) is true trivially. From now on we assume $V(x) > 0$ $m_G$-a.e. and $m_G\{x \in G; V(x) = \infty \text{ and } f(x) \neq 0\} = 0$. Let

$$I = \int_\Gamma |\hat{f}(\gamma)|^{p'} T(\gamma)\, dm_\Gamma(\gamma) \le 2^{p'}(I_1 + I_2) \tag{2.2}$$

where

$$I_1 = \sum_{j=-\infty}^{\infty} \int_{2^j B < T(\gamma) \le 2^{j+1} B} T(\gamma) \left| \int_{V(x) \ge 2^{jp-j-1}} f(x)\overline{\gamma(x)}\, dm_G(x) \right|^{p'} dm_\Gamma(\gamma)$$

and

$$I_2 = \sum_{j=-\infty}^{\infty} \int_{2^j B < T(\gamma) \leq 2^{j+1}B} T(\gamma) \left| \int_{V(x) < 2^{jp-j-1}} f(x)\overline{\gamma(x)}\, dm_G(x) \right|^{p'} dm_\Gamma(\gamma).$$

$$I_1 \leq 2B \sum_{j=-\infty}^{\infty} 2^j \int_\Gamma \left|(f\chi_{V \geq 2^{jp-j-1}})^\wedge(\gamma)\right|^{p'} dm_\Gamma(\gamma)$$

$$\leq 2B \sum_{j=-\infty}^{\infty} 2^j \left( \int_G |f(x)|^p \chi_{V \geq 2^{jp-j-1}}(x)\, dm_G(x) \right)^{p'/p}$$

(by the Hausdorff-Young inequality)

$$\leq 2B \left( \int_G |f(x)|^p h(x)\, dm_G(x) \right)^{p'/p}$$

(since $p'/p \geq 1$) where $h(x) = \sum_{j=-\infty}^{\infty} 2^{jp/p'} \chi_{V \geq 2^{jp-j-1}}(x)$. For any $x$, $h(x) = \sum_{j=-\infty}^{J} 2^{jp/p'}$ where $J$ is the largest integer satisfying $V(x) \geq 2^{Jp-J-1}$. Estimating the geometric series defining $h$, we see $h(x) \leq 2V(x)/(1 - 2^{1-p})$. Therefore, there exists a constant $c$ such that

$$I_1 \leq c \left( \int_G |f(x)|^p V(x)\, dm_G(x) \right)^{p'/p}. \tag{2.3}$$

Now, $I_2 \leq \int_\Gamma T(\gamma) (\int_{2V(x) < (T(\gamma)/B)^{p-1}} |f(x)|\, dm_G(x))^{p'}\, dm_\Gamma(\gamma)$. By Lemma 2.2, there exists a measurable function $W$ satisfying (1.1). Let $J$ be the least integer such that $2^J \geq \|f\|_1$. Let $r_J = \infty$ and for $j < J$, choose $r_j$ such that

$$3 \cdot 2^{j-2} < \int_{2W(x) < r_j^{p-1}} |f(x)|\, dm_G(x) \leq 2^j.$$

$$I_2 \leq \sum_{j=-\infty}^{J} \int_{Br_{j-1} < T(\gamma) \leq Br_j} T(\gamma) \left( \int_{2W(x) < r_j^{p-1}} |f(x)|\, dm_G(x) \right)^{p'} dm_\Gamma(\gamma).$$

Let

$$D_j = \left\{ x \in G;\, r_{j-2}^{p-1} \leq 2W(x) < r_{j-1}^{p-1} \right\}.$$

Then

$$\int_{2W(x) < r_{j-1}^{p-1}} |f(x)|\, dm_G(x) \leq 8 \int_{D_j} |f(x)|\, dm_G(x) \quad \text{and}$$

$$I_2 \leq 8^{p'} \sum_{j=-\infty}^{J} \int_{Br_{j-1} < T(\gamma) \leq Br_j} T(\gamma) \left( \int_{D_j} |f(x)|\, dm_G(x) \right)^{p'} dm_\Gamma(\gamma).$$

