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INTERSECTION HOMOLOGY OF WEIGHTED PROJECTIVE SPACES AND PSEUDO-LENS SPACES

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In this paper we calculate the integral intersection homology groups of weighted projective spaces and pseudo-lens spaces. Most computations of intersection homology have been for the rational groups. The groups calculated here have interesting torsion.

1. Introduction. Let $b = (b_0, \dots, b_n)$ be an $(n+1)$-tuple of positive integers. The weighted projective space is by definition

$$\mathbf{P}_n(b_0, \dots, b_n) = \{(z_0, \dots, z_n) \in \mathbf{C}^{n+1} - \{0\}\}/\sim$$

where $(z_0, \dots, z_n) \sim (\lambda^{b_0} z_0, \dots, \lambda^{b_n} z_n)$, $\lambda \in \mathbf{C}^\times = \mathbf{C} - \{0\}$.

Let $b' = (b_1, \dots, b_n)$. The pseudo-lens space is by definition

$$L_n(b_0; b_1, \dots, b_n) = \left\{ (z_1, \dots, z_n) \in \mathbf{C}^n \,\middle|\, \sum_{i=1}^{n} |z_i|^2 = 1 \right\} / \sim$$

where $(z_1, \dots, z_n) \sim (\zeta^{b_1} z_1, \dots, \zeta^{b_n} z_n)$, $\zeta \in \mathbf{Z}/b_0 \subset \mathbf{C}^\times$.

Note that $\mathbf{P}_n(b_0, \dots, b_n)$ is naturally identified with $\mathbf{P}_n(rb_0, \dots, rb_n)$. Furthermore, if $(r, b_i) = 1$, $\mathbf{P}_n(rb_0, \dots, b_i, \dots, rb_n)$ is identified with $\mathbf{P}_n(b_0, \dots, b_n)$ via the map $[z_0, \dots, z_n] \mapsto [z_0, \dots, z_i, \dots, z_n]$. Also, $L_n(b_0; rb_1, \dots, rb_n) \cong L_n(b_0; b_1, \dots, b_n)$, and $L_n(b_0; b_1, \dots, b_n) \cong L_n(b_0; rb_1, \dots, b_i, \dots, rb_n)$ if $(r, b_i) = 1$ $(i = 1, \dots, n)$. Thus, we may assume $\gcd(b_0, \dots, \check{b}_i, \dots, b_n) = 1$ for all i.

Since both weighted projective spaces and pseudo-lens spaces are rational homology manifolds, intersection homology of these spaces with rational coefficients is isomorphic to ordinary rational homology, which is the same as the homology of ordinary projective space or the ordinary sphere. Thus our focus goes to torsion phenomena in intersection homology, which are much more complicated than torsion phenomena of ordinary homology and not yet well understood.

We first discuss the intersection homology of pseudo-lens spaces, then we will determine the intersection homology of weighted projective spaces by using the results about pseudo-lens spaces. We also determine the Goresky-Siegel invariant which indicates how much Poincaré duality fails over integers.

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2. Preliminaries. In order to calculate intersection homology, we first stratify $\mathbf{P}_n(b)$ and $L_n(b_0; b')$. Let p be a prime number. Decompose each b_i into primes:

$$b_i = \prod_{p\colon \text{prime}} p^{\alpha_{p,i}} \qquad i = 1, \dots, n.$$

Define the index $i_p(k) \in \{0, 1, \dots, n\}$, $0 \leq k \leq n$, so that we have

$$\alpha_{p,i_p(0)} \geq \alpha_{p,i_p(1)} \geq \cdots \geq \alpha_{p,i_p(n-1)} = \alpha_{p,i_p(n)} = 1.$$

We also define $j_p(k) \in \{1, 2, \dots, n\}$, $1 \leq k \leq n$, such that

$$\alpha_{p,j_p(1)} \geq \alpha_{p,j_p(2)} \geq \cdots \geq \alpha_{p,j_p(n-1)} = \alpha_{p,j_p(n)} = 1.$$

Define:

$$\begin{aligned} &X_{2n-2}(p) \\ &\quad = \{[z_0, \dots, z_n] \in \mathbf{P}_n(b) | z_{i_p(n)} = z_{i_p(n-1)} = \cdots = z_{i_p(n-l-1)} = 0\} \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad 2 \leq l \leq n. \end{aligned}$$

