GROUPS OF ISOMETRIES OF A TREE
AND THE KUNZE-STEIN PHENOMENON

Claudio Nebbia

In this paper we prove that every group of isometries of a homogeneous or semihomogeneous tree which acts transitively on the boundary of the tree is a Kunze-Stein group. From this, we deduce a weak Kunze-Stein property for groups acting simply transitively on a tree (in particular free groups on finitely many generators).

1. Introduction. Let $G$ be a locally compact group, then $G$ is said to satisfy the “Kunze-Stein property” or sometimes $G$ is called a “Kunze-Stein group” if $L^p(G) * L^2(G) \subset L^2(G)$ for every $1 < p < 2$.

This property was discovered by R. A. Kunze and E. M. Stein for the group $SL_2(R)$ [15]. Later the same property was proved for every connected semisimple Lie group with finite center by M. Cowling [6]. In this paper we prove that every locally compact group of isometries of a homogeneous or semihomogeneous tree has the Kunze-Stein property provided that $G$ acts transitively on the boundary of the tree. The proof of our Theorem is based on M. Cowling’s proof of the Kunze-Stein phenomenon for $SL_2(R)$ [6]. A weaker property is deduced for discrete groups acting simply transitively on the tree but not on the tree boundary.

It is known that the group $SL_2(\kappa)$, where $\kappa$ is a local field, may be realized as a closed subgroup of the group of all isometries of a homogeneous tree in such a way that $SL_2(\kappa)$ acts transitively on the boundary [17]. In particular our result implies that $SL_2(\kappa)$ is a Kunze-Stein group for every local field. This was proved by Gulizia [13] for a local field $\kappa$ such that the finite residue class field associated with $\kappa$ is not of characteristic 2.

We follow the terminology and definitions of [6]. In particular $A(G)$ is the Fourier algebra of $G$ as defined in [7]; $C_{00}(G)$ denotes the space of continuous functions with compact support and $L^p(G), 1 \leq p \leq \infty$, the usual $L^p$-space with respect to a fixed left Haar measure. As observed in [6], a locally compact group $G$ is a Kunze-Stein group if and only if $A(G) \subset L^q(G)$ for every $q > 2$. We will also use the theory of representations for groups acting on a tree developed by P. Cartier.
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2. Notations. We shall give a concise description of the tree and of the group of isometries. We refer the reader to [3, 17, 18] for undefined notions and terminology. Let $X$ be a homogeneous tree of order $r$; the distance $d(x, y)$ is defined as the length of the unique geodesic $[x, y]$ connecting $x$ to $y$. Let $Aut(X)$ be the group of all isometries of $X$. We assume also $r \geq 3$ (otherwise, for $r = 2$, $Aut(X)$ is amenable and noncompact, hence it is not a Kunze-Stein group). $Aut(X)$ is a locally compact separable group and the stability subgroup $K$ of a vertex of $X$ is compact and open in $Aut(X)$. A subgroup $\Gamma$ of $Aut(X)$ is called simply transitive if it acts transitively on the vertices and $\Gamma \cap K = \{1\}$. In other words, $\Gamma$ acts simply transitively on $X$ iff the map $\gamma \in \Gamma \to \gamma(x_0) \in X$ is bijective for a fixed vertex $x_0$ in $X$. It is known that every such group is isomorphic to the free product of $t$ copies of the integers and $s$ copies of the group of order 2 with $2t + s = r$ [1, 4]. Since $K$ is open, $\Gamma$ is discrete in $Aut(X)$. Moreover $\Gamma \cdot K = Aut(X)$ and $\Gamma$ is a lattice. As usual, let $\langle f, h \rangle = \int f(g)h(g)\,dg$.

Let $\Omega$ be the boundary of the tree, that is the set of equivalence classes of sequences of distinct vertices $\{s_n: n = 0, 1, 2, \ldots\}$ such that $d(s_i, s_{i+1}) = 1$ for every $i = 0, 1, 2, \ldots$; two such sequences are said to be equivalent if they have infinitely many common vertices.

