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SHIFTS OF INTEGER INDEX ON THE HYPERFINITE II_1 FACTOR

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In this paper we consider shifts on the hyperfinite II_1 factor arising as a generalization of a construction of Powers. We determine the conjugacy classes of certain of these shifts.

1. Introduction. Let R be the hyperfinite II_1 factor with normalized trace tr. A shift α on R is an identity-preserving $*$-endomorphism which satisfies $\bigcap_{m\geq 1}\alpha^m(R) = \mathbf{C}1$. We say that α has shift index n if the subfactor $\alpha(R)$ has the same index $n = [R\colon \alpha(R)]$ in R as defined by Jones, in **[2]**.

In **[3]** Powers considered shifts of index 2 on R. These were constructed using functions $\sigma\colon \mathbf{N}\cup\{0\}\to\{-1,1\}$ and sequences $\{u_j\colon j\in \mathbf{N}\}$ of self-adjoint unitaries satisfying $u_iu_j = \sigma(|i-j|)u_ju_i$. If $A(\sigma)$ is the $*$-algebra generated by the $\{u_j\}$ and tr is the normalized trace on $A(\sigma)$ defined by $\mathrm{tr}(w) = 0$ for any non-trivial word in the u_i, the GNS construction $(\pi_{\mathrm{tr}}, H_{\mathrm{tr}}, \Omega_{\mathrm{tr}})$ gives rise to the von Neumann algebra $M = \pi_{\mathrm{tr}}(A(\sigma))''$. Different characterizations were given in **[3]** and **[4]** for M to be the hyperfinite factor R. In **[4]** it was shown this is the case if and only if the sequence $\{\dots,\sigma(2),\sigma(1),\sigma(0),\sigma(1),\sigma(2),\dots\}$ is aperiodic. For this case, the shift α on $M = R$ defined by the relations $\alpha(\pi_{\mathrm{tr}}(u_i)) = \pi_{\mathrm{tr}}(u_{i+1})$ has index 2. In **[3]** it was shown that the σ-sequence above is a complete conjugacy invariant for α. (We say shifts α,β are conjugate if there exists an automorphism γ of R such that $\alpha = \gamma\cdot\beta\cdot\gamma^{-1}$.)

Motivated by **[3]**, Choda in **[1]** considered shifts of index n, defined on R by $\alpha(u_j) = u_{j+1}$, for a sequence of unitaries $\{u_j\}$ generating R, and satisfying $(u_j)^n = 1, u_1u_{j+1} = \sigma(j)u_{j+1}u_1$, where $\sigma\colon \mathbf{N}\cup\{0\}\to\{1,\exp(2\pi i/n)\}$. In this setting and under the assumption $\alpha(R)'\cap R = \mathbf{C}1$ she characterizes the normalizer $N(\alpha)$ of α (see Definition 3.4) and the unitary α-generators of R.

In this paper we generalize some of the results of **[1,3,4]**. In §2 we consider, for a fixed n, algebras generated by sequences $\{u_j\}$ of unitaries, of order n, and satisfying $u_1u_{j+1} = \sigma(j)u_{j+1}u_1$ for functions $\sigma\colon \mathbf{N}\cup\{0\}\to\Omega_n$, the set of nth roots of unity. We determine necessary and sufficient conditions for these algebras, under the GNS representation for a certain trace, to generate the hyperfinite II_1 factor R in the weak closure [Theorem 2.6]. If α is the shift determined by the equations $\alpha(u_i) = u_{i+1}$, then $[R: \alpha(R)] = n$. If $n = 2$ or 3 it follows from [2] that $\alpha(R)' \cap R = \mathbf{C}1$. Here we show the somewhat surprising result that $\alpha(R)' \cap R = \mathbf{C}1$ regardless of the index (Theorem 3.2), so that Choda's assumption holds automatically. Finally we use this result to determine $N(\alpha)$ and show how Powers' techniques generalize to characterize the conjugacy classes of shifts of prime index n.

2. Factor condition. We begin by considering in more detail the construction of the last section. Fix an integer $n > 1$. Let Ω_n be the nth roots of unity, and $\sigma\colon \mathbf{Z} \to \Omega_n$ a function with $\sigma(0) = 1$ and $\sigma(j)^{-1} = \sigma(-j)$. Consider the sequence $\{u_j\colon j \in \mathbf{N}\}$ of distinct unitary operators, each of order n, and satisfying

$$u_i u_j = \sigma(i-j) u_j u_i. \tag{1}$$

Then the u_j generate a $*$-algebra, $A(\sigma)$, consisting of linear combinations of words w of the form $w = u_1^{i_1} u_2^{i_2} \cdots u_m^{i_m}$. From (1) one observes that for words w, w' in $A(\sigma)$ there is a $\lambda \in \Omega_n$ such that $ww' = \lambda w' w$.

