

ABSTRACT RIEMANNIAN STRATIFICATIONS

GIACOMO MONTI BRAGADIN

Let A be an abstract stratification. Assume that every stratum X is a riemannian manifold. We first give some conditions, under which it is possible to endow A with a suitable metric extending a well known technique of riemannian manifolds. Then stronger conditions are introduced in order that the metric space A becomes a G -space. An abstract stratification with all the above conditions is called an abstract riemannian stratification. Whitney stratifications are, in a natural way, riemannian.

Introduction. An abstract stratification is the disjoint union of a locally finite set of smooth manifolds, the strata, glued together according to certain rules (see [9], [13], [15], [16]).

The basic tools of differential calculus, smooth functions, vector fields and differential forms, can be also defined for an abstract stratification by means of “controlled” collections of “tools”. Classical results can be extended to abstract stratifications (e.g. de Rham’s theorems, see [15], [8]) in this way.

On the other hand a controlled approach seems to be useless when we take into account a riemannian structure: control conditions on the metric of the strata are too strict (bundle-like metrics, see [11], are not controlled in general).

The purpose of this paper is to obtain weaker conditions in order that riemannian metrics of the strata glue together in an appropriate manner. In §1 sufficient conditions are given for extending the fundamental theorem of the riemannian geometry; in §2 a definition of compatibility among the riemannian structures of the strata is given and connections with the structure of G -space are studied. It is shown that the axioms of finite compactness, convexity, local prolongability are satisfied in mild hypotheses, whereas local uniqueness and existence of geodesic coordinates must be required (the former is an open question even in manifolds with boundary, see [1]).

0. Notation. 1. By a manifold we shall always mean a smooth manifold; topologically it is a connected, paracompact space. For any

manifold M let $\mathcal{C}^\infty(M)$ be the ring of smooth real valued functions on M , and let $T_p M$ be the tangent space at $p \in M$.

If g is a riemannian metric on M , for each $v \in T_p M$ the length $\|v\| = g_p(v, v)$ is defined; if g_{ij} are the components of g with respect to local coordinates centered at p and v^i are those of v , then

$$\|v\| = \left(\sum g_{ij}(p) v^i v^j \right)^{1/2}.$$

The length of a differentiable curve L parametrized by $\varphi: [a, b] \rightarrow M$ is the integral of its velocity vector:

$$|L| = \int_a^b \|\varphi_t\| dt.$$

An abstract stratification is a triple $(A, \mathcal{S}, \mathcal{T})$ where

- A is a connected, locally compact topological space with a countable base;
- \mathcal{S} is a locally finite partition of A into locally closed subsets, the strata, such that if $X, Y \in \mathcal{S}$ and $X \cap \bar{Y} \neq \emptyset$, then $X \subset \bar{Y}$; moreover every $X \in \mathcal{S}$ is a manifold without boundary with the induced topology;
- $\mathcal{T} = \{T_X, \pi_X, \rho_X\}$ is the family of the *control data*:
 - T_X is an open neighbourhood of X in A ,
 - $\pi_X: T_X \rightarrow X$ is a continuous retraction,
 - $\rho_X: T_X \rightarrow [0, +\infty)$ is the *tubular function* (see [9], [13]).

We shall often denote the triple $(A, \mathcal{S}, \mathcal{T})$ by A for short.

For each integer $r \geq 0$ define $\mathcal{S}^r = \{X \in \mathcal{S} \mid \dim X \leq r\}$, $A^r = \bigcup_{X \in \mathcal{S}^r} X$. Then it is easy to define, by restriction, control data \mathcal{T}^r so that $A^{(r)} = \{A^r, \mathcal{S}^r, \mathcal{T}^r\}$ is an abstract stratification, called the r -skeleton of A . If $A = A^{(r)}$ for some $r \in \mathbf{N}$ we shall say that A is finite dimensional and the smallest r with this property is called $\dim A$. We shall always suppose that A is finite dimensional; if $n = \dim A$ then $\sum A = A^{(n-1)}$ is non-empty and $A^* = A - \sum A$ is dense in A ; each stratum X is endowed with a riemannian metric g_X .

2. Let (X, d) be a metric space.

We use the notation (xyz) to indicate that x, y, z are distinct points of X and $d(x, y) + d(y, z) = d(x, z)$ (“ y lies between x and z ”).