Since $V(x) > 0$ $m_G$-a.e. and $f = 0$ $m_G$-a.e. on the set where $V = \infty$, we may write

$$|f(x)| = \left(|f(x)|V(x)^{1/p}\right)V(x)^{-1/p}$$

and apply Hölder's inequality on $D_j$ to get

$$I_2 \leq 8^{p'} \sum_{j=-\infty}^{J} \int_{Br_{j-1} < T(\gamma)} T(\gamma)\, dm_\Gamma(\gamma) \left( \int_{D_j} V(x)^{-p'/p}\, dm_G(x) \right)$$
$$\times \left( \int_{D_j} |f(x)|^p V(x)\, dm_G(x) \right)^{p'/p}.$$

However, $D_j \subseteq \{x \in G; V(x) < r_{j-1}^{p-1}\}$, so

$$(2.4) \quad I_2 \leq 8^{p'} \sum_{j=-\infty}^{J} \int_{Br_{j-1} < T(\gamma)} T(\gamma)\, dm_\Gamma(\gamma) \left( \int_{V(x) < r_{j-1}^{p-1}} V(x)^{-p'/p}\, dm_G(x) \right)$$
$$\times \left( \int_{D_j} |f(x)|^p V(x)\, dm_G(x) \right)^{p'/p}$$
$$\leq 8^{p'} A \left( \sum_{j=-\infty}^{J} \int_{D_j} |f(x)|^p V(x)\, dm_G(x) \right)^{p'/p}$$

(by the condition (1.3) and the fact that $p'/p \geq 1$)

$$= 8^{p'} A \left( \int_G |f(x)|^p V(x)\, dm_G(x) \right)^{p'/p}.$$

Combining (2.2), (2.3) and (2.4) gives (1.4). □

**3. Compact abelian case.** The decomposition (2.1) is trivial if $G$ is compact in the sense that no $\mathbf{R}^n$ factor appears. A different argument is required.

LEMMA 3.1. *If $G$ is a compact, connected, abelian group with Haar measure $m_G$ and $V$ is greater than $0$ $m_G$-a.e., then there exists a measurable function $W$ on $G$ satisfying* (1.1).

*Proof*. Choose a non-trivial $\gamma \in \Gamma$ and let $\ker\gamma = H$. $H$ is a proper closed subgroup of $G$, hence $m_G(H) = 0$ (since $G$ is compact and connected). Now $\gamma(x) = \gamma(y)$ if and only if $xH = yH$ so that $\gamma$ is constant on cosets of $H$ and assumes different values on different cosets. Let $\phi(x) = \mathrm{Re}(\gamma(x))$, i.e. $\phi(x) = \frac{1}{2}(\gamma(x) + \overline{\gamma(x)})$. If $\phi(x) = \cos\theta$ for some $\theta \in [0, 2\pi)$,

$$\begin{aligned} &m_G\{x \in G; \phi(x) = \cos\theta\} \\ &\quad = m_G\{x \in G; \gamma(x) = \cos\theta + i\sin\theta \text{ or } \cos\theta - i\sin\theta\} \\ &\quad = m_G\{z_1H \cup z_2H\} \quad \text{with } z_1H,\ z_2H \text{ cosets of } H \text{ in } G \\ &\quad \le m_G(z_1H) + m_G(z_2H) = 0. \end{aligned}$$

Let $B_r = \{x \in G; V(x) = r\}$ $(r > 0)$ and $E = \bigcup_{m_G(B_r) > 0} B_r$. $E$ and each $B_r$ are measurable. Let

$$W(x) = \begin{cases} 3V(x)/4 & \text{on } G \setminus E \\ 3r/4 + r\phi(x)/8 & \text{on } B_r. \end{cases}$$

$W$ satisfies (1.1) and the measurability of $V$ and continuity of $\phi$ imply the measurability of $W$. □

*Proof of Theorem* 1.2. This is exactly as for Theroem 1.1 except that the measure on $\Gamma$ is now counting measure so that all integrals over $\Gamma$ (and its subsets) are replaced by the corresponding sums.