Note that $\overline{X}_{2n-2l}(p)$ is homeomorphic to

$$\mathbf{P}_n(b_{i_p(n)}, b_{i_p(n-1)}, \dots, b_{i_p(n-l-1)}).$$

Now we define the stratification of $\mathbf{P}_n(b)$ as follows:

$$\begin{aligned} X_{2n} &= \mathbf{P}_n(b), \\ X_{2n-2l} &= \bigcup_{p\colon \text{prime}} X_{2n-2l}(p), \qquad 2 \leq l \leq n-1, \\ X_{2n-2l-1} &= X_{2n-2l}, \qquad 2 \leq l \leq n-1, \\ X_{2n-1} &= X_{2n-2} = X_{2n-3} = X_{2n-4}. \end{aligned}$$

Also define:

$$Y_{2n-2l-1} = \begin{cases} \{[z_1, \dots, z_n] \in L_n(b_0; b') | z_{j_p(n)} = \cdots = z_{j_p(n-l-1)} = 0\} \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{if } p | b_0, \\ \varnothing \quad \text{if } (p, b_0) = 1. \end{cases}$$

$\overline{Y}_{2n-2l-1}(p)$ is homeomorphic to $L_n(d; b_{i_p(n-l-1)}, \ldots, b_{i_p(n)})$, where $d = \gcd(b_{i_p(0)}, \ldots, b_{i_p(n-l)})$. Define the stratification of $L_n(b_0; b')$ as follows:

$$\begin{aligned}
&Y_{2n} = \mathbf{P}_n(b), \\
&Y_{2n-2l-1} = \bigcup_{p:\text{ prime}} Y_{2n-2l-1}(p), \qquad 2 \le l \le n-1, \\
&Y_{2n-2l-2} = Y_{2n-2l-1}, \qquad 2 \le l \le n-1, \\
&Y_{2n-1} = Y_{2n-2} = Y_{2n-3} = Y_{2n-4}.
\end{aligned}$$

LEMMA 2.1. (i) *Suppose x is contained in $X_{2n-2l}(p)$. Then the link of the stratum L_x at x is homeomorphic to $L_n(d; b_{i_p(n-l-1)}, \ldots, b_{i_p(n)})$, where $d = \gcd(b_{i_p(0)}, \ldots, b_{i_p(n-l)})$. Furthermore, the tubular neighborhood of $X_{2n-2l}(p)$ is isomorphic to $X_{2n-2l}(p) \times \mathrm{Cone}(L_x)$.*

(ii) *Suppose x is contained in $Y_{2n-2l-1}(p)$. Then the link of the stratum L_x at x is homeomorphic to $L_n(d'; b_{j_p(n-l-1)}, \ldots, b_{j_p(n)})$, where $d' = \gcd(b_{j_p(0)}, \ldots, b_{j_p(n-l)})$. Furthermore, the tubular neighborhood of $Y_{2n-2l}(p)$ is isomorphic to $Y_{2n-2l}(p) \times \mathrm{Cone}(L_x)$.*

Proof. The proof is elementary and we omit it.

3. Intersection homology of pseudo-lens spaces. To state our result, we introduce some notation. A perversity $\bar{p} = (p(2), \ldots, p(n-2))$ is said to be a two-step perversity if $p(2l) \equiv 0 \bmod 2$ for all $l \ge 1$. The associated two-step perversity $\bar{p}^a$ of $\bar{p}$ is the one defined by

$$p^a(c) = \begin{cases} 2\left[\frac{p(c)}{2}\right], & c = 2l, \\ [p(c-1)] + [p(c+1)], & c = 2l+1, \\ p(n), & c = 2l+1 = n. \end{cases}$$

We define $m_p(k) = \max(\alpha_{p,0} - \alpha_{p,j_p(k)})$.

By $A(n, j, k)$ $(1 \le k \le n-1)$ we denote the point on the plane whose coordinates are $(2n-k+1, 2n-2j+1)$ and by $B(n, j, k)$ $(1 \le k \le n-j)$ the point whose coordinates are $(2n-2k+1, 2n-2j-2k)$. The graph of a two-step perversity $\bar{p}$ passes through one and only one of the $A(n, j, k)$'s or $B(n, j, k)$'s. Finally we define the perversity $\bar{p} - 2$ as $(p(4)-2, p(5)-2, \ldots, p(n-2)-2)$ when $p(4) \ge 2$.

