PACIFIC JOURNAL OF MATHEMATICS
Vol. 132, No. 2, 1988

ON DEFORMING G-MAPS TO BE FIXED POINT FREE

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When $f: M \to M$ is a self-map of a compact manifold and $\dim M \geq 3$, a classical theorem of Wecken states that f is homotopic to a fixed point free map if, and only if, the Nielsen number $n(f)$ of f is zero. When M is simply connected, and $\dim M \geq 3$ the NASC becomes $L(f) = 0$, where $L(f)$ is the Lefschetz number of f. An equivariant version of the latter result for G-maps $f: M \to M$, where M is a compact G-manifold, is due to D. Wilczyński, under the assumption that M^H is simply connected of dimension ≥ 3 for any isotropy subgroup H with finite Weyl group WH. Under these assumptions, f is G-homotopic to a fixed point free map if, and only if, $L(f^H) = 0$ for any isotropy subgroup H (WH finite), where $f^H = f|M^H$ and M^H represents those elements of M fixed by H. A special case of this result was also obtained independently by A. Vidal via equivariant obstruction theory. In this note we prove the analogous equivariant result without assuming that the M^H are simply connected, assuming that $n(f^H) = 0$, for all H with WH finite. There is also a codimension condition. Here is the main result.

THEOREM. *Let G denote a compact Lie group and M a compact, smooth G-manifold. Let $(H_1), \dots, (H_k)$ denote an admissible ordering of the isotropy types of M, $M_i = \{x \in M: (G_x) = (H_j),\ j \leq i\}$ the associated filtration. Also, let $\mathcal{F}$ denote the set of integers i, $1 \leq i \leq k$, such that the Weyl group $WH_i = NH_i/H_i$ is finite. Suppose that for each $i \in \mathcal{F}$, $\dim M^{H_i} \geq 3$ and the codimension of $M_{i-1} \cap M^{H_i}$ in M^{H_i} is at least 2. Then, a G-map $f: M \to M$ is G-homotopic to a fixed point free G-map $f': M \to M$ if, and only if, the Nielsen number $n(f^{H_i}) = 0$ for each $i \in \mathcal{F}$.*

1. Preliminaries. Throughout this note G will denote a compact Lie group and M will denote a compact, smooth G-manifold. For any closed subgroup H in G, we denote by NH the normalizer of H in G and by $WH = NH/H$, the Weyl group of H in G. The conjugacy class of H, denoted by (H), is called the orbit type of H. If $x \in M$ then G_x denotes the isotropy subgroup of x, i.e. $G_x = \{g \in G | gx = x\}$. For each subgroup H of G, $M^H = \{x \in M | hx = x \text{ for all } h \in H\}$ and $M_H = \{x \in M | G_x = H\}$. Let $\{(H_j)\}$ denote the (finite) set of isotropy types of M. If (H_j) is subconjugate to (H_i), we write $(H_j) \leq (H_i)$. We can choose an *admissible* ordering on $\{(H_j)\}$ so that $(H_j) \leq (H_i)$ implies $i \leq j$ (see **[1]**). Then we have a filtration of G-subspaces $M_1 \subset M_2 \subset \cdots \subset M_k = M$, where $M_i = \{x \in M \colon (G_x) = (H_j), j \leq i\}$.

