PACIFIC JOURNAL OF MATHEMATICS
Vol. 132, No. 1, 1988

REPRESENTING POLYNOMIALS BY POSITIVE LINEAR FUNCTIONS ON COMPACT CONVEX POLYHEDRA

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If K is a compact polyhedron in Euclidean d-space, defined by linear inequalities, $\beta_i \geq 0$, and if f is a polynomial in d variables that is strictly positive on K, then f can be expressed as a positive linear combination of products of members of $\{\beta_i\}$. In proving this and subsidiary results, we construct an ordered ring that is a complete $\mathrm{AGL}(d, \mathbf{R})$-invariant for K, and discuss some of its properties. For example, the ordered ring associated to K admits the Riesz interpolation property if and only if it is $\mathrm{AGL}(d, \mathbf{R})$-equivalent to a product of simplices. This is exploited to show that certain polynomials are not in the positive cone generated by the set $\{\beta_i\}$.

Let L be a subfield of the real numbers, and let $\beta_i = \sum_{1\leq j\leq d} a_{ij}X_j + a_{i,d+1}$ $(i = 1, 2, 3, \ldots, s)$ be linear polynomials ("linear forms") in the d variables $\{X_j\}$, with coefficients from L. Suppose the convex polyhedron in $\mathbf{R}^d$ defined by $K = \bigcap(\beta_i)^{-1}([0, \infty))$ is compact and has interior. Let f be a polynomial in the d variables with entries from L, such that the restriction, $f \mid K$, is strictly positive. Then our first result (I.3) asserts that f may be represented as a combination with coefficients from $L \cap \mathbf{R}^+$ (that is, *positive numbers* in L) of terms that are products of the original set of β's that determine K. If f vanishes at only a vertex of K (and is strictly positive elsewhere), this decomposition does not hold in general (§III).

Our second principal result concerns the Riesz decomposition property in an ordered ring naturally associated to K, and leads to some interesting geometric characterizations of those polytopes that are affinely homeomorphic to products of simplices. With K defined as above, define a *monomial* (in the β_i's) to be a polynomial in the X's that can be expressed as a product of the form

$$\beta^w = \beta_1^{w(1)}\beta_2^{w(2)}\cdots\beta_s^{w(s)}$$

where $w(k)$ is a non-negative integer, and w is the s-tuple $(w(1), w(2), \ldots, w(s))$. Define $R_L[K]$ (or simplv $R[K]$ if there is no ambiguity about the coefficient field L) to be the polynomial ring,

$R[K] = L[X_1, X_2, \ldots, X_d]$, with the following positive cone:

$$R[K]^+ = \left\{ \sum \alpha_w \beta^w \mid \alpha_w \in L \cap \mathbf{R}^+ \right\}.$$

This imposes the structure of a partially ordered ring on $R[K]$. To see this, note that if f belongs to $R[K]^+$, then $f \mid K \geq 0$; thus if f belongs to $R[K]^+ \cap -R[K]^+$, we would have that $f \mid K = 0$; as K contains interior, this forces f to be identically zero.

In [**H1**] and [**H3**], there is studied a class of partially ordered rings (emanating from the study of K_0 of fixed point C^*-algebras), which when tensored with L (over $\mathbf{Z}$) sometimes are order-isomorphic to a ring of the form $R[K]$. In particular, those that arise in this fashion must satisfy the Riesz interpolation property [**EHS**]. Our second main result is that for general K, $R[K]$ will satisfy the Riesz interpolation property if and only if K is, up to affine homeomorphism, a (cartesian) product of simplices (Corollary to II.6). In the course of proving this, we also show that products of simplices are characterized by properties such as: If $\{F_k\}$ is a collection of faces of K and $\bigcap F_k$ is either empty or a singleton, then $\bigcap F_k = \bigcap (F_k)^{\sim}$, where $F^{\sim}$ denotes the affine span of the face F (II.5).

In the context of this article, affine homeomorphism is the natural equivalence relation between polytopes in $\mathbf{R}^d$. This is implemented by elements of $\mathrm{AGL}(d, \mathbf{R})$; i.e., the group of transformations of $\mathbf{R}^d$ generated by translations and $\mathrm{GL}(d, \mathbf{R})$. Although the assignment $K \mapsto R[K]$ is not functorial (viewing K as a compact convex set, and $R[K]$ is a partially ordered ring), it is true that $R[K]$ is a complete $\mathrm{AGL}(d, \mathbf{R})$-invariant for K; that is, if K, K' are compact d-dimensional polytopes (we use polyhedra, polytopes interchangeably and to mean *convex* polyhedra), then K is affinely homeomorphic to K' if and only if $R[K]$ is isomorphic to $R[K']$ as ordered rings (II.6).

In general, it is extremely difficult to tell if an element f of $R[K]$ is positive in the sense described above. When Riesz interpolation holds, it is quite a bit easier to decide this; indeed, a consequence of the result that this property entails K is a product of simplices, allows us in principle to decide if a given element is positive (by using [**H2**; Theorem B]). In §III, we use some easier consequences of interpolation, to construct for every d-dimensional compact polytope K with $d > 1$, an element f that is strictly positive (except at a vertex of K) as a function on K, and yet is not in the positive cone of $R[K]$. This contrasts with the first cited result.

Section IV discusses some connections between the original K_0-theoretic questions and the problems dealt with here.

I. Strict positivity. In this section, we lay the basis for the rest of the paper, and prove the strict positivity result; namely, if f is a polynomial in the d variables $\{X_j\}$, and the restriction of f to K does not vanish at any point, and is positive, then f can be written as a positive combination of the monomials that came out of the definition of K. We use standard results both on convex polytopes, and on partially ordered rings.

The principal result in I.1 (specifically, part (a) with $L = \mathbf{R}$) goes back at least to Minkowski (see **[Ce**; p. 14**]** and the discussion therein.)

I.1. PROPOSITION. *Let K be a compact convex polyhedron with interior in $\mathbf{R}^d$, defined by $\beta_i \geq 0$, where $\beta_i = \sum_{1\leq j\leq d} a_{ij}X_j + a_{i,d+1}$ ($i = 1, 2, 3, \ldots, s$) are linear forms.*

(a) *If β is a linear form and $\beta \mid K \geq 0$, then there exist non-negative $\lambda_1, \lambda_2, \ldots, \lambda_s$ in L with*

$$\beta = \sum \lambda_i \beta_i.$$

(b) *If the β of part* (a) *is strictly positive as a function on K, then all of the λ_i obtained there may be chosen to be strictly greater than zero.*

(c) *Each of the variables X_j is an L-linear combination of the β_i's.*

Proof. (a). We first deal with the case of $L = \mathbf{R}$. This is well-known but we provide a self-contained proof for the convenience of the reader.

Let $\langle\langle , \rangle\rangle$ denote the usual inner product on $\mathbf{R}^d$, and define

$$K^* = \{x \in \mathbf{R}^d \mid \langle\langle x, y\rangle\rangle \leq 1 \text{ for all } y \in K\}.$$

For k in K, $\langle\langle x, k\rangle\rangle \leq 1$ for all x in K^*, so that $K^{**} \supset K$. Now we prove the reverse inclusion.

Without loss of generality, we may assume that a neighbourhood of $\mathbf{0}$ (the origin) lies in K. Writing $\beta_i = \sum_{1\leq j\leq d} a_{ij}X_j + a_{i,d+1}$, from $\beta_i \mid K \geq 0$, we deduce that $a_{i,d+1} > 0$ (if equality held, β_i would be identically zero, and there would be nothing to do). We may thus assume that $a_{i,d+1} = 1$ for all i. Now for each i, define b_i in $\mathbf{R}^d$ via $b_i = -(a_{i1}, a_{i2}, \ldots, a_{id})$. As $\beta_i(z) = -\langle\langle b_i, z\rangle\rangle + 1$, we see that $\beta_i \mid K \geq 0$ is equivalent to $\langle\langle b_i, y\rangle\rangle \leq 1$ for all y in K. In particular, each b_i belongs to K^*. Now let z be an element of K^{**}. Then $\langle\langle b_i, z\rangle\rangle \leq 1$ for all i, so that $\beta_i(z) \geq 0$. Thus z belongs to K, whence $K = K^{**}$.

Next, we claim that $K^* = \mathrm{cvx}\{b_i\}$. Set $K' = \mathrm{cvx}\{b_i\}$. Then

$$(K')^* = \{y \in \mathbf{R}^d \mid \langle\langle b_i, y\rangle\rangle \leq 1\} = \{y \in \mathbf{R}^d \mid \beta_i(y) \geq 0\} = K.$$

From $(K')^{**} = K'$, we deduce that $K' = (K')^{**} = K^*$.

Now given our original β with $\beta \mid K \geq 0$, say $\beta = \sum_{1\leq j\leq d} a_j X_j + 1$, define the corresponding $b = -(a_1, a_2, \ldots, a_d)$. From $\beta \mid K \geq 0$, we deduce $\langle\langle b, y\rangle\rangle \leq 1$ for all y in K, so that b belongs to K^*. As $K^* = \mathrm{cvx}\{b_i\}$, there exist non-negative real numbers λ_i with $\sum \lambda_i = 1$ such that $b = \sum \lambda_i b_i$. This translates back to $\beta = \sum \lambda_i \beta_i$; hence, when $L = \mathbf{R}$, (a) is proved.