Define a trace tr on $A(\sigma)$ by setting $\mathrm{tr}(1) = 1$ and $\mathrm{tr}(w) = 0$ if w is a word not a scalar multiple of the identity. Passing to the GNS construction $(\pi_{\mathrm{tr}}, H_{\mathrm{tr}}, \Omega_{\mathrm{tr}})$ we see that the representation π_{tr} is faithful (note that for distinct words $w_1, w_2, \ldots, w_m$, and $A = \sum_{i=1}^m a_i w_i$, $a_i \in \mathbf{C}$, $\mathrm{tr}(A^*A) = \sum_{i=1}^m |a_i|^2$) so that we shall identify $A(\sigma)$ with its image $\pi_{\mathrm{tr}}(A(\sigma))$ under π_{tr}. Let $\|\ \|_2$ be the trace norm on $A(\sigma)$ given by $\|A\|_2^2 = \mathrm{tr}(A^*A)$. Then we observe that H_{tr} is the space of l^2-summable series $\sum_{i=1}^\infty a_i \delta_{w_i}$, where $\{w_i\colon i \in \mathbf{N}\}$ is a sequence consisting of distinct words in the u_j, and $\delta_w(w') = 0$ if $w^*w' \neq \lambda 1$, $\delta_w(w') = \lambda$ if $w^*w' = \lambda 1$. Let A lie in the center of $A(\sigma)''$, and suppose $A\delta_1 = \sum a_i \delta_{w_i}$. Then for all words $w \in A(\sigma)$,

$$w^* A w \delta_1 = \sum a_i \delta_{w^* w_i w}.$$

Since δ_1 is separating for $A(\sigma)''$ we have $w_i w = w w_i$ for all i with $a_i \neq 0$. From this relation it follows immediately that $A(\sigma)''$ has non-trivial center if and only if there are non-trivial words in the center. We record this in the following (cf. [3, Theorem 3.9], [4, Theorem 3.4]).

LEMMA 2.1. *Let $A(\sigma)$ and* tr *be as above. Then $A(\sigma)''$ has non-trivial center if and only if there exists a non-trivial word in $A(\sigma)$ such that $w'w = ww'$ for all words w' in $A(\sigma)$.*

We may uniquely define a $*$-endomorphism α on $A(\sigma)''$ by setting $\alpha(u_i) = u_{i+1}$. To show α is a shift, let $A \in \bigcap \alpha^m(A(\sigma)'')$, with $\operatorname{tr}(A) = 0$ and $\|A\| \leq 1$. Then given $\varepsilon > 0$ there are positive integers $N < M$ and a B in the unit ball of the algebra $\mathscr{B}$ generated by $u_1, \dots, u_N$, (resp., C in the unit ball of the algebra $\mathscr{C}$ generated by $u_{N+1}, \dots, u_M$) such that $\|(A - B)\delta_1\| < \varepsilon$ (resp., $\|(A - C)\delta_1\| < \varepsilon$). Then there are distinct non-trivial words $w_i \in \mathscr{B}$ (resp., $w'_j \in \mathscr{C}$) so that

$$B = b_0 1 + \sum_{i=1}^{k} b_i w_i \quad \left(\text{resp., } C = c_0 1 + \sum_{j=1}^{1} c_j w'_j\right).$$

From $|\operatorname{tr}(A - B)| \leq \|(A - B)\delta_1\| < \varepsilon$ we have $|b_0| < \varepsilon$, and similarly, $|c_0| < \varepsilon$. Then

$$\begin{aligned} \|A\|_2^2 = \operatorname{tr}(A^*A) &= (A\delta_1, A\delta_1) \\ &\leq |((A - B)\delta_1, A\delta_1)| + |(B\delta_1, (A - C)\delta_1)| + |(B\delta_1, C\delta_1)| \\ &< \varepsilon + \varepsilon + |\operatorname{tr}(C^*B)| = 2\varepsilon + |\overline{c_0} b_0| < 2\varepsilon + \varepsilon^2. \end{aligned}$$

Since ε is arbitrary, $\|A\|_2 = 0$, so $A = 0$. thus $\bigcap \alpha^m(A(\sigma)'')$ consists of scalar multiples of the identity, and we have verified the following.

LEMMA 2.2. *Let α be the $*$-endomorphism defined on $A(\sigma)''$ by $\alpha(u_i) = u_{i+1}$. Then α is a shift.*

DEFINITION 2.3. Let $w = \lambda u_{j_1}^{k_{j_1}} \cdots u_{j_l}^{k_{j_l}}$, with $|\lambda| = 1, k_{j_1} \neq 0 \bmod n$, $k_{j_l} \neq 0 \bmod n$, and $j_1 < j_2 < \cdots < j_l$. Then the length of w is $j_l - j_1 + 1$. If $w = \lambda 1$ then w has length 0.