**4. Compact non-abelian case.** If $G$ is compact and non-abelian, there is a further complication due to the fact that the representations of $G$ are not necessarily one-dimensional, so that an argument like that of Lemma 3.1 will not hold. We rely instead on a decomposition of $G$ involving Lie groups.

LEMMA 4.1. *Let $G$ be a compact, connected Lie group and $d$ the invariant Riemannian metric on $G$. For each $y \in G$, let $S_r(y) = \{x \in G; d(x, y) = r\}$ $(r \ge 0)$. Then $m_G(S_r(y)) = 0$.*

*Proof*. Omitted.

LEMMA 4.2. *If $G$ is a compact, connected, non-abelian group with Haar measure $m_G$ and $V$ is a non-negative, measurable function on $G$ which is greater than 0 $m_G$-a.e., then there exists a measurable function $W$ on $G$ satisfying* (1.1).

*Proof*. If $G$ is compact and connected, then

$$G \simeq \Big(T \times \prod_{i \in I} G_i\Big)/H$$

where $T$ is a compact, connected, abelian group, $\{G_i; i \in I\}$ is a family of compact, simply connected, simple Lie groups and $H$ is a totally disconnected, closed subgroup of $Z(T \times \prod_{i \in I} G_i)$ (the centre of $T \times \prod_{i \in I} G_i$). For a discussion of this decomposition, see §6.5.6 of [7].

$G$ is non-abelian, so $I$ must be non-empty and at least one $G_i$ must be non-trivial. Let $G_1$ be non-trivial, $T_1 = T \times \prod_{i \in I, i \neq 1} G_i$ and $G_2 = G_1 \times T_1$. $G_1$, being a connected, compact Lie group, possesses an invariant Riemannian metric $d$ yielding the topology.

Define an invariant pseudo-metric $\bar{d}$ on $G_2$ by

$$\bar{d}(x, y) = d(x_1, y_1)$$

where $x_1$, $y_1$ are the $G_1$-components of $x, y$ ($\in G_2$) respectively. Define $D$: $G \times G \to \mathbf{R}^+$ by

$$D(xH, yH) = \inf_{a,\, b \in H} \bar{d}(xa, yb).$$

In particular, $D(yH, H) = \inf_{a \in H} d(y_1, a_1)$ (by the invariance of $d$). Now $H \subseteq Z(G_1 \times T_1) \simeq Z(G_1) \times Z(T_1)$ and $Z(G_1)$ is finite since $G_1$ is simple. Therefore, the infimum in the definition of $D(yH, H)$ is taken over a finite set (i.e. $d(y_1, a_1)$ takes only finitely many values as $a$ varies over $H$). So we may write

$$D(yH, H) = \min_{a \in H} d(y_1, a_1).$$

We shall need Theorem C §63 of [3] which implies that if $\pi$: $G_2 \to G_2/H = G$ is the canonical homomorphism then $m_{G_2} \circ \pi^{-1}$ is a Haar measure on $G$. (There exists a constant $\alpha > 0$ for which $m_G = \alpha m_{G_2} \circ \pi^{-1}$.) So

$$\begin{aligned}
&m_G\{yH \in G;\, D(yH, H) = \beta\} \\
&\quad = \alpha m_{G_2}\Big\{y \in G_2;\, \min_{a \in H} d(y_1, a_1) = \beta\Big\} \\
&\quad \leq \alpha m_{G_2}\{y \in G_2;\, d(y_1, a_1) = \beta \text{ for some } a \in H\} \\
&\quad \leq \alpha m_{G_2}\left\{\bigcup_{i=1}^{n} \left(S_\beta(a_1^i) \times T_1\right)\right\} \\
&\qquad\quad \Big(\text{where } \{a_1^1, \ldots, a_1^n\} \text{ is the finite set consisting} \\
&\qquad\qquad\qquad\qquad \text{of the projection of } H \text{ onto } Z(G_1)\Big) \\
&\quad \leq \alpha \sum_{i=1}^{n} m_{G_1}\left(S_\beta(a_1^i)\right) m_{T_1}(T_1) = 0
\end{aligned}$$

where the last step follows by Lemma 4.1 and the compactness of $T_1$. Let $g(yH) = \sin D(yH, H)$. As before, let $B_r = \{x \in G;\, V(x) = r\}(r > 0)$ and $E = \bigcup_{m_G(B_r) > 0} B_r$. $E$ and $B_r$ are measurable. Let

$$W(x) = \begin{cases} 3V(x)/4 & \text{on } G \setminus E \\ 3r/4 + rg(x)/8 & \text{on } B_r. \end{cases}$$