For a general subfield L of $\mathbf{R}$, we observe that the β_i's all have coefficients from L. Writing $\beta = \sum \lambda_i \beta_i$ with non-negative real λ_i, let $I = \{i \mid \lambda_i > 0\}$. Let $\{Y_i \mid i \in I\}$ be a set of real variables, and consider the system of linear equations $\sum_I Y_i \beta_i u = \beta$. Let V_L be the (affine) space of solutions $Y = (Y_i)$ over L, and let V be the real solution space. We note that since the system of equations is defined over L and has a real solution, it also has a solution Y^0 defined over L (use the reduced echelon form—Gaussian elimination requires only operations with coefficients form L). As all the coefficients of β_i, β belong to L, we have that $(V_L - Y^0)\mathbf{R} + Y^0 = V$. In particular, V_L is dense in V (as L is dense in $\mathbf{R}$). Let $\lambda_0 = (\lambda_i)_{i\in I}$ be the strictly positive (recall the definition of I!) solution to $\beta = \sum \lambda_i \beta_i$ obtained previously. We may approximate this by solutions over L (that is, in V_L), and since λ^0 is strictly positive, we may find an approximant in V_L which is itself strictly positive, concluding the proof of (a).

(b) Let $\gamma = \sum \beta_i$. Select an integer N so that $\beta \mid K \geq 1/N$, and choose an integer M so that $\gamma \mid K < M/N$. Then $(\beta - \gamma/M) \mid K \geq 0$, so that by (a), $\beta - \gamma/M = \sum \lambda_i \beta_i$ (with λ_i in L^+). Thus

$$\beta = \sum (\lambda_i + 1/M)\beta_i.$$

(c) Let γ be an arbitrary linear form (e.g., $\gamma = X_j$). Let M be a positive integer such that $\gamma \mid K \geq -M$. Then $\gamma = (\gamma + M) - M$, and each of $\gamma + M, M$ is positive on K. By (a), each is an L-linear combination of the β_i's, whence so is γ. □

I.2. Corollary. *With the hypotheses as in* I.1, *the constant linear form,* 1, *is a strictly positive L-linear combination of the β_i's.*

Proof. As $1 > 0$ on K, I.1(b) applies. □

An element u of a partially ordered abelian group G is an *order unit* if it is positive, and for all elements g of G, there exists a positive integer N such that $g \leq Nu$. The partially ordered group is *unperforated* if for all g in G, if there is a positive integer n such that $ng \geq 0$, then g is positive. It is routine to verify that in our case, with $R[K]^+$ generated additively and multiplicatively by the β_i's and $L \cap \mathbf{R}^+$, that $R[K]$ is unperforated (just use that the positive rationals are in $L \cap \mathbf{R}^+$). An immediate consequence of I.2 is that the constant function 1 is an order unit for $R[K]$. We are now in a position to prove the first main result of this article.

I.3. THEOREM. *Let L be a subfield of $\mathbf{R}$, and let*

$$\beta_i = \sum_{1 \leq j \leq d} a_{ij} X_j + a_{i,d+1} \qquad (i = 1, 2, 3, \ldots, s)$$

be linear forms in the d variables $\{X_j\}$, with coefficients from L. Suppose that the convex subset of $\mathbf{R}^d$ defined by

$$K = \bigcap (\beta_i)^{-1}([0, \infty))$$

*is compact and has interior. If f is a polynomial in d variables with coefficients from L such that $f(k) > 0$ for all k in K, then f may be represented as a **positive** L-linear combination (that is, coefficients from $L \cap \mathbf{R}^+$) of monomials in the $\{\beta_i\}$.*

Proof. We have previously observed that $R_L[K]$ is a partially ordered L-algebra that has 1 as an order unit, and is unperforated. By e.g., [**H1**; I.2], the extremal positive group homomorphisms $R_L[K] \to \mathbf{R}$ must be multiplicative, hence must correspond to a point in $\mathbf{R}^d$, $(t_1, t_2, \ldots, t_d)$, given by the values of the variables $\{X_1, \ldots, X_d\}$.

Since the image of each β_i must be non-negative under such a map, we must have that $\beta_i(t_1, t_2, \ldots, t_d) \geq 0$. Hence the point evaluations can only correspond to points in K.

The hypothesis on f thus guarantees that at all extremal positive homomorphisms α: $R[K] \to \mathbf{R}, \alpha(f) > 0$. By [**H1**; I.1], f belongs to $R[K]^+$. □

We may of course consider other subsets of $\mathbf{R}^d$, and ask if a similar result holds. Let $\{\alpha_i\}$ be a set of polynomials in d real variables, and define $K = \{r = (r_j) \in \mathbf{R}^d \mid \alpha_j(r) \geq 0 \text{ for all } i\}$ (so K is a general semi-algebraic set). Suppose that K is compact and d-dimensional (although whether these additional hypotheses are really needed is unclear), and define the ring $R[K, \alpha_i]$ to be the subring of $\mathbf{R}[X_1, \ldots, X_d]$ generated by the reals and $\{\alpha_i\}$. Equip $R[K, \alpha_i]$ with the positive cone generated multiplicatively and additively over the positive reals by $\{\alpha_i\}$. This approach to semi-algebraic sets might be of interest.

Given f in $R[K, \alpha_i]$ such that $f(k) > 0$ for all k in K, does it follow that f belongs to $R[K, \alpha_i]^+$? We give two examples in this context, one a perturbation of the other, with an affirmative answer for one and a negative answer for the other. We first note that it is sufficient that $R[K, \alpha_i] = \mathbf{R}[X_i]$ and that 1 be an order unit for $R[K, \alpha_i]$, for then the proof of I.3 works verbatim. (It is also true that 1 being an order unit is necessary as well, and the argument for this will come out of the first example.)

I.4. Example. Suppose $d = 1$, $\alpha_1 = X_1$, $\alpha_2 = X_2$, and $\alpha_3 = 1 - (X_1)^2 - (X_2)^2$; then K is the closed upper right quadrant of the unit disk. Clearly $R[K, \alpha_i] = \mathbf{R}[\alpha_i]$ is just the polynomial ring $\mathbf{R}[X_1, X_2]$. We first observe that the constant function 1 is not an order unit in $R[K, \alpha_i]$. Suppose it were an order unit. Then there would exist an integer N so that $N \geq X_1$ in $R[K, \alpha_i]$. This means there would exist P_1, P_2, P_3, in the three variable polynomial ring $\mathbf{R}[x_1, x_2, x_3]$ with positive coefficients, and a non-negative real number c, so that

$$(*) \qquad N - X_1 = X_1 P_1(X_1, X_2, \alpha_3) + X_2 P_2(X_1, X_2, \alpha_3) + \alpha_3 P_3(X_1, X_2, \alpha_3) + c.$$

We may assume that $P_1(0,0,0) = 0$ (or else it could be absorbed into the N on the left side); hence we may absorb all the terms from $X_1 P_1(X_1, X_2, \alpha_3)$ into the other two, except for (possibly) a polynomial of the form $Q = (X_1)^2\{q(X_1)\}$, where q is a polynomial in one real variable with no negative coefficients. Next, we may evaluate at $(0,0,1)$ (that is, $X_1 \mapsto 0, X_2 \mapsto 0$, $\alpha_3 \mapsto 1$), and deduce $N - c = P_3(0,0,1)$, so that $N \geq c$. If $N = c$, X_1 would have to vanish identically on K (as the expression $(*)$ would yield that $-X_1 \geq 0$ on K); thus $N > c$, and so we may absorb c into N. We are thus reduced to:

$$N - X_1 = Q + X_2 P_2(X_1, X_2, \alpha_3) + \alpha_3 P_3(X_1, X_2, \alpha_3).$$

For a real number a between 0 and 1, we may evaluate this last expression at $(a, \sqrt{(1-a^2)}, 0)$, and so obtain

$$\begin{aligned} N - a &= a^2 q(a) + (\sqrt{1-a^2})\{P_2(a, \sqrt{1-a^2}, 0)\} \\ &= a^2 q(a) + (\sqrt{1-a^2})\left\{\sum \lambda_{i,j} a^i (1-a^2)^{j/2}\right\} \quad (\lambda_{i,j} \in \mathbf{R}^+) \\ &= a^2 q(a) + (\sqrt{1-a^2})\left\{\sum \lambda_{i,2k} a^i (1-a^2)^k\right\} \\ &\quad + \sum \lambda_{i,2k+1} a^i (1-a^2)^{k+1}. \end{aligned}$$