The length $\lambda(x)$ of a curve $x(t)$ ($a \leq t \leq b$) is defined in the usual way: for any partition $\delta: a = t_0 < t_1 < \dots < t_k = b$ we put $\lambda(x, \delta) = \sum d(x(t_i), x(t_{i-1}))$ and $\lambda(x) = \sup \lambda(x, \delta)$; the metric d is *intrinsic* if $d(x, y)$ equals the infimum of the lengths of all curves from x to y .

A *segment* $T(y, z)$ is a curve from y to z of length $d(y, z)$: it is a shortest join of y to z and any subarc is a segment.

A *geodesic* is a local isometry $\sigma: R \rightarrow X$; in particular a geodesic is locally a segment.

The metric space (X, d) is called a *G-space* if

- (i) the closed balls are compact;
- (ii) the metric d is intrinsic;
- (iii) the axioms P and U (listed below) are satisfied.

Axiom P (local prolongability): each point p of X has a neighbourhood U_p such that for distinct x, y in U_p there exists $z \in X$ with (xyz) .

Axiom U (uniqueness of the prolongation): let $i = 1, 2$. If (xyz_i) holds and $d(z_1, x) = d(z_2, x)$, then $z_1 = z_2$.

Condition (i) is often called *finite compactness*. Condition (ii) is equivalent (for finitely compact metric spaces) with the so-called *Menger convexity*: For each $x, y \in X$ ($x \neq y$) there exists a point $z \in X$ such that (xyz) . Conditions (i), (ii) ensure that segments exist, almost locally; if axiom P holds then each segment can be extended to a geodesic, uniquely if axiom U holds.

The most important results on G -spaces are contained in [5], [6], [7]. G -spaces include complete riemannian manifolds; the theory developed in [5], [6], [7] leads one to recognize which theorems pertain to G -spaces theory and which to riemannian geometry.

1. Riemannian metrics on abstract stratifications. Let L be a curve parametrized by $\varphi: [a, b] \rightarrow A$. If $\text{im}(\varphi) \subset X$ for some stratum X and L is piece-wise differentiable (p.d.) curve of the manifold X , then $|L|$ is defined. If there exists a stratum X such that $\text{im}(\varphi) \subset \overline{X}$ and for $t \in [a, b)$ the restriction L_t of L to $[a, t]$ is p.d. in X , then we define $|L| = \lim_{t \rightarrow b} |L_t|$.

Finally if there exists a subdivision of $[a, b]$: $a = a_0 < a_1 < \dots < a_n = b$ such that the restriction L_i of L to $[a_i, a_{i+1}]$ has a length $|L_i|$ in the above sense, we put $\sum_{i=0}^{n-1} |L_i| = |L|$.

DEFINITION 1. The curve L is *summable* if $|L| \in \mathbf{R}$; for every pair x, y of points of A we define $d_A(x, y) = \inf\{|L| \mid L \text{ is summable with endpoints } x, y\}$.

We shall prove that in suitable hypotheses d_A is a metric on A which

induces the stratification topology. For this we need the following

DEFINITION 2. Let A be an abstract stratification and a topological manifold. The C^0 structure is compatible with the smooth structures of the strata if for every coordinate system (U, φ)

- (i) for each $x \in U$, if X is the stratum which contains x , then there exists a neighbourhood U_x of x in X such that $(U_x, \varphi|_{U_x})$ is a coordinate system of the smooth structure of X ;

the C^0 structure is compatible with the riemannian structures of the strata if in addition:

- (ii) let $\{U_{X,i}\}$ be an open covering of X with coordinate neighbourhoods given by (i); then fix a refinement $\{V_{X,i}\}$ and define

$$m_{X,i} = \inf \left\{ \sum_1^{\dim X} (g_{X,i})_{h,k}(x) \alpha^h \alpha^k \mid \sum_1^{\dim X} (\alpha^1)^2 = 1, x \in V_{X,i} \right\},$$

$$M_{X,i} = \sup \left\{ \sum_1^{\dim X} (g_{X,i})_{h,k}(x) \alpha^h \alpha^k \mid \sum_1^{\dim X} (\alpha^1)^2 = 1, x \in V_{X,i} \right\};$$

for every compact $K \subset A$ put

$$m_{X,K} = \liminf_{\bar{V}_{X,i} \cap K \neq \emptyset} m_{X,i}, \quad M_{X,K} = \limsup_{\bar{V}_{X,i} \cap K \neq \emptyset} M_{X,i}.$$

Then we require that for every $X \in \mathcal{S}$ and for every compact $K \subset A$

$$0 < m_{X,K}, \quad M_{X,K} \in \mathbf{R}.$$

THEOREM 1. Let A be as in Definition 2. Then d_A is a metric on A , which induces the underlying topology.