THEOREM 2.4. *Suppose $n = p$ where p is prime. Let $\{a_j \colon j \in \mathbf{Z}\}$ be a sequence of integers such that $a_0 = 0, a_{-j} = -a_j$. Define $\sigma \colon \mathbf{Z} \to \Omega_p$ by $\sigma(j) = \exp(2\pi i a_j/p)$. Then $A(\sigma)''$ is the hyperfinite* II_1 *factor if and only if $(\dots, a_{-1}, a_0, a_1, \dots)$ is aperiodic when viewed as a sequence over $\mathbf{Z}/p\mathbf{Z}$.*

Proof. The proof is similar to that of [**4**, Theorem 2.3]. If $A(\sigma)'' \neq R$ there is by Lemma 2.1 a non-trivial word $w = u_1^{l_0} \cdots u_{m+1}^{l_m}$ in its center. If $w = \alpha(w')$ for some word w' it is easy to show w' is also central, so we may assume $l_0 \neq 0 \bmod p$. We may also assume $l_m \neq 0$ (p)

and that $m+1$ is the minimum length among all central words. If $v = u_1^{q_0} \cdots u_{m+1}^{q_m}$ is another such word it is apparent using (1) that $v = \lambda w^c$ for some integer c, some $\lambda \in \mathbf{C}$. For let c satisfy $\mathrm{c}l_m = q_m\ (p)$, then by (1) one sees that $w^c v^{-1}$ is a central word of shorter length than w, and must therefore be a scalar multiple of 1.

Now $u_j w = w u_j$ for all j. Setting $j=1$, and using (1) repeatedly, one has

$$\begin{aligned} u_1 w &= u_1 u_1^{l_0} u_2^{l_1} \cdots u_{m+1}^{l_m} = \sigma(0)^{l_0} u_1^{l_0} u_1 u_2^{l_1} \cdots u_{m+1}^{l_m} \\ &= \sigma(0)^{l_0} \sigma(1)^{l_1} u_1^{l_0} u_2^{l_1} u_1 u_3^{l_2} \cdots u_{m+1}^{l_m} \\ &= [\sigma(0)^{l_0} \sigma(1)^{l_1} \cdots \sigma(m)^{l_m}] w u_1 = \exp\left(2\pi i \left(\sum_{s=0}^{m} l_s a_s\right) / p\right) w u_1, \end{aligned}$$

so that $\sum_{s=0}^{m} l_s a_s = 0\ (p)$. Making similar calculations for $u_j w = w u_j$ one obtains the following homogeneous system over $\mathbf{Z}/p\mathbf{Z}$:

$$\begin{array}{lccccccccc} & l_0 a_0 & + & l_1 a_1 & + & l_2 a_2 & + \cdots + & l_m a_m & = & 0\ (p) \\ & -l_0 a_1 & + & l_1 a_0 & + & l_2 a_1 & + \cdots + & l_m a_{m-1} & = & 0\ (p) \\ (2) & \vdots & & & & & & & & \\ & -l_0 a_m & - & l_1 a_{m-1} & - & l_2 a_{m-2} & - \cdots - & l_m a_1 & = & 0\ (p) \\ & \vdots & & & & & & & & \end{array}$$

Rewriting one has

$$AL = [0, 0, \ldots]^T \bmod p \tag{3}$$

where $L = [l_0, \ldots, l_m]^T$, and

$$A = \begin{bmatrix} a_0 & a_1 & a_2 & \cdots & a_m \\ -a_1 & a_0 & a_1 & \cdots & a_{m-1} \\ -a_2 & -a_1 & a_0 & \cdots & a_{m-2} \\ & & \vdots & & \end{bmatrix}. \tag{4}$$

Let $A_0, A_1, \ldots$ be the rows of A. From the symmetry of A it is straightforward to observe that for $q \geq m$,

$$l_0 A_1 + l_1 A_{q-1} + \cdots + l_m A_{q-m} = [0, 0, \ldots, 0],$$

so that the rank of A (over $\mathbf{Z}/p\mathbf{Z}$) coincides with the rank of the matrix A' consisting of the first $m+1$ rows of A. By the argument in the previous paragraph, central words of minimal length correspond to solutions K of $A'K = [0, 0, \ldots, 0]^T$, so the only solutions to this equation are of the form $K = cL, c \in \mathbf{Z}/p\mathbf{Z}$. Hence A has a rank m over $\mathbf{Z}/p\mathbf{Z}$.