$\Omega$ is a compact metric space; if $x_0 \in X$ and $\omega_0 \in \Omega$ there exists a unique sequence of distinct vertices $\{s_n\}$ in the class $\omega_0$ such that $s_0 = x_0$. In this way, $\Omega$ can be regarded as the set of infinite sequences starting from a fixed vertex $x_0$ in $X$. There exists a unique probability measure $\nu$ on $\Omega, Aut(X)$-quasi invariant and $K$-invariant. Let $P(g, \omega)$ be the Poisson kernel, that is, for $g \in Aut(X)$ and $\omega \in \Omega$, $P(g, \omega) = d\nu_g/d\nu(\omega)$, with $\nu_g(\omega) = \nu(g^{-1}\omega)$.

For every complex number $z$, we define the following representation of $Aut(X)$:

$$[\pi_z(g)f](\omega) = P^z(g, \omega)f(g^{-1}\omega).$$
It is known that, for $t \in \mathbb{R}$, $\pi_{1/2+it}$ are unitary irreducible representations on $L^2(\Omega)$; in fact even the restrictions to $\Gamma$ are irreducible [12, pg. 76; 1].

For a fixed vertex $x_0$ in $X$, let $X^+ = \{x \in X : d(x, x_0) \text{ is even}\}$ and $X^- = X \setminus X^+$. The partition $X^+, X^-$ is independent of the choice of $x_0$. If $G$ is a closed unimodular subgroup of Aut($X$) acting transitively on $X^+$ but not on the tree, then the representations $\pi_{1/2+it}|_G$ are irreducible for $t \neq (2m + 1)\pi/2\lg(r - 1)$, $m \in \mathbb{Z}$ [2, pg. 39, pg. 62]. Let $J$ be the interval $[0, \pi/\lg(r - 1)]$ and $c(z)$ the following complex function:

$$c(z) = [(r - 1)^{2-2z} - 1]/[(r - 1)^{1-2z} - 1].$$

Finally, let $dm$ be the following measure:

$$dm(t) = [(r - 1)\lg(r - 1)/4\pi r |c(\frac{1}{2} + it)|^2] dt.$$

3. The results. Let $G$ be a closed noncompact subgroup of Aut($X$) acting transitively on $\Omega$, and $K_0 = G \cap K$. Since $K_0$ is compact open in $G$ we can assume that its measure is one.

PROPOSITION 1. $K_0$ acts transitively on $\Omega$.

Proof. Since $G/K_0$ is countable, Baire's theorem implies that every orbit of $K_0$ on $\Omega$ is open. By [17, Prop. 3.4], there exist $g \in G$, a sequence $\{s_n\} \subset X$, $n \in \mathbb{Z}$ and $i_0 \in \mathbb{Z}$ $i_0 \neq 0$, such that $d(s_n, s_{n+1}) = 1$ and $g(s_n) = s_{n+i_0}$ for every $n \in \mathbb{Z}$. In this proof we realize $\Omega$ as the set of all infinite sequences $\{t_n\}$ issued from $t_0 = s_0$. Therefore the sets: $E(x) = \{\{t_n\} \in \Omega : t_j = x\}$ with $x \in X$ and $d(s_0, x) = j$ form a basis for the topology of $\Omega$. Let $\omega_1 = \{s_0, s_1, \ldots\}$ and $\omega_2 = \{s_0, s_{-1}, s_{-2}, \ldots\}$.

Since $K_0\omega_1$ and $K_0\omega_2$ are open, it follows that there exists $j > 0$ such that $E(s_j) \subset K_0\omega_1$ and $E(s_{-j}) \subset K_0\omega_2$. Using the automorphism $g$, it is not hard to show that $K_0$ acts transitively on $\mathcal{C}E(s_{-1})$ and $\mathcal{C}E(s_1)$, respectively. Obviously, $\mathcal{C}E(s_{-1}) \cap \mathcal{C}E(s_1) \neq \emptyset$ and $\mathcal{C}E(s_{-1}) \cup \mathcal{C}E(s_1) = \Omega$. This means that $K_0$ acts transitively on $\Omega$.

PROPOSITION 2. Let $G$ be a closed noncompact subgroup of Aut($X$) acting transitively on $\Omega$. Then either $G$ acts transitively on the vertices of $X$, or $G$ has two orbits $X^+$ and $X^-$. 