We now recall the definition of the *local* Nielsen number. If U is an open subset of M and $f \colon U \to M$ is a compactly fixed map (not necessarily a G-map), then the Nielsen number $n(f, U)$ is defined **[3]** using the equivalence relation on the fixed point set $\operatorname{Fix} f$ as follows. Two fixed points x and y in U are equivalent if there is a path α *in* U such that $f(\alpha)$ and α are endpoint homotopic in M. The remainder of the local theory proceeds along the lines of the global theory. Note that if $\dim M \geq 3$, the path α above may be taken as a simple path in U and assuming (as we may) that $\operatorname{Fix} f$ is finite, α may be chosen to avoid all fixed points different from x and y. Then in a small closed tubular nieghborhood $T \subset U$ of α f may be altered in the interior T^0 of T (via the Wecken method **[1]** or the "Whitney trick" see **[4]**) to obtain a map $f' \colon U \to M$ so that f' is an extension of $f|U - T^0$, $f' \sim f$ and f' has only one fixed point in U. This is the technique of coalescing fixed points. Note that T is a closed n-ball. If this remaining fixed point has *index* 0, f' may be altered within T^0 to remove it, thus obtaining $f'' \sim f$ such that $f''|U - T^0 = f|U - T^0$ and f'' has no fixed points in T. A key point here is that f is altered in the interior of a closed *contractible* neighborhood of α.

2. The Proof of Main Theorem. We first note that the G-map $f \colon M \to M$ preserves the filtration $M_1 \subset \cdots \subset M_k$, i.e., $f(M_i) \subset M_i$. Also, $W_i = WH_i$ acts on M^{H_i} and freely on $M^{H_i} - M_{i-1}$. Furthermore $f^{H_i} = f|M^{H_i} \colon M^{H_i} \to M^{H_i}$ is a W_i-map. We will set $f_i = f|M_i$. We then let $\mathscr{F}$ denote the indices i, $1 \leq i \leq k$, such that W_i is finite. Whenever A is a G-set, $\tilde{A}$ will denote the corresponding set of orbits, i.e., $\tilde{A} = A/G$. Similarly $\tilde{f} \colon \tilde{A} \to \tilde{B}$, will denote the map induced by a G-map $f \colon A \to B$. Finally, $\operatorname{Fix} f$ is a G-set, and each orbit in $\operatorname{Fix} f$ will be referred to as a *fixed orbit*.

2.1. Lemma. (*Controlled Homotopy Extension*). *Let (X, A) denote a G-pair such that all orbits in $X - A$ have the same orbit type G/H and $f \colon (X, A) \to (X, A)$ a G-map (of pairs). Let V denote a closed G-neighborhood of A and $f^H \colon X^H \to X^H$ the restriction of f to X^H. Then, any (NH/H)-homotopy f_t^H, relative to V^H, with $f_0^H = f^H$ extends to a G-homotopy f_t, relative to V, with $f_0 = f$. Furthermore if f_1^H is fixed point free on $(X - A)^H$, then so is f_1 on $X - A$.*

Proof. The proof is an easy consequence of II.5.12 in Bredon [**1**]. Let $W = NH/H$ and $Y = X - A$. The homotopy f_t is defined on Y by setting

$$f_t(y) = gf_t^H(g^{-1}y), \qquad G_y = gHg^{-1}.$$

Then, f_t is defined on Y and on $V \cap Y$, $f_t = f_0$, $0 \le t \le 1$. Thus, setting $f_t = f$ on V extends f_t to all of X. Note that

$$gf_1^H(g^{-1}y) = y \Leftrightarrow f_1^H(g^{-1}y) = g^{-1}y, \qquad y \in Y,$$

which verifies the last assertion of the lemma.

2.2. PROPOSITION (*Inductive Step when W_i is finite*). *Let $f\colon M \to M$ denote a G-map such that*

(1) *$f_{i-1}\colon M_{i-1} \to M_{i-1}$ is fixed point free,*
(2) *$W_i = NH_i/H_i$ is finite,*
(3) *f_i has a finite number of fixed orbits on $U_i = M_i - M_{i-1}$,*
(4) *$n(f_i^H, U_i^H) = 0$.*

Then, f is G-homotopic to $f'\colon M \to M$, relative to M_{i-1} so that f_i' is fixed point free.