From the symmetry of A' one observes $A'\tilde{L} = [0, 0, \ldots, 0]^T$, where $\tilde{L} = [l_m, \ldots, l_0]^T$. Hence $\tilde{L} = cL$ for some c in $\mathbf{Z}/p\mathbf{Z}$. Hence if $(A_0)_j$ is the row vector obtained from A_j by reversing the order of the entries then $(A_0)_j$ has inner product 0 with L. This fact, and the property that rows $A_{m+1}, A_{m+2}, \ldots$ are in the span of rows $A_1, \ldots, A_m$ imply that $BL = [0, 0, \ldots, 0]^T$, where B is a row consisting of any $m+1$ consecutive entries of the sequence $(\ldots, a_{-2}, a_{-1}, a_0, a_1, a_2, \ldots)$. Therefore, for any $j \in \mathbf{Z}$, if $B_j = [a_j, \ldots, a_{j+m}]$, $B_{j+1}^T = C(B_j^T)$, where

$$C = \begin{bmatrix} 0 & 1 & 0 & & \ldots & 0 \\ 0 & 0 & 1 & 0 & \ldots & 0 \\ \vdots & & & & & \\ cl_0 & cl_1 & & \cdots & & cl_{m-1} \end{bmatrix}$$

and c is an integer such $cl_m = -1$ (p). C is invertible over $\mathbf{Z}/p\mathbf{Z}$, so $C^s = I$ for some s, and therefore $B_{j+s} = B_j$, all $j \in \mathbf{Z}$, so that $(\ldots, a_{-2}, a_{-1}, a_0, a_1, a_2, \ldots)$ is periodic.

Conversely, suppose the sequence is periodic, with period length m. Consider the homogeneous system $AX = [0, 0, \ldots]^T$, where $X = [x_0, x_1, \ldots, x_m]^T$ and A is as above. Using the periodicity $a_j = a_{j+m}$ one observes that the $(m+j)$th equation coincides with the jth equation, for all j, so the system $AX = 0$ reduces to m equations in $m+1$ unknowns. Let $L = [l_0, \ldots, l_m]^T$ be a non-trivial solution. Then repeated use of (1) shows that the (non-trivial) word $w = u_1^{l_0} \cdots u_{m+1}^{l_m}$ lies in the center, so that $A(\sigma)''$ is not a factor. □

COROLLARY 2.5. *Suppose $n = p^r$ where p is prime. Let $\{a_j : j \in \mathbf{Z}\}$ be a sequence of integers such that $a_0 = 0, a_{-j} = -a_j$ and $\sigma : \mathbf{Z} \to \Omega_n$ the function defined by $\sigma(j) = \exp(2\pi i a_j/p^r)$. Then $A(\sigma)''$ is the hyperfinite* II$_1$ *factor if and only if $(\ldots, a_{-1}, a_0, a_1, \ldots)$ is an aperiodic sequence over $\mathbf{Z}/p\mathbf{Z}$.*

Proof. Suppose $A(\sigma)''$ has non-trivial center. Then there is a non-trivial word w in the center. Since $w^{p^r} = \lambda 1$, some $\lambda \in \mathbf{C}$, we may assume by replacing w with an appropriate power w^{p^k} if necessary, that w is a non-trivial word such that $w^p = \lambda 1$. As in the proof of the theorem we may assume further that w has minimal length among all such central words and that

$$w = u_1^{k_0} \cdots u_{m+1}^{k_m},$$

where $k_0 \neq 0 \bmod p^r$. Moreover, since w^p is a scalar it follows from (1) that p^{r-1} divides k_j, for all j.

We have $u_j w = w u_j$ for all $j \in \mathbf{N}$. Calculating as in the preceding proof one derives the system

$$k_0 a_0 + k_1 a_1 + \cdots + k_m a_m = 0 \ (p^r)$$
$$-k_0 a_1 + k_1 a_0 + \cdots + k_m a_{m-1} = 0 \ (p^r)$$
$$\vdots$$

Let $l_j = k_j / p^{r-1}$, then we obtain the same system as in (3), where $L = [l_0, \ldots, l_m]^T$. Hence the sequence $(\ldots, a_{-1}, a_0, a_1, \ldots)$ is periodic over $\mathbf{Z}/p\mathbf{Z}$, as before.

Conversely, if the sequence is periodic, with period m, we showed there is a non-trivial solution L to the system $AL = 0 \pmod p$. Let $k_j = l_j p^{r-1}$. Since $l_0 \neq 0\ (p), k_0 \neq 0\ (p^r)$ so that $K = [k_0, \ldots, k_m]^T$ is a non-trivial solution to the system $AK = 0 \pmod{p^r}$. It is then straightforward to show that the corresponding word $w = u_1^{k_0} \cdots u_{m+1}^{k_m}$ commutes with the $\{u_j\}$ so that w is central and $A(\sigma)''$ is not a factor. □

The corollary allows us to proceed to the general case. Let n have prime factorization $p_1^{r_1} \cdots p_s^{r_s}$. Let Ω_n be the nth roots of unity. Let

$$\phi\colon \mathbf{Z}/n\mathbf{Z} \to \mathbf{Z}/p_1^{r_1}\mathbf{Z} \oplus \cdots \oplus \mathbf{Z}/p_s^{r_s}\mathbf{Z}$$

be the isomorphism given by $k \to (kn_1P_1, \ldots, kn_sP_s)$ where $P_q = n/(p_q^{r_q})$ and $n_1, \ldots, n_s$ satisfy $\sum n_q P_q = 1$. We denote by $\phi(k)_q$ the qth entry of $\phi(k), \phi(k)_q \in \mathbf{Z}/p_q^{r_q}$.