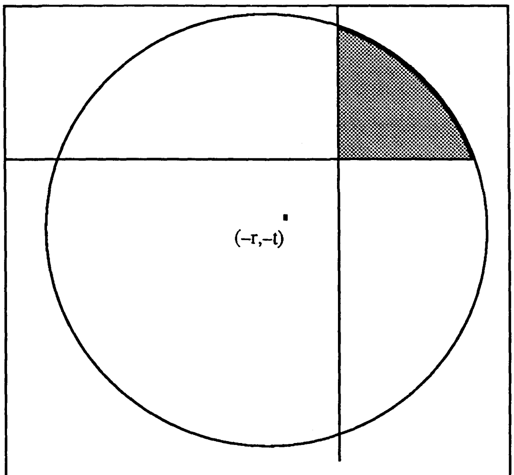
As this equation holds for $0 \le a \le 1$, it is true as an identity in the variable a. Observing that $\{\sqrt{(1-a^2)}, 1\}$ is linearly independent over $\mathbf{R}[a]$, the expression in brace brackets in the last line of the equation above must vanish identically, that is, $\sum \lambda_{i,2k} a^i (1-a^2)^k = 0$. We then have the identity of polynomials,

$$N - a = a^2 q(a) + \sum \lambda_{i,2k+1} a^i (1-a^2)^{k+1}.$$

The coefficient of a on the left is -1. That on the right is $\sum \lambda_{1,2k}$, which is non-negative; this is the desired contradiction, and so we have shown that 1 is not an order unit by showing that $N - X_1$ is not in $R[K, \alpha_i]^+$ for any positive integer N. In particular, $2 - X_1$ is strictly positive as a function on K, but is not positive in $R[K, \alpha_i]$. □

More generally, in an $R[K, \alpha_i]$ obtained from the *polynomials* $\{\alpha_i\}$ (as opposed to linear forms as in the Theorem), if there exists an integer N so that $N \ge \alpha_i$ for all i, then 1 is an order unit (since the positive cone is generated additively and multiplicatively by the α_i's). Thus, if 1 were *not* an order unit, there would exist i so that for all N, $N - \alpha_i$ would not belong to $R[K, \alpha_i]^+$. However, 1 is strictly positive as a function on K, so that there exists an integer M such that $(M - \alpha_i) \mid K > 0$. Thus 1 being an order unit for $R[K, \alpha_i]$ is *necessary* for an affirmative answer to the question.

I.5. Example. Set $\alpha_1 = X_1$, $\alpha_2 = X_2$, $\alpha_3 = 1 - (X_1 + r)^2 - (X_2 + t)^2$, where r, t are positive real numbers. Then K is a portion of a disk (Illustration I.1), and again $R[K, \alpha_i] = R[X_1, X_2]$. This time 1 *is* an order unit, as $\alpha_3 \ge 0$ yields $1 \ge 2rX_1$, $1 \ge 2tX_2$; and $\alpha_3 + (X_1 + r)^2 + (X_2 + t)^2 \le 1$ entails $\alpha_3 \le 1$. As $\{\alpha_i\}$ generates the positive cone (multiplicatively and additively, over $\mathbf{R}^+$), these three inequalities ensure that 1 is an order unit. Thus in this case, if f is any



Circle of radius 1 centred at (–r,–t). The shaded area represents the region determined by X ≥ 0,Y ≥ 0, and 1– (X+r)^2 – (Y+t)^2 ≥ 0.

ILLUSTRATION I.1

polynomial in X_1, X_2 such that $f \mid K > 0$, then f is a positive linear combination of products of the α_i's. □

The upshot of the methods used in these examples is that (for general K, defined via polynomials) 1 is an order unit of $R[K, \alpha_i]$ if and only if for all f in $R[K, \alpha_i]$ (*not* necessarily the polynomial ring) satisfying $f \mid K > 0$, f lies in $R[K, \alpha_i]^+$. If we replace "f in $R[K, \alpha_i]$" by "f in the polynomial ring", then of course we also require that $R[K, \alpha_i] = \mathbf{R}[X_i]$. □

Now let C be an arbitrary convex body (a d-dimensional convex subset of $\mathbf{R}^d$) that is compact. Since C may be approximated from within by convex polyhedra, we have the following consequences of I.2:

Let f be a real polynomial in d variables, such that f is positive on Int C. Given $\varepsilon > 0$, there exists a representation of f as a positive linear combination of monomials in linear forms $\{\beta_i\}$ such that if $K = \bigcap(\beta_i)^{-1}([0, \infty))$, then $\operatorname{Int} C \supset K$, and the measure (Lebesgue) of $C \backslash K$ is less than ε.

II. Interpolation. A partially ordered abelian group has the *Riesz interpolation property* (see e.g., [**EHS**]) if for a quadruple of elements

a, b, c, d in the group satisfying $a \le c, a \le d, b \le c, b \le d$, (we abbreviate this, $a, b \le c, d$), there exists e in the group such that $a, b \le e \le c, d$. This is equivalent to Riesz decomposition, which asserts that whenever a_i, b_j are positive elements in the group and $\sum a_i = \sum b_j$, then there exist positive a_{ij} such that for all i, $a_i = \sum_j a_{ij}$, and for all j, $b_j = \sum_i a_{ij}$.

A partially ordered ring having interpolation is much easier to deal with than one that does not, for it is often possible to decide if an element is positive or not by using interpolation (as we shall see in §III). Moreover, there are a number of properties implied by interpolation that do not necessarily hold in rings without it, and this has turned out to be important in some results. It is thus of interest to decide which rings of the form $R[K]$ (returning to our original formulation, for compact convex polyhedra) satisfy the interpolation property.

The main theorem of this section asserts that if $R[K]$ satisfies interpolation, then up to affine homeomorphism, K is a cartesian product of simplices. The converse is true but the proof is indirect. Along the way, we obtain geometric characterizations of products of simplices, e.g., if $\{F_k\}$ is a collection of faces, with corresponding affine spans $\{F_k^\sim\}$, and $\bigcap F_k$ is either a singleton or empty, then $\bigcap (F_k)^\sim = \bigcap F_k$ (II.5).

We also show (for arbitrary convex compact polytopes with interior), that $R[K]$ as an *ordered* ring is a complete $\mathrm{AGL}(d, \mathbf{R})$-invariant for K, even though the assignment $K \mapsto R[K]$ is not functorial.

If K is defined by the set of (linear) forms $\{\beta_i\}$, that is, $k = \bigcap (\beta_i)^{-1}([0, \infty))$, and β is a form which is non-negative on K, we say β is *redundant* if the affine dimension of $\beta^{-1}(0) \cap K$ is less than $d-1$; β is *irredundant* if the dimension is exactly $d - 1$. If β is irredundant, then it must belong to $\{\beta_i\}$ (at least up to scalar multiples); if it is redundant, by I.1, it must be an L-linear combination of the β_i's. We can assume that $\{\beta_i\}$ consists of irredundant forms. An *irredundant monomial* in the β_i's is simply a product of them (assuming all the β_i's are irredundant). Finally a set of linear forms $\{\beta_i\}$ is *irredundant* if each member is irredundant and no member is a scalar multiple of any other.

We define the affine dimension of a null set to be -1, and that of a singleton set to be 0.

II.1. LEMMA. (a) *For β an irredundant linear form on K and s an element of $R[K]$, $0 \le s\beta$ (in $R[K]$) implies $0 \le s$.*

(b) *Let r be an element of $R[K]$, β^w $(=\prod(\beta_i)^{w(i)}$, where $w=(w(1), w(2),\dots))$ an irredundant monomial, and N a positive integer. Then $-N\beta^w \leq r \leq N\beta^w$ entails that β^w divides r (in $R[K]$).*

(c) *Every order-ideal of $R[K]$ is an ideal, and is generated (as an ideal of $R[K]$) by irredundant monomials.*

(d) *If $R[K]$ admits interpolation, then the ideal generated by a set of irredundant monomials is an order-ideal, and finitely many will suffice to generate the ideal.*

Proof. (a) Write $s\beta = \sum \lambda_w \beta^w$ where $\lambda_w > 0$, and the β^w's are monomials in the β_i's. Since β vanishes on the $d-1$-dimensional face $\beta^{-1}(0)\cap K$ of K, so must each β^w appearing in this expression for $s\beta$. Any polynomial of d variables that vanishes on a $d-1$-dimensional subset of a zero set of an irreducible polynomial (in this case β) must be divisible by that polynomial in the polynomial ring. Hence each β^w is divisible by β, and by unique factorization, each of the quotients β^w/β is an irredundant monomial, so is a product of the β_i's. Thus $s=\sum \lambda_w\beta^w/\beta$, and so belongs to $R[K]^+$.

(b) The zero set of r contains $(\beta^w)^{-1}(0)\cap K$, a union of $d-1$-dimensional faces. Each $d-1$-dimensional face corresponds uniquely (up to scalar multiplication) to one of the β_i, so as in (a), each such β_i divides β^w. Applying (a) and induction on the sum of the entries of the tuple w, we obtain the desired conclusion.

(c) By [**H1**; 1.2(a)], every order-ideal is an ideal, as 1 is an order unit (I.1). Let I be an order-ideal; then $I=I^+-I^+$. Pick r in I^+; it is sufficient to show that r is a sum of monomials each of which belongs to I. Write $r=\sum \lambda_w\beta^w$ $(\lambda_w>0)$. Then $0\leq \beta^w \leq (1/\lambda_w)r$, and thus β^w belongs to I.