Proof. By Proposition 1, $K_0$ acts transitively on $\Omega$, that is, $K_0$ acts transitively on the set $S_0^n = \{y \in X : d(s_0, y) = n\}$ for every $n \geq 0$. Moreover for every $g \in G$, $gK_0g^{-1}$ acts transitively on $S_n^x$ for every $n \geq 0$ and $g(s_0) = x$. In particular for every $x \in G(s_0)$, $G(s_0)$ is an
infinite union of sets $S^X_m$. This implies that if $x, y \in G(s_0)$ then $S^X_m \cup S^X_m \subset G(s_0)$. Therefore $S^X_m \subset G(s_0)$ implies that:

$$
\bigcup_{j=0}^{+\infty} S^X_{jm} \subset G(s_0).
$$

If $G(s_0)$ contains vertices $x$ and $y$ with $d(x, y) = 1$, then $G(s_0) = X$ and $G$ is transitive on $X$. Suppose now $G(s_0) \neq X$; thus $G(s_0) \subset X^+$. Let $t = \min\{m > 0 : S^X_m \subset G(s_0)\}$. It follows that $G(s_0) \cap S^X_m = \emptyset$ for $0 < m < 2t m \neq t$ and $\bigcup_{j=0}^{+\infty} S^X_{jm} \subset G(s_0)$. Let $x \in S^s_t$ and $[s_0, x] = \{s_0, x_1, x_2, \ldots, x_{t-1}, x\}$ the geodesic connecting $s_0$ to $x$; we can choose $y \in X$ in such a way that $d(y, x) = t$, $d(y, s_0) = 2t - 2$ and $[x, s_0] \cap [x, y] = [x, x_{t-1}] = \{x, x_{t-1}\}$. Since $d(x, y) = t$, $y \in G(s_0)$ but $y \in S^X_{2t-2}$ so that $S^X_{2t-2} \subset G(s_0)$. This implies that $2t - 2 = t$, that is, $t = 2$ and $G(s_0) = X^+$. Similarly, we can prove that $G(s_1) = X^-$, with $d(s_0, s_1) = 1$.

The aim of this note is to prove the following Theorem.

**Theorem 1.** Every closed subgroup $G$ of Aut$(X)$ acting transitively on $\Omega$ is a Kunze-Stein group.

It is enough to prove the Theorem for noncompact groups. First, we observe that:

$$
\int_{\Omega} \|\pi_{1/2+it}|G(u)\|_{HS}^2 dm(t) \leq \|u\|^2_2 
$$

for every $u \in C_{00}(G)$.

Indeed $(G, K_0)$ is a Gelfand pair because $K_0$ acts transitively on $\Omega$ and $g^{-1} \in K_0 g K_0$ for every $g \in G$ [9, Prop. 1.2]. The representations $\pi_{1/2+it}|G$ are irreducible iff $1$ (the function identically one on $\Omega$) is a cyclic vector. By Proposition 2, we have two possibilities: if $G$ is transitive on $X$, then the representations $\pi_{1/2+it}|G$ are irreducible for every $t \in \mathbb{J}$ [12, pg. 76; 1]; otherwise for $t \in \mathbb{J}$, $t \neq \pi/2lg(r - 1)$ [2, pg. 39, pg. 62].

Since, for Gelfand pairs, the Plancherel measure on the irreducible unitary representations of $G$ having a $K_0$-fixed vector depends only on the right $K_0$-invariant functions [9, Th. 4.2; 16, pg. 65], to prove the inequality, it is enough to prove that

$$
\int_{\Omega} \|\pi_{1/2+it}|G(u)\|_{HS}^2 dm(t) = \|u\|^2_2
$$

for every right $K_0$-invariant function $u$ in $C_{00}(G)$. To show this, let $T$ be the following projection on $L^2(\Omega)$: $Tf = \int_{\Omega} f(\omega) dv(\omega)]1$, for
We have \( T = \int_{K_0} \pi_{1/2+it}(k) \, dk \) (recall that \( K_0 \) is transitive on \( \Omega \)). Let \( \text{Aut}(X) = \Gamma K \); every function \( u \) right \( K_0 \)-invariant on \( G \) corresponds to a function \( \tilde{u} \) on \( \Gamma \) in such a way that \( \|u\|_2 = \|\tilde{u}\|_2 \) and \( \pi_{1/2+it}|_G(u) = [\pi_{1/2+it}|_{\Gamma}(\tilde{u})]T \).