Proof. Let $p \in A$ and let (U, φ) be a coordinate system centered at p . We can choose a number $\alpha > 0$ such that $B_\alpha = D(0, \alpha) \subset \varphi(U)$. Then for every $r \in (0, \alpha]$ we put $K_r = \varphi^{-1}(B_r)$ and $K = K_\alpha$.

Now let L be any summable curve contained in K , with endpoint p . Let q be the startpoint of L and let $x: [a, b] \rightarrow A$ be a parametrization of L . Let $\|L\|$ be the euclidean length of $\varphi^0 L$.

We suppose at first that $p \in X$ and there exists a stratum Y such that $X < Y$ and $L - \{p\} \subset Y$.

Then we can cover $L - \{p\}$ with coordinate neighbourhoods Y_{Y_i} as in the Definition 2. If $\{V_i\}$ is the corresponding refinement, then there exist curves L_i such that $L_i \subset V_i$ and $L_i \cap L_{i+1}$ is a point, for every i .

If we put $\alpha_i = |L_i|$, $\beta_i = \|L_i\|$, $A_n = \sum \alpha_i$, $B_n = \sum \beta_i$ we get (see [4])

$$\begin{aligned}\frac{A_n}{B_n} &= \frac{1}{B_n} \sum \frac{\alpha_i}{\beta_i} \beta_i \leq \frac{1}{B_n} \sum M_{Y_i} \beta_i \leq \max_{1 \leq i \leq n} M_{Y_i}, \\ \frac{A_n}{B_n} &= \frac{1}{B_n} \sum \frac{\alpha_i}{\beta_i} \beta_i \geq \frac{1}{B_n} \sum m_{Y_i} \beta_i \geq \min_{1 \leq i \leq n} M_{Y_i},\end{aligned}$$

hence:

$$(1) \quad 0 < m_{Y,K} \leq \frac{|L|}{\|L\|} \leq M_{Y,K}.$$

The above (1) also holds in the general case. It is enough to give the proof for the case $L = L_1 \cup L_2$, $L_1 \cap L_2 = \{q\}$, $L_2 - \{q\} \subset Y$, $L_1 - \{p\} \subset Z$ and $X < Z < Y$. It follows from (1) that

$$m_{Y,K} \leq \frac{|L_2|}{\|L_2\|} \leq M_{Y,K} \quad \text{and} \quad m_{Z,K} \leq \frac{|L_1|}{\|L_1\|} \leq M_{Z,K},$$

hence

$$\begin{aligned}\frac{|L|}{\|L\|} &= \frac{|L_1| + |L_2|}{\|L_1\| + \|L_2\|} \\ &\leq \frac{\|L_1\| M_{Z,K} + \|L_2\| M_{Y,K}}{\|L_1\| + \|L_2\|} \leq \max(M_{Z,K}, M_{Y,K}), \\ \frac{|L|}{\|L\|} &\geq \min(m_{Z,K}, m_{Y,K}).\end{aligned}$$

All together, if

$$M = \max_{X \cap K \neq \emptyset} M_{X,K} \quad \text{and} \quad m = \min_{X \cap K \neq \emptyset} m_{X,K}$$

one has for any summable curve L contained in K

$$(2) \quad 0 < m \leq \frac{|L|}{\|L\|} \leq M.$$

For every $r \in (0, \alpha]$, if

$$m_r = \min_{X \cap K_r \neq \emptyset} m_{X,K_r} \quad \text{and} \quad M_r = \max_{X \cap K_r \neq \emptyset} M_{X,K_r}$$

one has $m = m_a \leq m_r$ and $M = M_a \geq M_r$, so that for every summable curve $L \subset K_r$ with $x(a) = q$, $x(b) = p$

$$(3) \quad 0 < m \|\varphi(q)\| \leq m_r \|\varphi(q)\| \leq |L| \leq M_r \|L\| < M \|L\|$$

where $\|\varphi(q)\|$ is the euclidean norm of $\varphi(q)$.

The theorem follows from (3) with a standard argument (see [2] or [4]).

REMARKS. 1. The long spiral of \mathbf{R}^2 and more generally any euclidean curve with infinite length, equipped with the geodesic distance, does not satisfy Definition 2(ii).