(d) (cf., [**H1**; §VII]) Let $\{\beta^w\}_{w\in\Omega}$ be a set of irredundant monomials generating an ideal I. Pick t in $I\cap R[K]^+$, and write $t=\sum\lambda_v\beta^v$ (λ_v in $L\cap\mathbf{R}^+$, nonzero), with β^v being irredundant monomials (not necessarily in I). We wish to show that each β^v belongs to I. We may also write $t=\sum r_w\beta^w$, with the r_w in $R[K]$, and w varying over a finite subset of $\mathbf{\Omega}$. As 1 is an order unit for $R[K]$, there exists a positive integer N such that each r_w satisfies $\pm r_w\leq N$. Thus $t\leq N(\sum\beta^w)$. By Riesz decomposition, there exist $t_{v,w}$ in $R[K]^+$ so that for all v, $\lambda_v\beta^v=\sum_w t_{v,w}$ and for all w, $\sum_v t_{v,w}\leq N\beta^w$. By (b), β^w divides each $t_{v,w}$, so $t_{v,w}$ belongs to I. Thus each of the β^v do.

Now $I = I^+ - I^+$ follows as in the preceding paragraph, that is, a typical $s = \sum r_w \beta^w$ satisfies $\pm s \leq N(\sum \beta^w)$, for some positive integer N, so that s is a difference of two positive elements of I.

Suppose that $0 \leq r \leq t$, with $t = \sum r_w \beta^w$ lying in I. Then each β^w lies in I by the first paragraph of this part of the proof. By Riesz decomposition, there exist t_w in $R[K]$ such that $r = \sum t_w$, and $t_w \leq \lambda_w \beta^w$ for each w. Part (b) implies that there exist t'_w in $R[K]$ with $t_w = \beta^w \cdot t'_w$; thus $r = \sum \beta^w \cdot t'_w \in I$, as desired. Hence I is an order-ideal.

Finally $R[K]$ is noetherian, so I may be generated as an ideal by a finite subset $\{r_\alpha\}$ of I. Write $r_\alpha = r_{\alpha,1} - r_{\alpha,2}$ as the difference of two positive elements of I. For each of the $r_{\alpha,j}$, find an expression as a positve sum of irredundant monomials, and let $\{\beta^z\}$ be a finite set of monomials that results by letting α and $j = 1, 2$ vary. Clearly, $\{\beta^z\}$ generates I as an ideal (and $\sum \beta^z$ is an order unit). □

If F is a face of the compact convex polyhedron K, then $F^\sim$ denotes the affine span of F.

II.2 Proposition. *If $R[K]$ satisfies interpolation, then for a family of faces, $\{F_k\}$,*

$$\bigcap F_k = \varnothing \quad \textit{implies} \quad \bigcap F_k^\sim = \varnothing.$$

Proof. Index the β_i's so that $i = 1, 2, \dots, s$. For $k = 1, 2, \dots,$ there exist subsets $J(k)$ of $\{1, 2, \dots, s\}$ so that

$$F_k = K \cap \left(\bigcap_{J(k)} \beta_i^{-1}(0) \right).$$

Set $\psi_k = \sum_{J(k)} \beta_i \in R[K]^+$, so that $F_k = (\psi_k)^{-1}(0) \cap K$. Since the zero set (intersected with K) of $\sum \psi_k$ is empty, by I.1(b) or I.3, there exists an integer N such that $1 \leq N(\sum \psi_k)$ in $R[K]$.

Since $R[K]$ admits interpolation, there exist positive r_j (for j in $\bigcup_k J(k)$), such that $1 = \sum r_j$ and $r_j \leq N\beta_j$. Thus each r_j vanishes on $F_j^\sim = (\beta_j)^{-1}(0)$, so that $\bigcap F_j^\sim$ $(j \in \bigcup J(k))$ must be empty (as 1 would have to vanish on it), and thus $\bigcap F_k^\sim = \varnothing$. □

Already the facial hypothesis in II.2 is quite strong; if $d = 2$, the only polygons satisfying it are triangles and parallelograms (and the corresponding $R[K]$'s do satisfy interpolation). In higher dimensions, more complications arise.

II.3. Proposition. *Suppose $R[K]$ has interpolation. Let $\{\beta_t\}_{t\in T}$ be a linearly independent set of irredundant linear forms on K such that*

$$\beta^{-1}(0) \supseteq \bigcap_T \beta_t^{-1}(0) = \varnothing$$

for some linear form β such that $\beta \mid K \geq 0$. Then β can be expressed as a non-negative linear combination of $\{\beta_t\}$ (and if K is definable over the subfield L, then the coefficients may be chosen additionally to lie in L).

Proof. By translation, we may assume that 0 lies in $\bigcap_T(\beta_i)^{-1}(0)$, so the latter is a subspace. A standard linear algebra argument yields the existence of real numbers (in L, if K is defined over L) $\{\alpha_t\}$, such that $\beta = \sum \alpha_t\beta_t$. We show that the α_t's are all non-negative.

From $\beta \mid K \geq 0$, from I.1, we deduce that β lies in $R[K]^+$. Let I, J be subsets of T defined as $I = \{t \in T \mid \alpha_t < 0\}$, and $J = \{t \in T \mid \alpha_t > 0\}$. Assume that I is not empty. We have that

$$\beta + \sum_{t\in I}(-\alpha_t)\beta_t = \sum_{t\in J}\alpha_t\beta_t.$$

Hence if i belongs to I, $\beta_i \leq N(\sum_J \beta_t)$ for some positive integer N. Riesz decomposition yields the existence of $\{r_t'\}_{t\in J}$ in $R[K]^+$ so that $\beta_i = \sum r_t'$ and $r_t' \leq N\beta_t$. By II.1(b), there exist r_t in $R[K]$ such that $r_t' = \beta_t r_t$ for all t in J. Thus β_i belongs to the ideal generated by the set $\{\beta_t\}_J$, and so

$$\ker\beta_i \supseteq \bigcap_j \ker\beta_t$$

(the β_t's being linear, it is reasonable to talk of their kernels). However, since the set $\{\beta_i\} \cup \{\beta_t\}_J$ is a linearly independent set of linear forms on $\mathbf{R}^d$, we have that $\bigcap_J \ker\beta_t$ is not contained in $\ker\beta_i$, a contradiction. Thus I is empty. □

II.4. Proposition. *Suppose that $R[K]$ has interpolation. Let $\{\beta_i\}_T$ be an irredundant set of linear forms (that are non-negative) on K such that $(\bigcap_T \beta_i)^{-1}(0) \neq \{0\}$. Then the affine dimension of $(\bigcap_T \beta_i)^{-1}(0)$ is at most $d - |T|$.*

Proof. Suppose the dimension exceeds $d - |T|$. By translation, and II.2, we may assume that $\mathbf{0}$ belongs to $K \cap \{\bigcap_T(\beta_i)^{-1}(0)\}$; in particular, the β's become linear functions on $\mathbf{R}^d$. They cannot be linearly independent, so there exists a relation of the form

$$\sum_S \alpha_s \beta_s = \sum_U \alpha_u \beta_u$$

where S, U are disjoint subsets of T and the α's are all strictly positive, and moreover, $\{\beta_s\}_S$, $\{\beta_u\}_U$ are each linearly independent sets. For s in S, $\beta_S \mid K \geq 0$ (by definition), and as in the proof of II.3, each β_s belongs to the ideal of $R[K]$ generated by $\{\beta_u\}_U$. There thus exist polynomials $\{r_u\}$ such that $\beta_s = \sum_U r_u \beta_u$. Hence $\bigcap_U \ker \beta_u$ is contained in $\ker \beta_s$. By II.3, β_s is redundant or else equal to a multiple of one of the β_u's, contradicting irredundance of the original set $\{\beta_t\}_T$. □

The following gives two geometric characterizations for a compact convex polyhedron to be affinely equivalent to a direct product of simplices. Moreover, the proof contains an algorithm for constructing the affine transformation (via certain elementary row and column operations).

Conventionally, the dimension of a singleton set is zero, while that of the empty set is -1.

II.5. THEOREM. *Let K be a d-dimensional compact convex polyhedron in $\mathbf{R}^d$. The following are equivalent*:

(a)(i) *If $\{F_k\}$ is a collection of faces of K and $\bigcap F_k = \emptyset$, then $\bigcap (F_k)^\sim = \emptyset$.*

(ii) *Every vertex is contained in exactly d (and no more) faces of dimension $d-1$.*

(b) *If $\{F_k\}$ is a collection of faces of K and* $\dim \cap F_k \leq 0$, *then $\bigcap F_k = \bigcap (F_k)^\sim$.*

(c) *K is affinely homeomorphic to a product of simplices.*

Proof. (a)⇒(b). If $\dim \cap F_k = -1$, this is (a)(i). Suppose $\bigcap F_k$ is a singleton; then it must be a vertex of K. By translation, we may assume the vertex is $v = \mathbf{0}$, so that the $(F_k)^\sim$ are all subspaces.

Now any face F of dimension $d-t$ can be wrtten as an irredunant intersection of t faces of dimension $d-1, \{G_i\}$. It follows that $\bigcap (G_i)^\sim$ has dimension $d-t$, and as $F^\sim$ has the same dimension and is contained in $\bigcap (G_i)^\sim$, we obtain $F^\sim = \bigcap (G_i)^\sim$. We may thus assume that each of the F_k are already of codimension 1.