As before, let $\{u_j\colon j \in \mathbf{N}\}$ be unitaries, each of order n, satisfying $u_i u_j = \sigma(i-j) u_j u_i$, for some function $\sigma\colon \mathbf{Z} \to \Omega_n$ satisfying $\sigma(0) = 1$ and $\sigma(j)^{-1} = \sigma(-j)$. For fixed $j \in \mathbf{N}$ and $q \in \{1, 2, \ldots, s\}$ set $u_{jq} = u_j^{n_q P_q}$. The following properties are easily verified:

$$u_j = \prod_{q=1}^{s} u_{jq}, \tag{5.1}$$

$$\alpha(u_{jq}) = u_{j+1,q}, \qquad j \in \mathbf{N}. \tag{5.2}$$

Also, using (1) we have the properties

$$u_{iq} u_{jq'} = u_{jq'} u_{iq} \qquad \text{if } q \neq q', \tag{5.3}$$

$$u_{iq} u_{jq} = \sigma(i-j)^{(n_q P_q)^2} u_{jq} u_{iq}. \tag{5.4}$$

Let $A(\sigma)_q$, $1 \leq q \leq s$ be the subalgebra of $A(\sigma)$ generated by the $\{u_{jq}\colon j \in \mathbf{N}\}$.

THEOREM 2.6. *$A(\sigma)''$ is a factor if and only if $A(\sigma)''_q$ is a factor, for each q.*

Proof. Suppose $A \in A(\sigma)'_{q_0} \cap A(\sigma)''_{q_0}$. Then $A \in A(\sigma)'_q$ for all $q \neq q_0$, by (5.3). Hence $A \in A(\sigma)' \cap A(\sigma)''$ since the algebras $A(\sigma)_q$ generate $A(\sigma)$. So if A is non-trivial, $A(\sigma)''$ cannot be a factor.

Conversely, suppose $A(\sigma)''$ is not a factor. Then there is a non-trivial word $w = u_1^{l_1} \cdots u_m^{l_m}$ in $A(\sigma)$, by Lemma 2.1. Using (1) and (5) there is a λ of modulus 1 such that

$$w = \lambda \prod_{q=1}^{s} \left(\prod_{j=1}^{m} u_{jq}^{l_j} \right).$$

Choose q_0 such that $w_{q_0} = \prod_{j=1}^{m} u_{jq_0}^{l_j}$ is non-trivial. Since $u_{kq} w = w u_{kq}$ for all $k \in \mathbf{N}$, $q \neq q_0$, it follows from (5.3) that $u_{kq_0} w_{q_0} = w_{q_0} u_{kq_0}$. Hence w_{q_0} is central in $A(\sigma)_{q_0}$ and $A(\sigma)''_{q_0}$ is not a factor. □

REMARK. It is straightforward to show that if each $A(\sigma)''_q$ is a factor then $A(\sigma) \cong \otimes_q A(\sigma)_q$. We omit the proof since we do not require this result.

THEOREM 2.7. *Let $\{k_j : j \in \mathbf{Z}\}$ be a sequence in $\mathbf{Z}/n\mathbf{Z}$ such that $k_{-j} = -k_j$ and $\sigma : \mathbf{Z} \to \Omega_n$ the function given by $\sigma(j) = \exp(2\pi i k_j/n)$. Let $\phi : \mathbf{Z}/n\mathbf{Z} \to \mathbf{Z}/p_1^{r_1}\mathbf{Z} \oplus \cdots \oplus \mathbf{Z}/p_s^{r_s}\mathbf{Z}$ be the mapping defined above. Then $A(\sigma)''$ is a factor if and only if, for each q, $1 \leq q \leq s$, the sequence*

$$(\ldots, \phi(k_{-2})_q, \phi(k_{-1})_q, \phi(k_0)_q, \phi(k_1)_q, \phi(k_2)_q, \ldots)$$

is aperiodic over $\mathbf{Z}/p_q\mathbf{Z}$.