Proof. Consider the map $f_i\colon M_i \to M_i$ and let $H = H_i$ and $W = W_i$ for notational convenience. Now focus attention on $f_i^H\colon M_i^H \to M_i^H$ and let $\mathcal{O}_1$ and $\mathcal{O}_2$ denote two fixed W-orbits in U_i^H. Call the fixed *orbits $\mathcal{O}_1$ and $\mathcal{O}_2$ Nielsen equivalent* if for some $x \in \mathcal{O}_1$ and $y \in \mathcal{O}_2$, x and y are Nielsen equivalent in U_i^H (see [**2**]). We will coalesce two Nielsen equivalent orbits into one fixed orbit as follows. Suppose $\mathcal{O}_1 = Wx$ and $\mathcal{O}_2 = Wy$ with x and y Nielsen equivalent in U_i^H, i.e., there is a path α from x to y in U_i^H so that $f\alpha \sim \alpha$ (in M_i^H and with ends fixed). Because of assumption (3) we may assume that α avoids all other points of $\operatorname{Fix} f_i$ other than x and y. Project α to $\tilde{\alpha}$ in U_i^H/W by the orbit map $\eta\colon U_i^H \to U_i^H/W$ and let $\tilde{\beta}$ denote a *simple* path homotopic (relative to end points) to $\tilde{\alpha}$. Then $\tilde{\beta}$ lifts to a simple path β from x to y. If $N(\tilde{\beta})$ is a closed ball neighborhood of $\tilde{\beta}$ in U_i^H/W, then, since $N(\tilde{\beta})$ is contractible, $\eta^{-1}(N(\tilde{\beta})) = WN(\beta)$, where $N(\beta)$ is the corresponding ball neighborhood of β. Thus, $\eta^{-1}(N(\tilde{\beta}))$ consists of disjoint translates of $N(\beta)$ by W. The local Nielsen number $n(f_i^H, N(\beta))$ is at most one (see [**3**]). Applying the local Wecken theorem or the "Whitney trick" in $N(\beta)$ (see [**3**] or [**4**]), we can obtain a homotopy $H\colon N(\beta) \times I \to M_i^H$ such that $H_t|\partial N(\beta) = f_i$ for all t, $0 \le t \le 1, H_0 = f_i|N(\beta)$ and H_1 has at most one fixed point in the interior of $N(\beta)$. H has the extension $H(wx, t) = wH(x, t)$ to

$(WN(\beta)) \times I$ and to all of $M_i^H \times I$ by using f_i outside of $WN(\beta) \times I$. Then $H: M_i^H \times I \to M_i$ is a W-homotopy, with $H_1 = \varphi_i: M_i^H \to M_i^H$ having one less or two less fixed orbits. Continuing in this manner we obtain a W-map $\varphi_i': M_i^H \to M_i^H$, W-homotopic to f_i with finitely many fixed orbits no two of which are Nielsen equivalent. If x belongs to one of the remaining orbits Wx and D is a sufficiently small neighborhood of x, then the local indices $i(\varphi_i', D)$ and $i(\varphi_i', wD)$, $w \in W$, are the same and $i(\varphi_i', WD) = |W| i(\varphi_i', D)$. Since $n(f_i^H, U_i^H) = 0$, we must have $i(\varphi_i', D) = 0$, since Wx is the union of Nielsen classes. We can now remove x as a fixed point via a homotopy relative to ∂D and extend (as above) to a W-map $\varphi_i'': M_i^H \to M_i^H$, W-homotopic to φ_i', with Wx eliminated as a fixed orbit. Continuing in this manner we arrive at a W-map $\psi_i: M_i^H \to M_i^H$ which is fixed point free and W-homotopic to f_i relative to some closed neighborhood of M_{i-1}^H. By Lemma 2.1 this map ψ_i extends to a fixed point free G-map $f_i': M_i \to M_i$, G-homotopic to f_i (relative to M_{i-1}) and the G-homotopy extension theorem provides the required extension f' of f_i'.