Hypothesis (a)(ii) ensures that K is "locally" (that is, at any vertex) a simplex— this means that given a vertex v there exists a hyperplane given by $\alpha = 0 \in \mathbf{R}$, α being a linear form, with $\alpha(v) > 0$, so that K_v defined via $K_v = \{w \in K \mid 0 \leq \alpha(w) \leq \alpha(v)\}$ is a simplex, and v is one of its vertices; moreover, the assignment $F \mapsto F \cap K_v$ is a bijection between the faces of dimension $d-1$ of K that contain v, and the corresponding faces of K_v. (The inverse map is $F \mapsto F^{\sim} \cap K$.) As K_v is a simplex and $\bigcap(F_k \cap K_v) = \{\mathbf{0}\}$, it follows easily that $\bigcap(F_k \cap K_v) = \bigcap(F_k \cap K_v)^{\sim}$. Hence $\bigcap(F_k)^{\sim} = \{\mathbf{0}\}$, and thus $\bigcap(F_k)^{\sim} = \bigcap F_k$.

(b)$\Rightarrow$(c). Let the linear forms that define K (that is,

$$K = \bigcap(\beta_i)^{-1}([0,\infty)))$$

be given as

$$\beta_i = \sum_{j=1}^{d} a_{ij}X_j + a_{i,d+1}, \qquad 1 \leq i \leq s.$$

Form the $s \times (d+1)$ matrix

$$B = \{a_{ij}\}_{i=1,j=1}^{i=s,j=d+1}.$$

Then inside $\mathbf{R}^{d+1}$,

$$K \times \{1\} = \{r = (r_1, r_2, \dots, r_d, 1)^T \mid r_i \in \mathbf{R},\ Br \text{ has no negative entries }\}.$$

By performing certain row and column operations on B, we will construct an element of $\mathrm{AGL}(d, \mathbf{R})$ sending K to a product of simplicies in standard position.

The compactness of K yields:

(i) There exists a positive real number N so that if

$$s = (s_1, \dots, s_d, 1)^T \in \mathbf{R}^{d+1},$$

and $\max\{|s_i|\} \geq N$, then Bs contains a negative entry.

That K contains interior (in $\mathbf{R}^d$) allows us to conclude:

(ii) There exists s in $K \times \{1\}$ such that all entries of Bs are strictly positive (which we write as $Bs >> 0$).

If P is an elementary matrix of size $d+1$ whose bottom row is $(0\,0\dots0\,1)$, then P induces an affine homeomorphism $\psi_P\colon K \times \{1\} \to K' \times \{1\}$, where $K' = \{r = (r_i) \in \mathbf{R}^d \mid BP^{-1}(r_1, \dots, r_d, 1)^T \geq 0\}$, is given by sending $(r_1, \dots, r_d, 1)$ to $P(r_1, \dots, r_d, 1)^T$.

If Q is an elementary matrix of size the number of rows of B, and if Q implements either a row permutation or multiplication by a positive scalar, then the polytope determined by QB (instead of B) is equal to that of B, i.e., it is unchanged.

We call any concatenation of transformations of the forms, right multiplication by P, left multiplication by Q, of the types described above, an *admissible* transformation. Note that the matrices P referred to above implement all the usual elementary column operations except those changing the $(d+1)$st column. If K'' is the polytope obtained from an admissible transformation (which means that there are matrices $P_1, P_2, \dots, P_f$, so that $K'' = \psi_f \cdots \psi_2\psi_1(K)$), then obviously K'' will still satisfy (i) and (ii) above.

By means of admissible transformations, we shall first put B in a form so that the matrix consisting of the first d columns is in column reduced echelon form. If at any stage in this process, a column of zeros arises, the (ii) fails. Thus at each stage, each of the first d columns contains a nonzero entry. Therefore after a first row interchange, we may assume a_{11} is not zero, hence (by multiplying the first column by -1), we may assume it is 1. Now by the obvious column operations, we reduce to $a_{12} = a_{13} = \cdots = a_{1,d+1} = 0$. Since the second column at this stage is not zero, it has a nonzero entry, which necessarily is not in the first row (as $a_{12} = 0$). Via a row interchange that does not involve the first row, we may assume that a_{22} is not zero, hence is 1. We can then continue in the same fashion until we have reduced to the following situation:

$$(*) \qquad B_1 = \left[\begin{array}{c|c} \mathbf{I}_d & \begin{matrix}0\\0\\0\\\vdots\\0\\0\end{matrix} \\ \hline \begin{matrix} * & * & \cdots & * & * & * & * \\ \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots \\ * & * & \cdots & * & * & * & * \end{matrix} & \begin{matrix} * \\ \vdots \\ * \end{matrix}\end{array}\right]$$

We call the column of size $s-d$ that consists of the $d+1$st through s entries of the ith column, the *truncated i-th column.*

When B_1 has the form $(*)$, properties (i) and (ii) jointly imply:

(iii) For all $i \leq d$, the truncated ith column contains a negative entry.

To see this, let $u = (r_1, r_2, \ldots, r_d, 1)^T$ be such that $B_1 u \gg 0$. From $(*)$, $r_i > 0$. For j fixed and $M > 0$, set u' to be u with r_j replaced by Mr_j. If all entries of the truncated jth column were non-negative, then $B_1 u' \gg 0$ irrespective of M, violating (i).

Now we show that every truncated column in $(*)$ contains exactly one nonzero entry. Index rows $d+1$ through s, as $c^1, c^2, \ldots, c^{s-d}$, respectively. The jth entry of the row corresponding to c^i will be denoted $c^{i,j}$. Considering the c^i as linear functionals, we may assume that $(c^i)^{-1}(0) \cap K$ is of dimension $d-1$, or else the corresponding linear functional would be redundant, and so could be deleted from the set.

The first d rows (or their corresponding linear functions) are called simply β_k, $k = 1, 2, \ldots, d$, and when the form $(*)$ applies, $\beta_k = X_k$.

A consequence of hypothesis (b) is that if $\{\beta_k\} \cup \{c^i\} \supset D$, and if $V = \bigcap\{\alpha^{-1}(0) \mid \alpha \in D\}$ is non-empty, then $V \cap K$ would also be non-empty. To see this, simply note that each of $\alpha^{-1}(0) \cap K$ is a face of K of codimension 1, so that its affine span is precisely $\alpha^{-1}(0)$. Now (b) applies, with $\dim \bigcap F_k = -1$.

Now the form of B_1 will be discussed in detail.

(1) **For** $j \le d$, $c^{i,j} \le 0$.

Fix i. Set $S = \{j \le d \mid c^{i,j} < 0\}$, $T = \{j \le d \mid c^{i,j} > 0\}$, and $U = \{j \le d \mid c^{i,j} = 0\}$.

By (iii), S is not empty.

Assume that T is not empty. Consider the intersection, $(c^i)^{-1}(0) \cap \{\bigcap_T (\beta_j)^{-1}(0)\}$; this is:

$$V = \left\{(r_1, r_2, \ldots, r_d, 1)^T \mid r_j = 0 \text{ for } j \in T, \text{ and } \sum_S c^{i,j} r_j + c^{i,d+1} = 0\right\}.$$

Hence $V \cap K$ (which is the intersection of the corresponding faces) is contained in

$$\left\{(r_1, r_2, \ldots, r_d, 1)^T \mid r_j = 0 \text{ for } j \in T; r_i \ge 0 \text{ for all } i; \text{ and } \sum_S c^{i,j} r_j + c^{i,d+1} = 0\right\}.$$

Since V is not empty, neither is $V \cap K$; since $c^{i,j}$ is negative for j in S, we must have that $c^{i,d+1} \ge 0$. If the latter were equality, $V \cap K$ would be a singleton, hence by (b), so would V be and thus S would be empty, a contradiction. Hence $c^{i,d+1} > 0$.

Now consider

$$W = (c^i)^{-1}(0) \cap \left\{ \bigcap_S (\beta_j)^{-1}(0) \right\}$$
$$= \left\{ (r_1, r_2, \ldots, r_d, 1)^T \mid r_j = 0 \text{ for } j \in S, \text{ and } \sum_T c^{i,j} r_j + c^{i,d+1} = 0 \right\}.$$

As T is not empty, neither is W. Intersect this with K to obtain the intersection of the corresponding faces; that is not empty for the same reasons as before and

$$W \cap K = \left\{ (r_1, r_2, \ldots, r_d, 1)^T \mid r_j = 0 \text{ for } j \in S; r_i \geq 0 \text{ for all } i; \text{ and } \sum_T c^{i,j} r_j + c^{i,d+1} = 0 \right\}.$$

As $c^{i,j} \geq 0$ for j in T, and $c^{i,d+1} > 0$, this last set must be empty, a contradiction. Hence T is empty.

(2) $c^{i,d+1}$ **is positive.**

Fix i. There exists j so that $c^{i,j}$ is not zero (else c^i would be a constant linear functional, hence redundant), so must be negative. The equation, $c^i(x) = 0$, for $x = (r_1, r_2, \ldots, r_d, 1)^T$, would have no non-negative solutions if $c^{i,d+1}$ were negative, and this would imply that $(c^i)^{-1}(0) \cap K$ were empty, violating irredundance.