Therefore \( \|\pi_{1/2+it}|_G(u)\|_{HS} = \|\pi_{1/2+it}|_{\Gamma}(\tilde{u})\|_{L^2(\Omega)} \); hence the equality follows from \([12, \text{pg. 86; 1}]\). The proof of Theorem 1 is based on the following two Lemmas.

In the next Lemma, we denote by \( G \) a locally compact group and by \( L^\infty_1(G) \) the space of all functions \( f \) in \( L^\infty(G) \) such that \( \|f\|_\infty \leq 1 \); we assume \( \phi \) to be a complex continuous function on the strip \( S = [\alpha, \beta] \times \mathbb{R} \) with \( 0 < \alpha < \frac{1}{2} < \beta < 1 \), analytic on \( S^0 = (\alpha, \beta) \times \mathbb{R} \) and such that (1) \( \phi \) is bounded on \( S \); (2) \( |\phi(x + it)| \geq h(x) > 0 \) for every \( t \in \mathbb{R} \) and \( \alpha \leq x \leq \beta, x \neq \frac{1}{2} \). With these notations, we have:

**Lemma 1 (M. Cowling [6]).** Let \( F: S \to L^\infty_1(G) \) be a continuous map, analytic on \( S^0 \) (i.e. \( \langle F_z, u \rangle \) is an analytic function for every \( u \) in \( C_0(G) \)). If there exists a positive constant \( c \) such that

\[
\int_{\mathbb{R}} |\langle F_{1/2+it}, u \rangle|^2 |\phi(1/2 + it)| \, dt \leq c \|u\|^2_2 \quad \text{for every } u \text{ in } C_0(G),
\]

then the function \( F_{1/2} \) is in \( L^q(G) \) for every \( q > 2 \).

**Proof.** This Lemma is obtained from Lemma 2.1 of \([6, \text{pg. 215}]\) where \( S = [\alpha, \beta] \times \mathbb{R}, q = q' = 2, X = G \) and \( X_0 \) is a singleton, observing that the function \((z/z-2)^n\) could be replaced with a general analytic function \( \phi \) with the properties (1) and (2).

**Lemma 2.** The coefficients of the quasi-regular representation on \( \Omega \), that is the functions:

\[
\langle \pi_{1/2}(g) \xi, \eta \rangle = \int_{\Omega} P^{1/2}(g, \omega) \xi(g^{-1}(\omega)) \overline{\eta(\omega)} \, d\nu(\omega)
\]

for \( \xi, \eta \) in \( L^2(\Omega) \) and \( g \) in \( G \)

are in \( L^q(G) \) for every \( q > 2 \).

**Proof.** Since \( |\langle \pi_{1/2}(g) \xi, \eta \rangle| \leq \langle \pi_{1/2}(g) |\xi|, |\eta| \rangle \) it is enough to prove the Lemma for \( \xi \geq 0, \eta \geq 0 \) and \( \|\xi\|_2 = \|\eta\|_2 = 1 \). Define \( \xi_z = \xi^{2z} \) and \( \eta_z = \eta^{2z} \) for \( \xi(\omega) \neq 0 \neq \eta(\omega), \xi_z(\omega) = 0 \) for \( \xi(\omega) = 0 \); similarly \( \eta_z(\omega) = 0 \) for \( \eta(\omega) = 0 \). In particular \( \xi_{1/2} = \xi \) and \( \eta_{1/2} = \eta \). Let \( z = \delta + it \in S \) and \( p = 1/\delta > 1, q = p/(p - 1) = 1/(1 - \delta) \) the
conjugate index of \( p \); it is easy to see that:

1. \( \xi_z \in L^p(\Omega), \|\xi_z\|_p = 1. \)
2. \( \eta_z \in L^q(\Omega), \|\eta_z\|_q = 1. \)
3. \( \|\pi_z(g)u\|_p = \|u\|_p \) for every \( u \) in \( L^p(\Omega) \) and \( g \) in \( G \).