2.3. PROPOSITION (*Inductive Step when* $\dim W_i > 0$). *Let* $f_i: M_i \to M_i$ *denote a G-map such that*

(1) f_i^H *is fixed point free on* M_{i-1}^H,

(2) $\dim W_i > 0$, $W_i = NH_i/H_i$.

Then, f_i *is G-homotopic relative to A to a G-map* $f_i': M_i \to M_i$ *such that* f_i' *is fixed point free.*

Proof. This proposition follows from Lemma 3.3 in [**6**].

Proof of Theorem. We assume (inductively) that $f: M \to M$ is a G-map such that $f_{i-1}: M_{i-1} \to M_{i-1}$ if fixed point free. As a first step, choose a closed G-neighborhood V of M_{i-1} in M_i so that M_{i-1} is a G-deformation retract of V. Then, f is G-homotopic, relative to M_{i-1}, to a map f' such that $f_i': M_i \to M_i$ has no fixed points in V_i. Thus, we may assume that f itself has this property so that f_i is compactly fixed on $U_i = M_i - M_{i-1}$. In particular f_i^H is compactly fixed on U_i^H and the local Nielsen number $n(f_i^H, U_i^H)$ is defined. We consider two cases.

Case 1. $W_i = NH_i/N_i$ is finite, i.e., $i \in \mathscr{F}$.

In this case, the codimension condition applies to yield $n(f_i^H, U_i^H) = n(f_i^H) = 0$. This is because any path in M_i^H from x to y, $x \cup y \in U_i^H$ may be deformed (ends fixed) to a path in U_i^H, i.e., to one avoiding the submanifold M_{i-1}^H. Let V' denote a closed G-neighborhood of M_{i-1}^H

in M_i^H so that the fixed points $\operatorname{Fix} f_i^H$ of f_i^H are in $M_i^H - V'$. Choose a closed G-neighborhood $Q \subset U_i^H$ of $\operatorname{Fix} f_i$ so that $f_i^H(Q) \subset U_i^H$. Working in the orbit space U_i^H/W_i, we deform $\tilde{f}_i|\tilde{Q}$, relative $\partial\tilde{Q}$, to a map $\tilde{f}_i'$ with finitely many fixed points. Since W_i acts freely on U_i^H we may apply the covering homotopy theorem to conclude that f_i^H is homotopic relative to V' to $\varphi\colon M_i^H \to M_i^H$ where φ has finitely many fixed W_i-orbits and the homotopy is compactly fixed. Thus, $n(\varphi, U_i^H) = 0$ and we may apply Proposition 2.2 to conclude that f is G-homotopic, relative to M_{i-1}, to a map $f'\colon M \to M$ with $f_i'\colon M_i \to M_i$ fixed point free.

Case 2. $\dim W_i > 0$. We apply Proposition 2.3 and then the G-homotopy extension theorem to conclude that f is G-homotopic, relative to M_{i-1}, to a map $f'\colon M \to M$ with $f_i'\colon M_i \to M_i$ fixed point free.

Applying induction completes the proof of the sufficiency. The necessity is clear.

References

[1] G. Bredon, *Introduction to Compact Transformation Groups*, Academic Press, New York 1972.

[2] R. F. Brown, *The Lefschetz Fixed Point Theorem*, Scott-Foresman 1971.

[3] E. Fadell and S. Husseini, *Local fixed point index theory for non-simply connected manifolds*, Illinois J. Math., **25** (1981), 673–699.

[4] Boju Jiang, *Fixed Point Classes From a Differential Viewpoint*, Lecture Notes #886, Springer-Verlag 1981, 163–170.

[5] A. Vidal, *Equivariant Obstruction Theory for G-Deformations* (in German), Dissertation 1985, Universität Heidelberg.

[6] D. Wilczyński, *Fixed point free equivariant homotopy classes*, Fund. Math., **123** (1984), 47–60.

Received April 6, 1987. The first author was supported in part by the National Science Foundation under Grant No. DMS-8320099.

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