THEOREM 3.1. *Let $L_n = L_n(b_0; b')$ be the pseudo-lens space and $\bar{p}$ be any perversity. Then*

$$IH_*^{\bar{p}}(L_n) \cong IH_*^{\bar{p}^a}(L_n).$$

Suppose the associated two-step perversity $\bar{p}^a$ passes through $A(n, j, k_j)$ or $B(n, j, k_j)$, then

$$IH_i^{\bar{p}^a}(L_n) = \begin{cases} \mathbf{Z}, & i = 0,\ 2n-1, \\ \mathbf{Z}/m_j, & i = 2j-1,\ 1 \le j \le n, \\ 0, & \text{otherwise.} \end{cases}$$

where $m_j = \prod_{p:\ \text{prime}} p^{m_p(k_j)}$ $(1 \le j \le n)$.

Proof. We proceed by inducting on n. Choose a prime number p and fix it once and for all. Let $L_{n-1}(p) = \overline{Y}_{2n-3}(p)$. We claim the following three assertions:

$(1)_n$ $IH_*^{\bar{p}}(L_n) \cong IH_*^{\bar{p}^a}(L_n)$,
$(2)_n$ $IH_{2i}^{\bar{p}}(L_n) = 0, \quad i < 2n-3$,
$(3)_n$ If $p(4) \ge 2$, the map $IH_i^{\bar{p}-2}(L_{n-1}(p)) \otimes \mathbf{Z}_p \to IH_i^{\bar{p}}(L_n) \otimes \mathbf{Z}_p$ induced by the inclusion is an isomorphism for $i < 2n-3$, where $\mathbf{Z}_p$ is the ring of p-adic integers.

Assume these are true for the moment. From $(1)_n$ we may assume $\bar{p}$ is a two-step perversity. From $(2)_n$ $L_n(b_0; b')$ is $\bar{p}$-locally torsion free in the sense of Goresky-Siegel when $\bar{p}$ is a two-step perversity (cf. **[GS]**). Therefore the universal coefficient theorem holds for intersection homology; i.e.

$$IH_i^{\bar{p}}(L_n) \cong \mathrm{Hom}(IH_{2n-i-1}^{\bar{t}-\bar{p}}(L_n), \mathbf{Z}) \oplus \mathrm{Ext}(IH_{2n-i-2}^{\bar{t}-\bar{p}}(L_n), \mathbf{Z}),$$
$$IH_i^{\bar{p}}(L_n; G) \cong IH_i^{\bar{p}}(L_n) \oplus \mathrm{Tor}(IH_{i-1}^{\bar{p}}(L_n), G).$$

This reduces the problem; we now only have to show

$$IH_i^{\bar{p}}(L_n; \mathbf{Z}_p) = \begin{cases} \mathbf{Z}_p/p^{m_p(k_j)}, & i = 2j-1,\ 1 \le j \le n, \\ 0, & i = 2j,\ 1 \le j \le n \end{cases}$$

for all two-step perversities such that $p(4) = 2$ and for all prime numbers p. If $p(4) < 2$, the theorem follows from the universal coefficient theorem and the fact that if $\bar{p}$ passes through $A(n, j, k_j)$ or $B(n, j, k_j)$, the perversity $\bar{t} - \bar{p}$ passes through $B(n, n-j, k_j)$ or $A(n, n-j, k_j)$ respectively. By induction $(3)_n$ establishes the calculation because of the fact that if $\bar{p}$ passes through $A(n, j, k_j)$ or $B(n, j, k_j)$, $\bar{p} - 2$ passes through $A(n-1, j, k_j)$ or $B(n-1, j, k_j)$ respectively and $m_p(k_j)$ for L_{n-1} is equal to the one for L_n.

Now we prove $(1)_n$ to $(3)_n$. When $n = 1$, these are trivial. Assume $(1)_k$ to $(3)_k$ are true for all k less than n. Consider the obstruction sequence (cf. **[GM2**; §5.5**]**);

$$\begin{tikzcd} \underline{\underline{\mathrm{IC}}}^{\bullet}_{\bar{p}^a} \arrow[rr] & & \underline{\underline{\mathrm{IC}}}^{\bullet}_{\bar{p}} \arrow[dl] \\ & \underline{\underline{\mathrm{S}}}^{\bullet} \arrow[ul, "{[1]}"] & \end{tikzcd}$$