Let \( \psi(z) = \exp(z^2 - 1); |\psi(z)| \leq 1 \) on \( S \) and the map \( F_z = \psi(z)\pi_z(\cdot, \xi_z, \eta_z) \) is a continuous map on \( S \) into \( L^\infty_G \), analytic on \( S^0 \). Since \( F_{1/2} = \exp(-\frac{3}{4})\langle \pi_{1/2}(\cdot, \xi, \eta) \rangle \), to prove the Lemma, it suffices to show that:

\[
\int_\mathbb{R} |\langle F_{1/2+it}, u \rangle|^2 |\phi(\frac{1}{2} + it)| |dt| \leq c \|u\|_2^2 \quad \text{for every } u \in C_0(\mathbb{G})
\]

and some analytic function \( \phi \).

Let \( \phi(z) = \frac{(r-1)lg(r-1)}{[4\pi rc(z)c(1-z)]} \) where \( c(z) \) is the function defined in the preliminaries. \( \phi(z) = \phi(z + \pi i/lg(r-1)) \) and so \( \phi \) is bounded. Since \( \phi(z) \neq 0 \) for \( \Re z \neq \frac{1}{2} \), it follows that:

\[
|\phi(x + it)| \geq \min\{|\phi(x + it)| : t \in \mathbb{R}\} > 0 \quad \text{for every } x \neq \frac{1}{2}, \alpha \leq x \leq \beta.
\]

We have \( |\phi(\frac{1}{2} + it)| \leq d \leq dm(t) \). Let \( J_k \) be the interval

\[
J_k = [k\pi/lg(r-1), (k+1)\pi/lg(r-1)] \quad \text{for } k \in \mathbb{Z};
\]

therefore \( J_0 = J \). The functions \( \|\pi_{1/2+it}(u)\|_\text{HS} \) and \( dm(t) \) are periodic; hence, for every \( k \in \mathbb{Z} \):

\[
\int_{J_k} \|\pi_{1/2+it}(u)\|_\text{HS}^2 dm(t) = \int_{J} \|\pi_{1/2+it}(u)\|_\text{HS}^2 dm(t).
\]

Let \( h_k \) be the maximum of the function

\[
|\psi(\frac{1}{2} + it)|^2 = \exp(-3/2 - 2t^2) \quad \text{on } J_k \quad \text{and } \sum_{-\infty}^{+\infty} h_k = c < +\infty.
\]

Finally, we have:

\[
\int_{\mathbb{R}} |\langle F_{1/2+it}, u \rangle|^2 |\phi(\frac{1}{2} + it)| dt
\]

\[
= \sum_{-\infty}^{+\infty} \int_{J_k} |\psi(\frac{1}{2} + it)|^2 |\langle \pi_{1/2+it}(u), \xi_{1/2+it}, \eta_{1/2+it} \rangle|^2 dm(t)
\]

\[
\leq \sum_{-\infty}^{+\infty} h_k \int_{J_k} \|\pi_{1/2+it}(u)\|_\text{HS}^2 dm(t) = c \int_{J} \|\pi_{1/2+it}(u)\|_\text{HS}^2 dm(t)
\]

\[
\leq c \|u\|_2^2,
\]

(recall that \( \|\xi_{1/2+it}\|_2 = \|\eta_{1/2+it}\|_2 = 1 \)).
Proof of Theorem 1. If $G$ acts transitively on $\Omega$, then $\Omega \simeq G/G_0$ where $G_0$ is the stability subgroup of a fixed point $\omega_0$ in $\Omega$. By the "principe de majoration" of C. Herz [14], for every $f$ in $A(G)$ there exists a coefficient of $\pi_{1/2}$ such that: $|f(g)| \leq \langle \pi_{1/2}(g)\xi, \eta \rangle$ for every $g$ in $G$. Hence, from Lemma 2, $A(G) \subset L^q(G)$ for every $q > 2$ and $G$ is a Kunze-Stein group.

Remark. We shall say that a vertex $v$ of a tree is of homogeneity $l$ if $v$ belongs to exactly $l$ edges. Let $X_{l,q}$ be a semihomogeneous tree, that is, a tree such that every vertex is of homogeneity $l$ or $q$ and two adjacent vertices are of homogeneity $l$ and $q$, respectively. We suppose $l \neq q$, otherwise $X$ is a homogeneous tree. Let $S_l$ and $S_q$ be the subsets of vertices of homogeneity $l$ and $q$, respectively. Theorem 1 is true for semihomogeneous trees, with the same proof.