For $1 \le i \le n$ define $h_i\colon G_2 \to \mathbf{R}^+$ by $h_i(y) = d(y, a_1^i)$. Each $h_i$ is continuous, hence measurable, and $D(yH, H) = \min_{1\le i\le n} h_i(y)$, so $D$ is measurable as a function on $G$ and hence $g$ is measurable also. The measurability of $V$ and $g$ now implies the measurability of $W$. Also, $W$ satisfies (1.1). □

*Proof of Theorem* 1.3. First note that if $U$ is a representation of $G$, then $d(U) \ge 1$. Suppose $V(x) = 0$ on a set of positive measure. Then the second integral in (1.7) is infinite for all $r > 0$. Therefore the first integral (sum) in (1.7) must be zero for all $r > 0$ and this implies that $T(U) = 0$ for all $U \in \hat{G}$. If this were the case then (1.8) would be true trivially.

If $m_G\{x \in G; V(x) = \infty \text{ and } f(x) \ne 0\} > 0$, then the second integral in (1.8) would be infinite and once again (1.8) would be true trivially. From now on we assume that $V(x) > 0$ $m_G$-a.e. and $m_G\{x \in G, f(x) \ne 0 \text{ and } V(x) = \infty\} = 0$. Let

$$I = \sum_{U\in\hat{G}} \|\hat{f}(U)\|_{\Phi_{p'}}^{p'} T(U)\, d(U) \le 2^{p'}(I_1 + I_2) \tag{4.1}$$

where

$$I_1 = \sum_{j=-\infty}^{\infty} \sum_{2^jB < T(U) \le 2^{j+1}B} T(U)\, d(U) \left\| \int_{V(x)\ge 2^{jp-j-1}} f(x)U(x)\, dm_G(x) \right\|_{\Phi_{p'}}^{p'}$$

and

$$I_2 = \sum_{j=-\infty}^{\infty} \sum_{2^jB < T(U) \le 2^{j+1}B} T(U)\, d(U) \left\| \int_{V(x)< 2^{jp-j-1}} f(x)U(x)\, dm_G(x) \right\|_{\Phi_{p'}}^{p'}$$

(using the fact that $\|A + B\|_{\Phi_{p'}}^{p'} \le 2^{p'}(\|A\|_{\Phi_{p'}}^{p'} + \|B\|_{\Phi_{p'}}^{p'})$ for $A, B \in \mathrm{End}(\mathscr{H}_U)$).

$$\begin{aligned} I_1 &\le 2B \sum_{j=-\infty}^{\infty} 2^j \sum_{U\in\hat{G}} \left\|(f\chi_{V\ge 2^{jp-j-1}})^\wedge(U)\right\|_{\Phi_{p'}}^{p'} d(U) \\ &\le 2B \sum_{j=-\infty}^{\infty} 2^j \left(\int_G |f(x)|^p \chi_{V\ge 2^{jp-j-1}}(x)\, dm_G(x)\right)^{p'/p} \end{aligned} \tag{4.2}$$

(by the Hausdorff-Young inequality)

$$\le c\left(\int_G |f(x)|^p V(x)\, dm_G(x)\right)^{p'/p}$$

(estimating the integral as in the proof of Theorem 1.1).

$$I_2 \le \sum_{U\in\hat{G}} T(U)\, d(U) \left\| \int_{2V(x)<(T(U)/B)^{p-1}} f(x)U(x)\, dm_G(x) \right\|_{\Phi_{p'}}^{p'}.$$

By Lemma 4.2, there exists a measurable function $W$ satisfying (1.1). Let $J$ be the least integer satisfying $2^J \geq \|f\|_1$, let $r_J = \infty$ and for $j < J$, choose $r_j$ such that

$$3 \cdot 2^{j-2} < \int_{2W(x) < r_j^{p-1}} |f(x)|\, dm_G(x) \leq 2^j.$$

Let $D_j = \{x \in G; r_{j-2}^{p-1} \leq 2W(x) < r_{j-1}^{p-1}\}$.