2. In general d_A does not induce on $X \in \mathcal{S}$ the metric g_X ; nevertheless the immersion $X \rightarrow A$ is a local isometry if X is maximal. In particular the geodesics of X (with respect to g_X) are the traces on X of those of A .

2. Compatibility. In the present section we shall consider only stratifications A with the following properties:

A is a topological manifold, the topological structure is compatible with the riemannian metrics g_X , $X \in \mathcal{S}$.

By Theorem 1 such a stratification is a metric space (A, d_A) ; we are looking for condition which make this metric space a G -space. Finite compactness is the first axiom of G -spaces. So we are led to consider only stratifications compact in the sense of [13]I.D: each stratum X is diffeomorphic with the interior of a manifold with boundary V_X (see [13], [15] or [16]), which admits a complete riemannian metric, by the Hopf-Rinow theorem. It is easily seen that the resulting metrics on the strata satisfy the conditions of Definition 2.

The other axioms need some compatibility conditions among the metrics g_X .

The first requirement is that

$$(4) \quad \lambda(L) = |L| \quad \text{for each summable curve } L.$$

If (4) holds, then

$$d_A(x, y) = \inf |L_{xy}| = \inf \lambda(L_{xy}) \quad \text{for every } x, y \in A$$

so that d_A is intrinsic (see [7]). It follows that (A, d_A) is convex (see [7] (1.5), also [6] (5.18)).

The second condition of compatibility is

$$(5) \quad \begin{array}{l} \text{for every } p \in A \text{ there exists a neighbourhood } U \text{ of } p \\ \text{such that if } \{x_n\}, \{y_n\} \text{ are sequences in } U \text{ which con-} \\ \text{verge respectively to some } x, y \in U, \text{ then the segments} \\ T(x_n, y_n) \text{ converge to the segment } T(x, y) \text{ in the Haus-} \\ \text{dorff metric (see [3] or [5]).} \end{array}$$

If (5) holds, it is sufficient to prove the axiom of local prolongability in the following situation:

x, y belong to an open convex subset $U \subset A$, $x \in A^*$, $y \in \sum A$.

For each $z \in U$, let t_z be the first point of $T(x, z) \cap \sum A$. Put $T = \{t \in U \cap \sum A \mid \exists z \in A \text{ such that } (xtz)\}$. We have to show that $T = U \cap \sum A$.

We shall divide the proof in two steps.

1st step: T is closed in $U \cap \sum A$.

Let $\{t_n\}$ be a sequence of points of T which converges to $t^* \in U \cap \sum A$. For each $n \in \mathbf{N}$ let be z_n such that $(xt_n z_n)$; because of the finite compactness of A , we can assume that $z^* = \lim z_n \in U$. Then by (5):

$T(t_n, z_n)$ converges to $T(t^*, z^*)$ and $T(x, z_n)$ to $T(x, z^*)$, so that it follows from $T(t_n, z_n) \subset T(x, z_n)$ that $T(t^*, z^*) \subset T(x, z^*)$. Hence $(xt^* z^*)$ holds and $t^* \in T$ as required.

2nd step. T is dense in $U \cap \sum A$.

Otherwise there exists an open set $V \subset U \cap \sum A$ such that no segment with endpoint in V has a prolongation. Let $a \in V$ be any point and let $\{a_n\} \{b_n\}$ be sequences in different components of $U - \sum A$ both converging to a . By (5) the sequence $\{T(a_n, b_n)\}$ converges to $\{a\}$, so that there exists a sequence $\{v_n\}$, with $v_n \in T(a_n, b_n) \cap \sum A$ and $\lim v_n = v$. Then $v_n \in V$ for great n , a contradiction.

The question of finding sufficient conditions for the uniqueness of the prolongation is still open, even for manifolds with boundary (see [1]). We just observe that the following

$$(4') \quad \lambda(L) = |L| \quad \text{for every segment } L$$

is necessary.

As a matter of fact, if L is a segment and $\lambda(L) \neq |L|$ then $\lambda(L) < |L|$. It follows that there exist points $p_1, p_2 \in L$ and a curve L_1 with endpoints p_1, p_2 such that $|L_1| < |L_{12}|$ (where L_{12} is the restriction of L). But L_{12} is a segment, so the last inequality gives a contradiction.

A compact stratification A with compatibility properties (4) and (5) and uniqueness of the prolongation is a G -space (as metric space (A, d_A) , see [6]).

REMARK. Let X, Y be strata of an abstract stratification A with $Y \subset \bar{X}$ and $\dim X = \dim A$.