If $c^{i,d+1} = 0$, then $c^i(x) = 0$ would have only one non-negative solution, and this would imply that $\dim(c^i)^{-1}(0) \cap K = 0$. This entails that c^i be redundant (unless $d = 1$, when the theorem would be trivial in any case), a contradiction.

By multiplying the bottom rows by positive constants, we may assume that $c^{i,d+1} = 1$ for all i.

(3a) **If for $i \neq i'$ and some j, $c^{i,j}$ are both nonzero, then $c^{i,j} = c^{i',j}$.**

Set $z = (0, 0, \ldots, 0 - 1/c^{i,j}, 0, 0, \ldots, 0, 1)^T$ (the nonzero entries are in the jth and the last positions only. Then

$$\{z\} = \left\{ \bigcap_{k \neq j} (\beta_k)^{-1}(0) \right\} \cap \{(c^i)^{-1}(0)\}.$$

As usual in these arguments, z belongs to the intersection of the corresponding faces by hypothesis, so in particular, z belongs to K. Thus $c^{i'}(z) \geq 0$, whence $-c^{i,j}/c^{i',j} + 1 \geq 0$; interchanging i and i', we obtain $c^{i,j} = c^{i',j}$.

(3b) **If $i \neq i'$, then there exist $j, j' \leq d$ such that $c^{i,j} \neq 0 = c^{i',j}$ and $c^{i',j'} \neq 0 = c^{i,j'}$.**

Otherwise, after a relabelling, there would exist i, i' with $c^{i,j} \neq 0$ implying $c^{i',j} \neq 0$ for all $j \leq d$. Set $S = \{j \leq d \mid c^{i',j} \neq 0 = c^{i,j}\}$; S may be empty. For all j not in S, $c^{i,j} = c^{i',j}$ by (3a). Then $c^i = c^{i'} + \sum_S (-c^{i',j})\beta_j$.

This contradicts the irredundance of c^i.

(3c) **For $i \neq i'$, for $j \leq d, c^{i,j} \cdot c^{i',j} = 0$.**

If not, in view of (3a,b), there would exist distinct i, i' and distinct j, j', j'' such that

$$c^{i,j} = c^{i',j} \neq 0; \quad c^{i,j'} \neq 0 = c^{i',j'}; \quad \text{and} \quad c^{i',j''} \neq 0 = c^{i,j''}.$$

Set $T = \{1, 2, \dots, d\} \backslash \{i, i'\}$. Form the intersection,

$$V = \left\{ \bigcap_{k \in T} (\beta_k)^{-1}(0) \right\} \cap \{(c^{i'})^{-1}(0)\}.$$

Then $V = \{(0, 0, \dots, 0, r_{j'}, 0, \dots, 0, -1/c^{i,j}, 0, \dots, 0, 1)^T\}$, that is, V has dimension 1 (the r_j term may vary freely). We show that $V \cap K$ contains at most a singleton. Select z in $V \cap K$; then the corresponding value of the free variable $r_{j'}$ must be nonnegative. Applying c^i to z, the result is $-c^{i,j}/c^{i,j} + c^{i,j} r_{j'} + 1$. In order for this to be non-negative (as z belongs to K), we must have that $r_{j'} = 0$ (as $c^{i,j'} < 0$). Thus $V \cap K$ is at most a singleton, contradicting (b).

(4) **Every truncated jth column ($j \leq d$) contains exactly one nonzero entry.**

Properties (iii) and (3c) yield this immediately.

Now we may multiply each of the first d columns by positive scalars so that the unique negative entry in that column becomes -1; then we multiply each of the first d rows by the corresponding reciprocals, and restore the size d identity block. Now permute the first d columns so that the $d + 1$st row is $(-1, -1, \dots, -1, 0, \dots, 0, 1)$, the $d + 2$nd row is $(0, \dots, 0, -1, \dots, -1, 0, \dots, 0, 1)$, and so on. Now perform the inverse permutation to the first d rows, to restore the identity matrix.

The form of the resulting matrix is then

$$B_2 = \left[\begin{array}{cccccccccccc|c} & & & & & & & & & & & & 0 \\ & & & & & & & & & & & & 0 \\ & & & & & & \mathbf{I}_d & & & & & & 0 \\ & & & & & & & & & & & & \cdot \\ & & & & & & & & & & & & \cdot \\ & & & & & & & & & & & & \cdot \\ & & & & & & & & & & & & 0 \\ & & & & & & & & & & & & 0 \\ \hline -1 & -1 & \dots & -1 & 0 & \dots & 0 & 0 & 0 & 0 & 0 & & 1 \\ 0 & 0 & \dots & 0 & -1 & \dots & -1 & 0 & 0 & 0 & 0 & & 1 \\ \cdot & \cdot & \dots & & & \dots & & & \dots & & & & \cdot \\ \cdot & \cdot & \dots & & & \dots & & \dots & & & \cdot & & \\ 0 & 0 & & \dots & \dots & \dots & \dots & \dots & -1 & \dots & -1 & & 1 \end{array}\right].$$

Let $n(i)$ be the number of -1's that appear in the row c^i in the matrix B_2. Define for an integer $m, S(m)$, the standard solid simplex in $\mathbf{R}^m$, that is $S(m) = \{(r_k) \in \mathbf{R}^m \mid r_k \geq 0 \text{ and } \sum r_k = 1\}$. Let K' be the polytope corresponding to B_2 in the sense that

$$K' \times \{1\} = \Big\{ r = (r_1, r_2, \dots, r_d, 1)^T \mid r_i \in \mathbf{R}, \quad B_2 r \text{ has no negative entries} \Big\}.$$

There is an obvious map $K \to \prod S(n(i))$, that sends the element $(r_1, \dots, r_2)^T$ to:

$$((r_1, \dots, r_{n(1)})^T, (r_{n(1)+1}, \dots, r_{n(2)+n(1)})^T, \dots, (r_{d-n(s-d)+1}, \dots, r_d)^T).$$

It is immediate from the form of B_2 that this is an affine homeomorphism.

(c)$\Rightarrow$(a). If K is a simplex, (a) clearly holds. In a product of polytopes, each face is a product of faces of the components, and (a) clearly follows. □

Now we show that affine equivalence is precisely the right notion of equivalence to implement order-isomorphism between the rings $R[K]$ that we are studying. There is no requirement that $R[K]$ have interpolation in what follows.

II.6. THEOREM. *Let K and K'' be d-dimensional compact convex polyhedra in $\mathbf{R}^d$, defined over a subfield L of $\mathbf{R}$. Every order-isomor-*

phism $R[K] \to R[K'']$ *of (partially ordered) rings induces an affine homeomorphism* $K'' \to K$*, and every affine homeomorphism* $K'' \to K$ *induces an ordered ring isomorphism* $R[K] \to R[K'']$.

Proof. Because of the full dimensional hypothesis, any affine homeomorphism from K'' to K extends uniquely to an affine homeomorphism $\phi\colon \mathbf{R}^d \to \mathbf{R}^d$. This induces a ring automorphism $\psi\colon \mathbf{R}[X_j] \to \mathbf{R}[X_j]$ which fixes the scalars, and because K and K'' are defined over L, their vertices have all of their coordinates lying in L. Thus ψ restricts to a ring automorphism $\psi\colon L[X_j] \to L[X_j]$. Inasmuch as the $d-1$-dimensional faces determine both the compact convex sets and the irredundant linear forms (which in turn determine the orderings), it follows that ψ is an order-isomorphism.

Now let $\psi\colon R[K], 1 \to R[K''], 1$ be an isomorphism of ordered groups that sends 1 to 1. This induces a homeomorphism between the corresponding pure state spaces (because the states are normalized with respect to 1), and thus a homeomorphism $\phi\colon K'' \to K$. Let $\{\beta_i\}$ be a full set of irredundant linear forms that determine K, and let $\{\alpha_j\}$ be a corresponding set for K''. We may attach to each of the β's and α's a positive real number, for example, to β_i, the number

$$t_i = \inf\{1/q \mid q \in \mathbf{Q}^+, q\beta_i \leq 1\}.$$

By **[EHS**; 1.4], $\sup\{\beta_i(k) \mid k \in K\} = t_i$. Let u_j denote the corresponding numbers for the α_j. Now consider $\psi(\beta_i)$. The zero set of β_i on K contains a $d-1$-dimensional ball, so that the zero set of $\psi(\beta_i)$ (on K'') also does. Since $\psi(\beta_i)$ is in $R[K'']^+$, its zero set is a union of faces, and thus at least one of these faces must be of dimension $d-1$; it thus corresponds to an α_j. By the argument in II.2(b) (based on zero sets) α_j must divide $\psi(\beta_i)$, say $\psi(\beta_i) = b_i\alpha_j$ (j is determined by i).