$\underline{\underline{\mathrm{H}}}^{-i}(\underline{\underline{\mathrm{S}}}^{\bullet})_x = 0$ unless $x \in X_{n-2l} - X_{n-2l-1}$ and $p(2l) > p^a(2l)$, in which case

$$\underline{\underline{\mathrm{H}}}^{-i}(\underline{\underline{\mathrm{S}}}^{\bullet})_x = \begin{cases} \underline{\underline{\mathrm{H}}}^{-i}(\underline{\underline{\mathrm{IC}}}^{\bullet}_{\bar{p}})_x, & i = (2n-1) - p(2l). \\ 0, & i \neq (2n-1) - p(2l). \end{cases}$$

On the other hand, $\underline{\underline{\mathrm{H}}}^{-(2n-1)+p(2l)}(\underline{\underline{\mathrm{IC}}}^{\bullet}_{\bar{p}})_x \cong IH^{\bar{p}}_{2l-p(2l)-1}(L_x)$ where L_x is the link of the stratum containing x, which is a pseudo-lens space of dimension less than $2n-1$. By definition of $\bar{p}^a$, $p(2l) > p^a(2l)$ only when $p(2l)$ is odd. Therefore $2l - p(2l) - 1$ is even, and from the induction assumption $IH^{\bar{p}}_{2l-p(2l)-1}(L_x) = 0$. As a result, $\underline{\underline{\mathrm{H}}}^{\bullet}(\underline{\underline{\mathrm{S}}}^{\bullet})_x = 0$ everywhere, and thus $\underline{\underline{\mathrm{S}}}^{\bullet} = 0$ in $D^b(L_x)$. This proves $(1)_n$.

Now we go to $(2)_n$ and $(3)_n$. $L_{n-1}(p)$ is stratified by $Y_{2n-k} \cap L_{n-1}$, $0 \leq k \leq 2n-3$. The codimension of the stratum $Y_{2n-k} \cap L_{n-1}$ is $k-2$. With respect to this stratification of $L_{n-1}(p)$, a $(\bar{p}, i)$-allowable chain in L_n with support in $L_{n-1}(p)$ is a $(\bar{p}-2, i)$-allowable chain in $L_{n-1}(p)$, and vice versa. In other words, the inclusion map $j\colon L_{n-1}(p) \to L_n$ induces $IC^{\bar{p}-2}_i(L_{n-1}(p)) \to IC^{\bar{p}}_i(L_n)$ and thus induces $j_*\underline{\underline{\mathrm{IC}}}^{\bullet}_{\bar{p}-2}(L_{n-1}(p); \mathbf{Z}_p) \to \underline{\underline{\mathrm{IC}}}^{\bullet}_{\bar{p}}(L_n; \mathbf{Z}_p)$. Consider the triangle in $D^b(L_n)$:

$$\begin{tikzcd} Rj_*\underline{\underline{\mathrm{IC}}}^{\bullet}_{\bar{p}-2}(L_{n-1}(p); \mathbf{Z}_p) \arrow[rr] & & \underline{\underline{\mathrm{IC}}}^{\bullet}_{\bar{p}}(L_n; \mathbf{Z}_p) \arrow[dl] \\ & \underline{\underline{\mathrm{A}}}^{\bullet}_{\bar{p}}(\mathbf{Z}_p) \arrow[ul, "{[1]}"] & \end{tikzcd}$$

For each x in L_n, we have a long exact sequence of stalks associated to the triangle above:

$$\cdots \to \underline{\underline{\mathrm{H}}}^{-i-1}\left(\underline{\underline{\mathrm{A}}}^{\bullet}_{\bar{p}-2}(\mathbf{Z}_p)\right)_x \to \underline{\underline{\mathrm{H}}}^{-i}\left(Rj_*\underline{\underline{\mathrm{IC}}}^{\bullet}_{\bar{p}-2}(L_{n-1}(p); \mathbf{Z}_p)\right)_x \to$$
$$\underline{\underline{\mathrm{H}}}^{-i}\left(\underline{\underline{\mathrm{IC}}}^{\bullet}_{\bar{p}}(L_n; \mathbf{Z}_p)\right)_x \to \underline{\underline{\mathrm{H}}}^{-i}\left(\underline{\underline{\mathrm{A}}}^{\bullet}_{\bar{p}}(\mathbf{Z}_p)\right)_x \to \cdots.$$