Proof. We have, for fixed q,

$$\begin{aligned} u_{1q} u_{j+1,q} &= u_1^{n_q P_q} u_{j+1}^{n_q P_q} = \sigma(j)^{(n_q P_q)^2} u_{j+1}^{n_q P_q} u_1^{n_q P_q} \\ &= \sigma(j)^{(n_q P_q)^2} u_{j+1,q} u_{1q} = \exp(2\pi i k_j/n)^{(n_q P_q)^2} u_{j+1,q} u_{1q} \\ &= \left[\prod_c \exp(2\pi i [k_j n_c/(p_c^r c)]) \right]^{(n_q P_q)^2} u_{j+1,q} u_{1q} \\ &= \exp(2\pi i n_q k_j/(p_q^{r_q}))^{(n_q P_q)^2} u_{j+1,q} u_{1q} \\ &= \exp(2\pi i \phi(k_j)_q/(p_q^{r_q}))^{n_q^2 P_q} u_{j+1,q} u_{1q}. \end{aligned}$$

By Theorem 2.5, therefore, the von Neumann algebra $A(\sigma)''_q$ is a factor if and only if the sequence $(\ldots, a_{-2}, a_{-1}, a_0, a_1, a_2, \ldots)$ is aperiodic mod p_q, where $a_j = \phi(k_j)_q(n_q^2 P_q)$. But $n_q^2 P_q$ is relatively prime to p_q, so the sequence above is aperiodic over $\mathbf{Z}/p_q\mathbf{Z}$ if and only if $(\dots, \phi(k_{-1})_q, \phi(k_0)_q, \phi(k_1)_q, \dots)$ is also. The preceding theorem now yields the result.

3. A conjugacy invariant for generalized shifts. In what follows we shall adhere to the following assumptions and notation. Let $n > 1$ be a fixed integer, and let $\sigma: \mathbf{N} \cup \{0\} \to \Omega_n$ be a mapping such that under the trace tr, the algebra $A(\sigma)$ generated by the words u_j, $j \in \mathbf{N}$, has weak closure $A(\sigma)''$ isomorphic to R, the hyperfinite II_1 factor. As before, α is the shift on R determined by the conditions $\alpha(u_i) = u_{i+1}$.

The following result justifies the terminology *shift of index n*.

THEOREM 3.1. *The subfactor $\alpha(R)$ of R has index $[R: \alpha(R)] = n$.*

Proof. For $i = 0, 1, \dots, n-1$, let V_i be the subspace $V_i = \overline{\alpha(R)u_1^i}$ in $L^2(R, \mathrm{tr})$. Then the V_i span $L^2(R, \mathrm{tr})$. Moreover, if w, w' are any words in $\alpha(R)$, we have $\mathrm{tr}([wu_1^i]^*[w'u_1^j]) = 0$ for $i \neq j$. Since $\alpha(R)$ is the strong closure of linear combinations of words we see that the V_i are orthogonal subspaces. The rest of the argument follows through exactly as in the proof of [2, Example 2.3.2]. □

THEOREM 3.2. *Let α be a shift on R constructed as above. Then $\alpha(R)' \cap R = \mathbf{C}1$.*

Proof. Let $\{w_i: i \in \mathbf{N}\}$ be a sequence of non-trivial words of $A(\sigma)$ such that $w_i^* w_j \neq \lambda 1$ for any $i \neq j$ and if w is a non-trivial word of $A(\sigma)$ then $w = \lambda w_i$ for some i and some λ of modulus 1.

Suppose $A \in \alpha(R)' \cap R$, then we have $A\delta_1 = a_0\delta_1 + \sum a_i \delta_{w_i}$, for some $a_i \in \mathbf{C}$, as in the discussion preceding Lemma 2.1. Then for $w \in \alpha(R)$,

$$a_0\delta_w + \sum a_i \delta_{w_i w} = Aw\delta_1 = wA\delta_1 = a_0\delta_w + \sum a_i \delta_{w w_i}.$$

Since δ_1 is separating for R there are non-trivial words in $\alpha(R)' \cap R$ if A is non-trivial.

Assuming $\alpha(R)' \cap R$ is non-trivial, and arguing as in Corollary 2.5, there exists a non-trivial word $w \in \alpha(R)' \cap R$ such that $w^p = \lambda 1$ for some prime p dividing $[R: \alpha(R)]$. Since $\alpha(R)$ is a factor, $w \notin \alpha(R)$, so w has the form $u_1^{k_0} u_2^{k_1} \cdots u_{m+1}^{k_m}$ with $k_0 \neq 0 \bmod n$. Moreover, we may assume that $m + 1$ is the minimal length among all words w in $\alpha(R)' \cap R$ such that w^p is a scalar multiple of 1.

Since $w^p = \lambda 1$ it follows from (1), then, that n/p divides each k_j. Hence w lies in the subalgebra A of $A(\sigma)$ generated by $u_1^{(n/p^r)}$ and its shifts, where p^r is the largest power of p dividing n. By Theorem 2.6, A'' is a subfactor of $A(\sigma)''$, and by hypothesis, $w \in \alpha(A)' \cap A''$. Set $v_1 = u_1^{(n/p^r)}$, and $v_{j+1} = \alpha^j(v_1)$. From the preceding paragraph, we have $w = v_1^{q_0} \cdots v_{m+1}^{q_m}$, where $q_j = k_j p^r/n$. Let $\sigma' \colon \mathbf{N} \cup \{0\} \to \Omega_{p^r}$ be the function satisfying $v_i v_j = \sigma'(|i-j|) v_j v_i$, and let $\{a_j \colon j \in \mathbf{N} \cup \{0\}\}$ be integers such that

$$\sigma'(j) = \exp(2\pi i a_j/p^r).$$

Since A'' is a factor, the sequence $(\dots, -a_2, -a_1, a_0, a_1, a_2, \dots)$ is aperiodic mod p, by Corollary 2.5.