Let g_X be a riemannian metric on X , such that: for every vector field v on Y , if \bar{v} is a controlled extension of v , then for every $y \in Y$, taken a sequence $\{x_n\}$ in X converging to y , the sequence

$$\|v_n\| = g_X(x_n)(\bar{v}, \bar{v})^{1/2}$$

converges to a limit, which does not depend on the choice of $\{x_n\}$. By results of [5] it is possible to show that there exists a riemannian metric g_Y on Y such that $g_Y(y)(v, v)^{1/2}$ is equal to the limit above. In other words a g_X as above induces a riemannian metric g_Y for every stratum $Y \subset X$. If the strata $X \subset A$ with $\dim X = \dim A$ are equipped with a g_X as above and for every $Y \subset \sum A$ the resulting g_Y is the same (i.e. does not depend on the choice of X such that $Y \subset \bar{X}$), the resulting collection $\{g_Z\}_{Z \in \mathcal{S}(A)}$ satisfies (4) and (5).

DEFINITION 3. We shall call the collection $\{g_Z\}$ of the previous remark *compatible* and the abstract stratification *riemannian*.

PROPOSITION. *Let $A \subset \mathbf{R}^N$ be a Whitney stratification, equip every stratum X with the induced riemannian metric. Then A is a riemannian stratification.*

COROLLARY 1. *Every abstract stratification has a riemannian structure. (Use a realization, see [10] or [12].)*

COROLLARY 2. *(b)-regularity implies 1-regularity.*

Proof. The geodesic distance is d_A and induces the subspace topology (see [14]).

Proof of the Proposition. It is sufficient to show that A is connected by arcs of finite length. To this purpose it is sufficient to show that for every $X \in \mathcal{S}(A)$ and every $x \in X$ it is possible to join x with a point on the boundary of X by a curve of finite length. This follows in turn from the following fact (whose proof is an exercise in standard calculus): if γ is a curve of \mathbf{R}^N such that $\{\gamma - \{\gamma(0), \gamma(1)\}, \gamma(0), \gamma(1)\}$ is a Whitney stratification, then γ has finite length.

REFERENCES

- [1] S. Alexander, *Distance Geometry in Riemannian Manifolds With Boundary*, Global diff. geometry and global analysis. Springer Lecture Notes **838** (1981).
- [2] R. L. Bishop and R. J. Crittenden, *Geometry of Manifolds*, Academic Press, New York (1964).
- [3] L. M. Blumenthal, *Theory and Applications of Distance Geometry*, Oxford (1953).
- [4] W. M. Boothby, *An Introduction to Differentiable Manifolds and Riemannian Geometry*, Academic Press, New York (1975).

- [5] H. Busemann, *Metric methods in Finsler spaces and in the foundations of geometry*, Ann. of Math. Studies, **8**, Princeton Univ. Press, Princeton (1942).
- [6] ———, *The Geometry of Geodesics*, Academic Press, New York (1955).
- [7] ———, *Recent synthetic differential geometry*, Erg. der Math., **54**, Springer, Berlin (1970).
- [8] A. Ferrari et G. Monti Bragadin, *Sur la cohomologie entière des prestratifications abstraites*, C.R.A.S. Paris, **279** (1974), 883–886.
- [9] J. Mather, *Notes on topological stability*, Mimeographed notes, Harvard (1970).
- [10] H. Natsume, *The realization of abstract stratified sets*, Kodai Math. J., **3** (1980), 1–7.
- [11] B. L. Reinhart, *Foliated manifolds with bundle-like metrics*, Ann. Math., **69** (1959), 119–131.
- [12] M. Teufel, *Abstract prestratified sets are (b)-regular*, J. Diff. Geom., **16** (1981), 529–536.
- [13] R. Thom, *Ensembles et morphismes stratifiés*, Bull. Amer. Math. Soc., **75** (1969), 240–284.
- [14] J. C. Tougeron, *Idéaux des fonctions différentiables*, Erg. der Math., **71** (1972), Berlin.
- [15] A. Verona, *Le théorème de deRham pour les prestratifications abstraites*, C.R.A.S. Paris, **273** (1971), 886–889.
- [16] ———, *Homological properties of abstract prestratifications*, Rev. Roum. Math., **XVII** (1972), 1109–1121.

Received January 24, 1986 and in revised form July 6, 1987.

UNIVERSITÀ DI GENOVA
VIA L.B. ALBERTI 4
I-16132 GENOVA (ITALY)