However, each β_i is an irreducible element in the polynomial ring. If we additionally assume that ψ is a ring isomorphism, then $\psi(\beta_i)$ must also be irreducible, so that b_i is a scalar (that is, in L). We also quickly see that $b_i = t_i/u_j$, so the latter belongs to L, and b_i is positive. Since there is an inverse isomorphism, we obtain that ψ induces a bijection $\{\beta_i\} \to \{\alpha_j b_i\}$. Since each of the variables (the X's) is a linear combination of $\{\beta_i\}$, $\{\alpha_j\}$ respectively (I.1(c)), we quickly deduce that ψ is affine, and thus induces an affine map $K'' \to K$. □

For an integer n, let $\mathbf{R}_n$ denote the polynomial ring in n variables over the field L, with positive cone generated additively and multiplicatively by $L \cap \mathbf{R}^+$ and $X_1, X_2, \dots, X_n, 1 - X_1 - \cdots - X_n$; in other words, $R_n = R[S(n)]$. Now $R_{\mathbf{Z}}[S(n)]$ is precisely the example in [**H1**; §VI], so it has Riesz interpolation. However, $R_n = R_{\mathbf{Z}}[S(n)] \otimes_{\mathbf{Z}} L$ (as ordered rings), so that R_n itself admits interpolation. This is the only proof I know of this fact(!).

II.7. Corollary. *Let $R[K]$ be the partially ordered ring obtained from compact convex d-dimensional polyhedron K in $\mathbf{R}^d$. If $R[K]$ satisfies the Riesz interpolation property, then it is order-isomorphic to exactly one (for each partition $\{n(i)\}$ of n) of*

$$\bigotimes_{\sum n(i)=d} R_{n(i)}.$$

(The tensor product is over L.) Conversely, any such tensor product satisfies interpolation.

Proof. From II.2, II.3, and II.5, K is affinely equivalent to a product of simplices, say $K \approx \prod S(n(i))$, and by dimension, $\sum n(i) = d$; by the previous result, $R[K] \approx R[\prod S(n(i))]$. It is easy to verify that the latter is order-isomorphic with the indicated tensor product.

If $L = \mathbf{Q}$, then the tensor product would just be the tensor product over $\mathbf{Z}$, and it is known and easy to check (e.g., [**EHS**; 2.2]) that having interpolation (in the presence of unperforation, which is automatic here) is preserved by tensor products. Now, $R_k \approx R_{\mathbf{Q}}[S(k)] \otimes_{\mathbf{Q}} L$, and as L is totally ordered and therefore a dimension group, so is R_k, and thus it satisfies interpolation. Hence,

$$R \approx (\otimes_{\mathbf{Q}} R[S(n(i))]) \otimes_{\mathbf{Q}} L$$

satisfies interpolation as well. □

In the two motivating examples [**H1**; Introduction and §VI], the polyhedra were respectively the simplex (in standard position) and the standard hypercube, and moreover, instead of allowing coefficients from a field, the coefficients were restricted to be integers. In both these cases, interpolation holds (just as when the coefficient ring is a field). However, these are exceptional examples, as $R_{\mathbf{Z}}[K]$ generally does not admit interpolation when $R[K]$ does. Here is a simple one-dimensional example:

II.8. EXAMPLE. If $d = 1$, as a ring $R_{\mathbf{Z}}[K] = \mathbf{Z}[X]$; set $\beta_1 = X$, $\beta_2 = 2 - X$, and $\beta_3 = 3 - X$. Note that $3 - X$ is not a positive combination of β_1 and β_2, as only integer coefficients are allowed. Now K is the interval $[0, 2]$. We observe that interpolation fails here. Point evaluation at 1 (that is $X \mapsto 1$) is a pure state with $\mathbf{Z}$ as image, but there are no positive elements in the kernel (a positive element can only vanish at an endpoint), which would violate [GH; 4.3 and 3.1] if $R_{\mathbf{Z}}[K]$ satisfied interpolation.

Similarly, if K is a compact convex polyhedron of dimension d inside $\mathbf{R}^d$, defined by integer forms, and K contains a point, not a vertex, with only integer coordinates, and one of them is ± 1, then $R_{\mathbf{Z}}[K]$ fails to satisfy interpolation. Just what conditions are necessary and sufficient for a general $R_{\mathbf{Z}}[K]$ to satisfy interpolation, are unclear.

This same example also fails to satisfy the unperforation property defined just after Corollary I.2. Observe that $2 = \beta_1 + \beta_2$, so that the constant function 2 is in the positive cone; however, 1 is not. Simply observe that the constant term in any word in the β's, if nonzero, is a product of a power of 2 with a power of 3, and not both exponents can be 0; thus 1 cannot be expressed as a sum of these words. On the other hand, every other positive integer is in the positive cone. □

III. An almost non-vanishing element that is not positive. Return to our original conventions concerning the d-dimensional polytope K in $\mathbf{R}^d$, and its corresponding ordered ring $R[K]$. Unless either $d = 1$, or the strict positivity hypothesized in I.2 holds, it is difficult to decide if a specific element, f, of $R[K]$ is in the positive cone. (In principle, if K were a product of simplices, then [H2; Theorem B] would give necessary and sufficient conditions for this to happen.) An obvious necessary condition is that the restriction of f to K be non-negative (as a function); almost as obvious, and following from the definition of $R[K]^+$, is that $f^{-1}(0) \cap K$ must be a union of faces. These are far from sufficient, as we shall demonstrate in this section. We exhibit for every polytope K of dimension d exceeding 1, an element f with the following properties:

(i) $f \mid K \geq 0$;

(ii) f is a sum of squares of polynomials;

(iii) $f^{-1}(0) \cap K$ consists of a single vertex of K;

(iv) f does *not* belong to $R[K]^+$.

Let K be a polytope of dimension d, with $d > 1$. Let $\{\beta_i\}_{1\le i\le s}$ be an irredundant set of linear forms that determine K. Pick a vertex v of K; by relabelling, we may assume

$$\{v\} = \left(\bigcap_{i\le d} \beta_i^{-1}(0)\right) \cap K$$

and in addition, $\{\beta_i\}_{i\le d}$ is affinely independent. For each positive integer N define the element of $R[K]$ (that is, the polynomial)

$$f_N = N\left(\sum_{i=1}^{d} \beta_i^d\right) - \prod_{i=1}^{d} \beta_i.$$

If $N > 1/d$, then by **[HLP**; p. 55, para. 2], f_N is a sum of squares of elements of $R[K]$, and f_N vanishes on K only at v. Thus (ii) and (iii) hold, and obviously (ii) implies (i). Now we show that f_N does not belong to $R[K]^+$.

The affine independence of the first d of the β's allows us to find a d-simplex K' inside K with v being a vertex of K', where K' is defined by $\beta_1, \beta_2, \ldots, \beta_d$, together with an additional linear form ψ. (This may also be done using Carathéodory's theorem.) Since $K \supset K'$, it follows from I.1(a) that $R[K']^+ \supset R[K]^+$. In particular, if f_N belonged to $R[K]^+$, it would also belong to $R[K']^+$.

As K' is a simplex, $R[K']$ admits interpolation. The ideal, I, of $R[K']$ generated by

$$\{\beta_i^d\}_{i=1}^{d}$$

is thus an order-ideal of $R[K']$ (II.1(d)). If f_N were in the positive cone,

$$\prod_{i=1}^{d} \beta_i$$

would consequently belong to I. However, the affine independence of $\{\beta_i\}_{i\le d}$ allows us to regard them as letters (variables) in which every element of $R[K']$ is uniquely expressible as a polynomial. It is well-known and easy to check that in the polynomial ring $\mathbf{R}[Y_j]$ in the variables $\{Y_j\}$, the element $Y_1 \cdots Y_d$ does not belong to the ideal generated by

$$\{Y_j^d\}_{j=1}^{d}$$

(there exists a commutative nilpotent algebra with every element nilpotent of index d or less, yet with a nonzero product of d terms). This contradicts f_N belonging to the positive cone of $R[K']$, and so it does not belong to the positive cone of $R[K]$, verifying (i). □

In the case of the standard simplex, $S(d)$, the argument above shows that the element

$$f_N = N\left(\sum_{i=1}^{d} X_i^d\right) - \prod_{i=1}^{d} X_i$$

is not positive in $R[S(d)]$ (for $N > 1/d$).

IV. Connections with K_0. We briefly describe the motivating examples occurring in **[H3**; §VII] and a rather odd map between the ordered rings considered here and those considered there. Let $x_1, \dots, x_d$ be variables (note: *lower case*) and let P be a polynomial in them with integer (real) coefficients in **[H1]** and **[H3]**, Laurent polynomials are considered, but this makes no essential difference). Write $P = \sum \lambda_w x^w$, where $w = (w(1), \dots, w(d))$ is a d-tuple of integers, and $x^w = x^{w(1)} \cdots x^{w(d)}$. Define $R_{\mathbf{Z},P}$ (R_P) as the ring (algebra, respectively) generated by $W = \{x^w/P \mid \lambda_w \neq 0\}$ (over $\mathbf{R}$) inside the rational functions. Define a positive cone to be that generated additively and multiplicatively (over $\mathbf{R}^+$) by W. This is equivalent to the original definition given in **[H1]**, by **[H1**; I.4].

For example, if $P = \lambda_0 + \sum \lambda_i x_i$, then upon setting

$$X_i = \frac{\frac{\lambda_i}{\lambda_0} x_i}{P},$$

we see that $R_P = \mathbf{R}[X_i]$, and this is a pure polynomial algebra, and moreover, R_P^+ is generated by $X_1, \dots, X_d, 1 - X_1 - \cdots - X_d$. In other words, R_P is order-isomorphic to $R[S(d)]$ ([op cit.; §VI]), and this was in fact the starting point for this article.