From $v_1 w \neq w v_1, v_j w = w v_j, j \geq 2$, we obtain, as in Corollary 2.5, the following system of equations over $\mathbf{Z}/p^r\mathbf{Z}$:

$$\begin{aligned} q_0 a_0 + q_1 a_1 + \cdots + q_m a_m \quad &\neq 0 \ (p^r) \\ -q_0 a_1 + q_1 a_0 + \cdots + q_m a_{m-1} &= 0 \ (p^r) \\ -q_0 a_2 - q_1 a_1 + \cdots + q_m a_{m-2} &= 0 \ (p^r) \\ &\vdots \end{aligned}$$

Since p^{r-1} divides each q_j we obtain the system

$$\begin{aligned} l_0 a_0 + l_1 a_1 + \cdots + l_m a_m \quad &\neq 0 \ (p) \\ -l_0 a_1 + l_1 a_0 + \cdots + l_m a_{m-1} &= 0 \ (p) \\ &\vdots \end{aligned} \tag{6}$$

by setting $l_j = q_j/p^{r-1}$.

Define a new sequence $z_1, \dots$ of unitaries of order p satisfying $z_i z_j = \sigma''(|i-j|) z_j z_i$, where $\sigma''(j) = \exp(2\pi i a_j/p)$. From Corollary 2.5 the z_j generate a factor M under the usual trace representation, with shift β satisfying $\beta(z_i) = z_{i+1}$ and $[M \colon \beta(M)] = p$. By [1, Theorem 3.7] $\beta(M)' \cap M$ is trivial. But (6) implies that $z_1^{l_0} \cdots z_{m+1}^{l_m}$ lies in $\beta(M)' \cap M$, a contradiction. Hence (6) cannot hold, and $\alpha(R)' \cap R$ is trivial. □

DEFINITION 3.3. Let α, β be shifts on R. Then α and β are conjugate if there is a $\gamma \in \operatorname{Aut}(R)$ such that $\alpha = \gamma \cdot \beta \cdot \gamma^{-1}$.

The preceding definition appears in [3], where it is shown, [3, Theorem 3.6], that for shifts of index 2 the corresponding functions $\sigma = \sigma_\alpha \colon \mathbf{N} \cup \{0\} \to \{-1, 1\}$ are a complete conjugacy invariant (cf. also [1]). Using techniques essentially the same as Powers' we prove an analogue for more general shifts.

We need the following definition.

DEFINITION 3.4. Let α be a shift of index n of R. The normalizer $N(\alpha)$ is the subset of unitary elements V of R such that $V\alpha^k(R)V^* = \alpha^k(R)$ for all k.

THEOREM 3.5. *A unitary $V \in R$ lies in $N(\alpha)$ if and only if V is a scalar multiple of a word in $A(\sigma)$.*

Proof. It is obvious that words lie in $N(\alpha)$. Suppose $V \in N(\alpha)$. Let $\theta \in \mathrm{Aut}(R)$ be defined by $\theta(u_1) = \zeta u_1$, where $\zeta = \exp(2\pi i/n)$, and $\theta(u_j) = u_j$ for $j > 1$ (see [**1**, Corollary 3.8]). It is straightforward to show that $\alpha(R)$ is the fixed point algebra of θ. We show that $\theta(V) = \zeta^k V$ for some k.

Let $W \in \alpha(R)$, then $V^*WV \in \alpha(R)$, so $V^*WV = \theta(V^*WV) = \theta(V^*)W\theta(V)$. Hence $V\theta(V^*) \in \alpha(R)' \cap R$. Therefore $V = \lambda\theta(V)$, by the preceding theorem. Since $\theta^n = \mathrm{id}$, $V = \theta^n(V) = \lambda\theta^{n-1}(V) = \cdots = \lambda^n V$, so λ is an nth root of unity, i.e., $\theta(V) = \zeta^{k_1} V$ for some k_1.