On the other hand

$$\underline{\underline{\mathrm{H}}}^{-i}\left(Rj_*\underline{\underline{\mathrm{IC}}}^{\bullet}_{\bar{p}-2}(L_{n-1}(p); \mathbf{Z}_p)\right)_x \cong IH^{\bar{p}-2}_{i+c-2n}(L'_x; \mathbf{Z}_p),$$
$$\underline{\underline{\mathrm{H}}}^{-i}\left(\underline{\underline{\mathrm{IC}}}^{\bullet}_{\bar{p}}(L_n; \mathbf{Z}_p)\right)_x \cong IH^{\bar{p}}_{i+c-2n}(L_x; \mathbf{Z}_p),$$
$$k \geq 2n-1-p(c),$$

where c is the codimension of the stratum in L_n containing x, L_x is the link of the stratum in L_n, and L'_x is the link of the stratum in $L_n(p)$. If we regard L_x as L_n and construct $L_{n-1}(p)$, then $L_{n-1}(p)$ is equal to L'_x. Thus, by $(3)_k$, $k < n$, $\underline{\underline{\mathrm{H}}}^{-i}(\underline{\underline{\mathrm{A}}}^\bullet_{\bar{p}}(\mathbf{Z}_p))_x = 0$ when $i < 2n-4$ regardless of the perversity $\bar{p}$. To look at $\underline{\underline{\mathrm{H}}}^{-i}(\underline{\underline{\mathrm{A}}}^\bullet_{\bar{p}}(\mathbf{Z}_p))_x$ when $2n-4 \leq i \leq 2n-1$, consider the following diagram:

$$\begin{array}{ccc} IH^{\bar{p}-2}_{i+c-2n}(L'_x) & \longrightarrow & IH^{\bar{p}}_{i+c-2n}(L_x) \\ {\scriptstyle\omega'}\big\downarrow & & \big\downarrow{\scriptstyle\omega} \\ IH^{\bar{t}-2}_{i+c-2n}(L'_x) & \longrightarrow & IH^{\bar{t}}_{i+c-2n}(L_x) \end{array}$$

where ω and ω' are the natural maps. If $i \geq 2n-4$, ω' is an isomorphism by definition of intersection homology. Also ω is an isomorphism if $i \geq 2n-4$ because $p(4) = 2$. Therefore $\underline{\underline{\mathrm{H}}}^{-i}(\underline{\underline{\mathrm{A}}}^\bullet_{\bar{p}}(\mathbf{Z}_p))_x$ and $\underline{\underline{\mathrm{H}}}^{-i}(\underline{\underline{\mathrm{A}}}^\bullet_{\bar{t}}(\mathbf{Z}_p))_x$ are isomorphic for all i and x, and thus $\underline{\underline{\mathrm{A}}}^\bullet_{\bar{p}}(\mathbf{Z}_p) = \underline{\underline{\mathrm{A}}}^\bullet_{\bar{t}}(\mathbf{Z}_p)$ in $D^b(L_n)$. Since L_n is a normal pseudo-manifold (cf. **[GM2**; §5.6]), $\mathscr{H}_c^{-*}(L_n, \underline{\underline{\mathrm{A}}}^\bullet_{\bar{t}}(\mathbf{Z}_p)) \cong H_*(L_n, L_{n-1}; \mathbf{Z}_p)$, where $\mathscr{H}_c$ is hypercohomology with compact support. Thanks to the excision property of homology we can calculate $H_i(L_n, L_{n-1}; \mathbf{Z}_p)$ easily:

$$H_i(L_n, L_{n-1}; \mathbf{Z}_p) = \begin{cases} \mathbf{Z}_p, & i = 2n-1, 2n-2, \\ 0, & \text{otherwise.} \end{cases}$$

Now $(2)_n$ and $(3)_n$ follow from the long exact sequence of hypercohomology associated to the triangle. □

4. Intersection homology of weighted projective spaces. In this section we determine the intersection homology of weighted projective spaces and the natural maps between them. Let

$$n_p(k) = \alpha_{p, i_p(k-1)} - \alpha_{p, i_p(k)},$$

and

$$l(\bar{p}, n, j, k) = \frac{1}{2} \max\{\min(p(2k), 2n-2j) - \max(0, 2k-2j-2), 0\}.$$

THEOREM 4.1. *Let $\mathbf{P}_n = \mathbf{P}_n(b)$ be a weighted projective space and $\bar{p}$ be any perversity. Then*

$$IH_i^{\bar{p}}(\mathbf{P}_n) = \begin{cases} \mathbf{Z}, & i = 2j, \quad 0 \leq j \leq n, \\ 0, & \text{otherwise.} \end{cases}$$