For general $P = \sum \lambda_w x^w$ (with λ_w all non-negative reals), define $\operatorname{Log} P = \{w \in \mathbf{Z}^d \mid \lambda_w \neq 0\}$; we associate to P a polytope, $K(P)$, namely the convex hull (in $\mathbf{R}^d$) of $\operatorname{Log} P$. Note that the vertices of this polytope are all lattice points (i.e., in $\mathbf{Z}^d$); such a polytope is called an *integral polytope* (sometimes lattice polytope is used). In our example above, $P = \lambda_0 + \sum \lambda_i x_i$, $K(P)$ is simply the standard d-polytope, and we have the not surprising isomorphism, $R_P \approx R[K(P)]$. Here is another example; it turns out that these examples are completely misleading!

Let $P = \prod_{1 \le i \le d}(1 + x_i)$. Set $X_i = x_i/(1 + x_i)$, so that $1 - X_i = 1/(1 + x_i)$; it is easy to see that R_P is ring isomorphic to $\mathbf{R}[X_i]$, the pure polynomial ring. We observe that $(R_P)^+$ is generated (additively and multiplicatively) by $\{X_i\} \cup \{1 - X_i\}$. Here $K(P)$ is the standard hypercube in $\mathbf{R}^d$, and again we have that $R_P \approx R[K(P)]$, even as ordered rings.

For any $P = \sum \lambda_w x^w$ (with λ_w all non-negative reals), R_P has interpolation. Thus we cannot expect a general theorem of the form $R_P \approx R[K(P)]$ (as ordered rings), since very few integral polytopes are $\mathrm{AGL}(d, \mathbf{R})$-equivalent to a product of polytopes. However, there is a vastly more stringent requirement. Suppose P is irreducible (over the real polynomial ring), and the set of differences $\operatorname{Log} P - \operatorname{Log} P$ generates the standard copy of $\mathbf{Z}^d$ inside $\mathbf{R}^d$ as an abelian group ($\operatorname{Log} P$ is "projectively faithful"; we can always reduce to this situation). Irreducibility of P is generic, in the sense that a randomly chosen P will be irreducible. Then by **[H3**; Theorem A8], if R_P is a unique factorization domain, up to the natural $\mathrm{AGL}(d, \mathbf{Z})$-action on $\mathbf{Z}^d$ (the exponents), $P = \lambda_0 + \sum \lambda_i x_i$—our first example. Since $R[K]$ is always a pure polynomial algebra, hence a unique factorization domain, we deduce that if P is irreducible (generic) and $\operatorname{Log} P$ is projectively faithful (innocuous), then $R_P \approx R[K(P)]$ even merely as rings implies that up to $\mathrm{AGL}(d, \mathbf{Z})$, P must be as in our first example. (Note that concerning rings of the form R_P, it is $\mathrm{AGL}(d, \mathbf{Z})$, *not* $\mathrm{AGL}(d, \mathbf{R})$, that implements natural isomorphisms.) So this isomorphism is an extremely rare event.

Even though one cannot expect an isomorphism between $R[K(P)]$ and R_P, there is always a natural order-preserving map that induces a homeomorphism on the pure state spaces. Define $\Phi_P \colon R[K(P)] \to R_P$ via:

$$X_i \mapsto \frac{x_i \frac{\partial P}{\partial x_i}}{P}.$$

It is readily verified that

$$\operatorname{Log}\left(x_i \frac{\partial P}{\partial x_i}\right) \subset \operatorname{Log} P,$$

so that the assignment has image in R_P, and thus extends to an algebra homomorphism. Recall the definition of pure state space from §I; we observed in the course of the proof of I.3, that the pure state space of $R[K]$ could always be identified with K, by means of point evaluations.

IV.1. Proposition. *If $K = K(P)$ contains interior, the map Φ_P is an order-preserving algebra embedding that induces a homeomorphism, $S(R_P, 1) \to S(R[K], 1) = K$, on the corresponding pure state spaces.*

Proof. To show that Φ_P is order-preserving, it is sufficient to show that if β is any linear form such that $\beta \mid K \geq 0$, then $\Phi_P(\beta) \in (R_P)^+$.

Write $\beta = \sum u_i X_i + u_0$, and $P = \sum \lambda_w x^w$. Then

$$x_i \frac{\partial P}{\partial x_i} = \sum_{w \in \operatorname{Log} P} \lambda_w w(i) x^w.$$

Thus:

$$\begin{aligned}\Phi_P(\beta) &= \sum_i \frac{u_i \Phi_P(X_i)}{P} + u_0 = \sum_{i,w} u_i w(i) \lambda_w \frac{x^w}{P} + u_0 \sum_w \lambda_w \frac{x^w}{P} \\ &= \sum_w \left(\sum_i u_i w(i) + u_0 \right) \lambda_w \frac{x^w}{P} = \sum_w \beta(w) \lambda_w \frac{x^w}{P}.\end{aligned}$$

As $\beta \mid K \geq 0, \beta(w) \geq 0$ for all w in $\operatorname{Log} P$. Thus $\Phi_P(\beta) \in (R_P)^+$.

Next we show that the induced map of pure states, $S(R_P) \to K$ is a homeomorphism. If γ is a pure state, then the induced map is:

$$\gamma \mapsto (\gamma \Phi_P(X_1), \gamma \Phi_P(X_2), \ldots, \gamma \Phi_P(X_d)) = \left(\gamma \left\{ \frac{x_i \frac{\partial P}{\partial x_i}}{P} \right\} \right).$$

It is not hard to check that the term on the right can be re-expressed as

$$\sum_{w \in \operatorname{Log} P} \lambda_w \gamma \left(\frac{x^w}{P} \right) w.$$

Hence this map on the state spaces is precisely the map Λ_P defined in **[H3**; §IV] (and also in **[H1**; §III]), where it is shown to be an (onto) homeomorphism $S(R_P) \to K$ (when K contains interior).

Next, we check that Φ_P is one to one. If f belongs to $R[K]$ and is not zero, there exists (as K contains interior) k in K such that $f(k) \neq 0$. There exists, by the ontoness of the map on the pure state spaces, γ in $S(R_P)$ such that $\gamma \mapsto k$. Then $\gamma(\Phi_P(f)) = f(k) \neq 0$, so that $\Phi_P(f)$ is not zero. □

The proof of course suggests how this mapping, Φ_P, came to mind!

Given an *integral polytope* K, an ordered ring R_K was defined, by choosing any P with no negative coefficients, such that $\operatorname{Log} P = K \cap \mathbf{Z}^d$, forming R_P, and inverting all of the order units therein. The set of order units in R_P is multiplicatively closed (and also closed under addition, although there seems to be no particular application for this). Let U denote the set of order units in $R[K]$; an element u of the ring belongs to U if and only if $u(k) > 0$ for all k in K (an easy consequence of I.3). Moreover, because Φ_P induces a homeomorphism on pure state spaces, for f in $R[K]$, f is an order unit, if and only if $\Phi_P(f)$ is such in R_p. In any event, Φ_P extends to an embedding (also called Φ_P) $R[K, U^{-1}] \to R_K$. Now $R[K, U^{-1}]$ admits a natural ordering, specifically, if f belongs to $R[K]$ and v to U, then $fv^{-1} \geq 0$ if there exists u in U such that fu belongs to $R[K]^+$. Similarly, R_K admits an analogous ordering, but it turns out that it is very easy to describe in terms of that of R_P because $(R_P)^+ = (R_K)^+ \cap R_P$ [**H3**; §2]. The extended map is still order-preserving, but it would be of interest to decide when it is an isomoprhism. It is unknown whether or not $R[K, U^{-1}]^+ \cap R[K] = R[K]^+$, and knowledge of when this occurs would be very helpful. For example, it might be that even though the original $R[K]$ did not have interpolation, the larger ordered ring $R[K, U^{-1}]$ might satisfy it.

On the other hand, there are still some limitations on K, for such an isomorphism to occur. For example, if R_K is a unique factorization domain, and $\operatorname{Log} P$ is projectively faithful, then K must be *integrally simple*:

For every vertex v of the integral polytope K, the convex hull of v together with the nearest lattice points along all the edges emanating from v is $\mathrm{AGL}(d, \mathbf{Z})$-equivalent to a standard solid polytope (in other words, this local polytope at v has volume $1/d!$).

To see just how strong this is, it is not hard to show that an integrally simple polytope that is also a simplex is *up to* $\mathrm{AGL}(d, \mathbf{Z})$ an integer multiple of the standard simplex, i.e., $mS(d)$ for some integer m. A careful analysis of the proof of II.5(b)$\Rightarrow$(c) reveals that it can be modified to prove:

An integrally simple polytope (with interior) that is $\mathrm{AGL}(d, R)$-equivalent to a product of simplices is $\mathrm{AGL}(d, Z)$-equivalent to a product of integer multiples of standard simplices.

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Received May 16, 1985 and in revised form, February 26, 1987. Supported by an operating grant, and a Steacie Memorial Fellowship from the Natural Sicences and Engineering Research Council of Canada.

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