Let $Z_1 = u_1^{-k_1} V$, then $\theta(Z_1) = Z_1$, so $Z_1 \in \alpha(R)$, and there is a $V_1 \in R$ such that $\alpha(V_1) = Z_1$. Hence $V = u_1^{k_1}\alpha(V_1)$. Also $V_1 \in N(\alpha)$, so that for some k_2, $\theta(V_1) = \zeta^{k_2} V_1$. Hence $Z_2 = u_1^{-k_2} V_1$ lies in $\alpha(R)$. There is then a V_2 in R such that $\alpha(V_2) = Z_2$, and therefore,

$$V = u_1^{k_1} Z_1 = u_1^{k_1}\alpha(V_1) = u_1^{k_1}\alpha(u_1^{k_2} Z_2) = u_1^{k_1} u_2^{k_2}\alpha^2(V_2).$$

Continuing in this fashion we find that for any m there are constants k_j and a unitary V_{m+1} such that

$$V = u_1^{k_1} u_2^{k_2} \cdots u_m^{k_m}\alpha^{m+1}(V_{m+1}).$$

Let $s = \sup\{m\colon k_m \neq 0 \bmod n\}$. We shall show that s is finite.

To do so, we make the following observation (cf. [**3**, Lemma 3.3]). If w is a non-trivial word generated by $u_1, \dots, u_q$ and w' is any word in R, then $\mathrm{tr}(w\alpha^l(w')) = 0$, for $l \geq q$. Since any $A \in R$ is a strong limit of linear combinations of words in R then $\mathrm{tr}(w\alpha^l(A)) = 0$, for $l \geq q$.

Given $\varepsilon > 0$ there is a $q \in \mathbf{N}$ and words w_i in the algebra generated by $u_1, \dots, u_q$ such that $\|V - V_0\|_2 < \varepsilon$, where $V_0 = \sum_{i=1}^c a_i w_i$. Let $m > q$ be an integer such that $k_m \neq 0 \bmod n$, then

$$\begin{aligned}\varepsilon &> |\mathrm{tr}(V^*[V - V_0])| \\ &= |1 - \mathrm{tr}(\alpha^{m+1}(V_{m+1}^*)u_m^{-k_m} \cdots u_1^{-k_1} V_0)| = 1,\end{aligned}$$

a contradiction if $\varepsilon < 1$. This yields the result.

Using the preceding characterization of the elements of $N(\alpha)$, we may obtain the following results on the conjugacy classes of shifts of prime index.

COROLLARY 3.6. *Let α be a shift of prime index p constructed as above. Let u, v be α-generators of R. Then $u = \mu v^k$ for some k relatively prime to p, and some μ in Ω_p.*

Proof. Since u and v are α-generators, and since each is an element of $N(\alpha)$, then by Theorem 3.5, $u = \mu v^{k_0}\alpha(u^{k_1})\cdots\alpha^m(v^{k_m})$, and $v = \nu u^{t_0}\alpha(u^{t_1})\cdots\alpha^m(u^{t_m})$, for some $m \in \mathbf{N}, \mu, \nu \in \Omega_p$, and integers $t_j, k_j, j = 1, 2, \dots, m$. Substituting the latter expression for v into the first equation, we obtain $u = \zeta u^{q_0}\alpha(u^{q_1})\cdots\alpha^{2m}(u)^{q_{2m}}$, for some $\zeta \in \Omega_p$, where $q_j = k_j t_0 + k_{j-1}t_1 + \cdots + k_0 t_j$ modulo (p). An argument similar to the proof of [3, Theorem 3.4] shows that $q_j = 0$ modulo (p), for $j > 1$. If t_r is the last non-zero exponent in the expression for v, then starting with the expression for q_{m+r} and working backwards to q_{r+1}, one observes successively that $k_m = k_{m-1} = \cdots = k_1 = 0$. Hence $u = \mu v^{k_0}$. □

REMARK. The result above does not hold for shifts of general index. Taking $n = 4$, for example, one checks that if u is an α-generator, then so is $v = u\alpha(u^2)$, since $u = \mu v\alpha(v^2)$, some $\mu \in \Omega_4$.

We omit the proof of the following result, which is virtually identical to the proof of [3, Theorem 3.6].

COROLLARY 3.7. *Let α, β be shifts of prime index p on R, constructed as above. Then α and β are conjugate if and only if they correspond to the same σ-function $\sigma\colon \mathbf{N} \cup \{0\} \to \Omega_p$.*

COROLLARY 3.8. *There are an uncountable number of non-conjugate shifts of R of prime index p constructed as above.*

Proof. This follows immediately since there are uncountably many functions σ satisfying the statement of Theorem 2.7. □

In [3] Powers introduced the notion of outer conjugacy for shifts. We say that shifts α and β are outer conjugate if there are a $\gamma \in \mathrm{Aut}(R)$ and a unitary $U \in R$ such that $\alpha \in \mathrm{Ad}(U) = \gamma \cdot \beta \cdot \gamma^{-1}$. The index of a shift is an outer conjugacy invariant, and so is the first positive

m ($m \in \{2, 3, \dots\} \cup \{\infty\}$, by Theorem 3.2) such that $\alpha^m(R)$ has nontrivial relative commutant. It is not known if this condition is also sufficient, even in the case of shifts of index 2 (cf. [**3**]).

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