$$I_2 \leq \sum_{j=-\infty}^{J} \sum_{Br_{j-1} < T(U) \leq Br_j} T(U)\, d(U) \left\| \int_{2W(x) < r_j^{p-1}} f(x) U(x)\, dm_G(x) \right\|_{\phi_{p'}}^{p'}$$

$$\leq \sum_{j=-\infty}^{J} \sum_{Br_{j-1} < T(U) \leq Br_j} T(U) d(U) \left( \int_{2W(x) < r_j^{p-1}} |f(x)| \|U(x)\|_{\phi_{p'}}\, dm_G(x) \right)^{p'}$$

$$= \sum_{j=-\infty}^{J} \sum_{Br_{j-1} < T(U) \leq Br_j} T(U)\, d(U)^2 \left( \int_{2W(x) < r_j^{p-1}} |f(x)|\, dm_G(x) \right)^{p'}$$

$$\left( \|U(x)\|_{\phi_{p'}} = d(U)^{1/p'} \text{ since } U(x) \in \mathscr{U}(\mathscr{H}_U) \right)$$

$$\leq 8^{p'} \sum_{j=-\infty}^{J} \sum_{Br_{j-1} < T(U) \leq Br_j} T(U) d(U)^2 \left( \int_{D_j} |f(x)|\, dm_G(x) \right)^{p'}.$$

We may write $|f(x)| = (|f(x)| V(x)^{1/p}) V(x)^{-1/p}$ and apply Hölder's inequality on $D_j$ to get

$$I_2 \leq 8^{p'} \sum_{j=-\infty}^{J} \sum_{Br_{j-1} < T(U)} T(U) d(U)^2 \left( \int_{D_j} V(x)^{-p'/p}\, dm_G(x) \right)$$
$$\times \left( \int_{D_j} |f(x)|^p V(x)\, dm_G(x) \right)^{p'/p}.$$

However $D_j \subseteq \{x \in G; V(x) < r_{j-1}^{p-1}\}$ and combining this with the assumption (1.7) gives

$$\text{(4.3)} \qquad I_2 \leq 8^{p'} A \sum_{j=-\infty}^{J} \left( \int_{D_j} |f(x)|^p V(x)\, dm_G(x) \right)^{p'/p}$$
$$\leq 8^{p'} A \left( \int_G |f(x)|^p V(x)\, dm_G(x) \right)^{p'/p}$$

(since $p'/p \geq 1$). Combining (4.1), (4.2) and (4.3) gives (1.8). □

The requirement in the above theorems that the group $G$ be connected may be relaxed slightly as follows.

PROPOSITION 4.3. *Suppose $G$ is a topological group with identity component $G_0$ such that $G/G_0$ is countable. If either of the conditions*

(i) *$G_0$ is non-compact LCA*

(ii) *$G_0$ is compact abelian*

(iii) *$G_0$ is compact non-abelian*

*hold, and $V$ is a non-negative, measurable function on $G$ which is greater than 0 $m_G$-a.e., then there exists a measurable function $W$ on $G$ satisfying* (1.1).

*Proof*. Choose coset representatives $x_j$ $(j \in \mathbf{Z}^+)$ for $G/G_0$. Let $x_jG_0$ be a coset of $G_0$ and define $V_j$: $G_0 \to \mathbf{R}^+$ by $V_j(x) = V(x_jx)$ $(x \in G_0)$. By either Lemma 2.2, 3.1 or 4.2 (depending on whether (i), (ii) or (iii) is satisfied), construct a measurable function $W_j$ on $G_0$ with the properties

$$V_j(x) \leq 2W_j(x) < 2V_j(x) \quad \text{for all } x \in G_0 \quad \text{and}$$
$$m_{G_0}\{x \in G_0; W_j(x) = \alpha\} = 0 \quad \text{for all } \alpha > 0.$$

Now let $W^j$: $x_jG_0 \to \mathbf{R}^+$ be given by $W^j(x_jx) = W_j(x)$, and $W$: $G \to \mathbf{R}^+$ by $W(y) = W^j(y)$ for $y \in x_jG_0$. $W$ is measurable and satisfies (1.1). $\square$

So if, in Proposition 4.3, $G$ satisfies (i), (ii) or (iii), then the conclusions of Lemmas 2.2, 3.1 or 4.2 (respectively) hold, and hence Theorems 1.1, 1.2 or 1.3 (respectively) will apply to $G$.