Indeed, if $G$ is a closed noncompact subgroup of $\text{Aut}(X_{l,q})$ acting transitively on the boundary of $X_{l,q}$, then $G \cap K_{v_0}$ acts transitively on the boundary for every vertex $v_0$. Moreover $G(v_0) = S_l$ and $G(w_0) = S_q$ for every $v_0 \in S_l$ and $w_0 \in S_q$. Hence, without loss of generality, we can suppose that $l < q$. The representations $\pi_{1/2+i\nu|G}$ are irreducible [2, pg. 62] and the Plancherel measure of the Gelfand pair $(G, G \cap K_{v_0})$ is a multiple of $|c(\frac{1}{2} + it)|^{-2}$ for an analytic function $c(z)$ [10, pg. 153]. The proof proceeds in the same fashion as for homogeneous trees.

4. Simply transitive subgroups. Let $\Gamma$ be a simply transitive subgroup of $\text{Aut}(X)$; for $\eta \in L^2(\Omega)$ we define, as in [11; 1], the Poisson transform of $\Gamma$: $\varphi(\eta)(x) = \langle \pi_{1/2}(x)\mathbf{1}, \eta \rangle$.

Corollary. $\varphi(L^2(\Omega)) \subset l^q(\Gamma)$ for every $q > 2$.

Proof. By Theorem 1, $f(g) = \langle \pi_{1/2}(g)\mathbf{1}, \eta \rangle \in L^q(\text{Aut}(X))$ for every $q > 2$. Let $\text{Aut}(X) = \Gamma K$ and $g = xk$ with $x \in \Gamma$ and $k \in K$; therefore $\pi_{1/2}(g)\mathbf{1} = \pi_{1/2}(x)\mathbf{1}$ because $\nu$ is $K$-invariant and so

$$\|\varphi(\eta)\|_{l^q(\Gamma)} = \|f\|_{L^q(\text{Aut}(X))}.$$ 

The Corollary follows.

$\Gamma$ is not a Kunze-Stein group (in a discrete Kunze-Stein group every amenable subgroup is finite); nevertheless, we can prove a "weak Kunze-Stein property":

$l^p(\Gamma) *_{\Gamma} l^2(\Gamma) \subset l^2(\Gamma)$ for every $1 < p < 2$, where $l^p$ is the space of radial functions in $l^p$, that is, the functions which depend only on the length of the words of $\Gamma$ and $*_{\Gamma}$ means the convolution product of...
It is easy to see that the "weak Kunze-Stein property" is equivalent to the following: $A_r(\Gamma) \subset l^q(\Gamma)$ for every $q > 2$. This was proved in [5] for free groups on finitely many generators. Notice that $A_r(\Gamma) = l^2(\Gamma) * \Gamma l^2(\Gamma)$.

**Theorem 2.** The following hold:

1. $l^2(\Gamma) * \Gamma l^2(\Gamma) \subset l^q(\Gamma)$ for every $q > 2$.
2. $l^p(\Gamma) * \Gamma l^2(\Gamma) \subset l^2(\Gamma)$ for every $1 < p < 2$.

**Proof.** It is enough to prove (2); (1) follows by duality argument. Putting $\hat{f}(xk) = f(x)$ with $x \in \Gamma$ and $k \in K$, it is possible to identify the functions $f$ on $\Gamma$ with the right $K$-invariant functions $\hat{f}$ on $\text{Aut}(X) = \Gamma K$. The radial functions on $\Gamma$ correspond to the bi $K$-invariant functions on $\text{Aut}(X)$. Let $f \in l^p(\Gamma)$ for $1 < p < 2$ and $\phi \in l^2(\Gamma)$, then the function $\hat{f} \ast \phi$ is right $K$-invariant; hence, by Theorem 1, the restriction to $\Gamma$ is in $l^2(\Gamma)$. Moreover: $(\hat{f} \ast \phi)|_{\Gamma} = f \ast \Gamma \phi$ and, from this, Theorem 2 follows.

**References**


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*Università degli Studi di Roma “La Sapienza”*  
Città Universitaria–00187 Roma, Italy