**5. General exponents.** In this section, a supplementary result is stated which loosens the restrictions on the exponents involved in the inequality at the expense of additional restrictions on the class of groups for which such an inequality may hold.

Given weight functions $V$ on $G$ and $T$ on $\hat{G}$ (the dual object of $G$), the proof requires the existence of measurable functions $W$ on $G$ and $S$ on $\hat{G}$ satisfying (1.1) and (1.2) respectively. If $G$ is compact, however, (1.2) is clearly impossible since $\hat{G}$ is discrete. The proof will work for

$$G \simeq H \times J \times \mathbf{R}^n \qquad (n > 0) \tag{5.1}$$

where $H$ is a countable abelian group and $J$ is a compact abelian metric group. The decomposition (5.1) then implies

$$\hat{G} = \Gamma \simeq \hat{H} \times \hat{J} \times \mathbf{R}^n \qquad (n > 0) \tag{5.2}$$

where $\hat{H}$ (the dual group of $H$) is a compact, abelian, metric group and $\hat{J}$ (the dual group of $J$) is a countable, abelian group. Hence, $H \times J$ and $\hat{H} \times \hat{J}$ are $\sigma$-compact and we may use Lemma 2.1 and the proof of Lemma 2.2 to construct functions $W$ and $S$ satisfying (1.1) and (1.2) respectively. (Note that Lemma 2.2 requires only that $G \simeq K \times \mathbf{R}^n$ with $n > 0$ and $K$ $\sigma$-compact.)

If $T$ is a measurable function on a measure space $M$, then $T^*$: $[0, \infty] \to [0, \infty]$ is the non-increasing rearrangement of $T$, a discussion of which may be found in Chapter 5 §3 of [**9**].

THEOREM 5.1. *Let $G$ be an LCA group of the form* (5.1) *with dual group $\Gamma$. Suppose $V$ is a non-negative, measurable function on $G$ and $T$ is a non-negative, measurable function on $\Gamma$. If $1 < p < 2$, $p \leq q < p'$ and there exist positive constants $A$, $B$ for which*

$$(5.3)\quad \left(\int_{(tT^*(t))^{p'/q} > Brt} T^*(t)\,dt\right)\left(\int_{V(x) < r^{p-1}} V(x)^{-1/(p-1)}\,dm_G(x)\right)^{q/p'} \leq A$$

*holds for all $r > 0$, then for every integrable $f$ on $G$*

$$(5.4)\quad \left(\int_\Gamma |\hat{f}(\gamma)|^q T(\gamma)\,dm_\Gamma(\gamma)\right)^{1/q} \leq c\left(\int_G |f(x)|^p V(x)\,dm_G(x)\right)^{1/p}$$

with $c$ independent of $f$. If $2 < q < \infty$, $q' < p \leq q$ and there exist positive constants $A, B$ for which

$$(5.5)\quad \left(\int_{(tW(t))^{q/p'} > Brt} W(t)\,dt\right)\left(\int_{rT(\gamma) > 1} T(\gamma)\,dm_\Gamma(\gamma)\right)^{p/q'} \leq A$$

holds for all $r > 0$ (with $W(t) = (V^{-1/(p-1)})^*(t)$), then for every integrable $f$ on $G$, (5.4) holds with $c$ independent of $f$.

By considering the case $G = \mathbf{R}^n$, the sufficiency conditions given here may be shown to be not necessary (as in [**6**]). However, if $T$ and $V$ are assumed to be radial and, as functions of $|x|$, $T$ is non-increasing while $V$ is non-decreasing, then the conditions are also necessary.

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