Suppose $\bar{p} \leq \bar{q}$. The natural map $IH_{2j}^{\bar{p}}(\mathbf{P}_n) \to IH_{2j}^{\bar{q}}(\mathbf{P}_n)$ is the multiplication by

$$\gamma = \prod_{p:\ \text{prime}} \prod_{k=1}^{n} p^{(l(\bar{q}^a,n,j,k)-l(\bar{p}^a,n,j,k))\cdot n_p(n-k+1)}.$$

Proof. Since the link of the stratum at any point in $\mathbf{P}_n$ is a pseudo-lens space, we have $IH_i^{\bar{p}}(\mathbf{P}_n) \cong IH_i^{\bar{p}^a}(\mathbf{P}_n)$ by the same argument as in the proof of Th. 3.1. Thus we may assume $\bar{p}$ is a two-step perversity. In that case we also have the universal coefficient theorem for intersection homology. To prove the first half, we only have to show when $p(4) = 2$. To prove the second half, we may assume $\bar{q} = \bar{t}$. Since $\mathbf{P}_n$ is a normal pseudo-manifold, $IH_i^{\bar{0}}(\mathbf{P}_n) \cong H^{2n-i}(\mathbf{P}_n)$ and $IH_i^{\bar{t}}(\mathbf{P}_n) \cong H_i(\mathbf{P}_n)$, and the map $IH_{2j}^{\bar{0}}(\mathbf{P}_n) \to IH_{2j}^{\bar{t}}(\mathbf{P}_n)$ coincides with the map $\bigcap[\mathbf{P}_n] : H^{2n-2j}(\mathbf{P}_n) \to H_{2j}(\mathbf{P}_n)$. The latter map has been calculated in **[K]** and **[M]**, and if we translate it into our notation carefully, we see that it is multiplication by

$$\gamma_0 = \prod_{p:\ \text{prime}} \prod_{k=1}^{n} p^{(l(\bar{t},n,j,k)-l(\bar{0},n,j,k))\cdot n_p(n-k+1)}.$$

Combining this result and the universal coefficient theorem, we can assume $p(4) = 2$. Let $\mathbf{P}_{n-1}(p) = \overline{X}_{2n-2l}(p)$. From now on, the proof proceeds along the same lines as Th. 3.1. We induct on n. Consider the triangle

$$\begin{tikzcd}
Rj_*\underline{\underline{\mathbf{IC}}}^\bullet_{\bar{p}-2}(\mathbf{P}_{n-1}(p);\mathbf{Z}_p) \arrow[rr] & & \underline{\underline{\mathbf{IC}}}^\bullet_{\bar{p}}(\mathbf{P}_n;\mathbf{Z}_p) \arrow[dl] \\
 & \underline{\underline{\mathbf{A}}}^\bullet_{\bar{p}}(\mathbf{Z}_p) \arrow[ul, "{[1]}"] &
\end{tikzcd}$$

Looking at the link of each point, we conclude $\underline{\underline{\mathbf{A}}}^\bullet_{\bar{p}}(\mathbf{Z}_p) = \underline{\underline{\mathbf{A}}}^\bullet_{\bar{t}}(\mathbf{Z}_p)$ in $D^b(L_n)$. Since

$$H_i(\mathbf{P}_n, \mathbf{P}_{n-1}(p);\mathbf{Z}_p) = \begin{cases} \mathbf{Z}_p, & i = 2n, \\ 0, & \text{otherwise,} \end{cases}$$

we have the following diagram when $0 \leq j \leq n-1$:

$$\begin{tikzcd}
IH_{2j}^{\bar{p}-2}(\mathbf{P}_{n-1}(p);\mathbf{Z}_p) \arrow[r, "\cong"] \arrow[d, "\omega'"'] & IH_{2j}^{\bar{p}}(\mathbf{P}_n;\mathbf{Z}_p) \arrow[d, "\omega"] \\
IH_{2j}^{\bar{t}-2}(\mathbf{P}_{n-1}(p);\mathbf{Z}_p) \arrow[r, "\cong"] & IH_{2j}^{\bar{t}}(\mathbf{P}_n;\mathbf{Z}_p) .
\end{tikzcd}$$

Taking into account that $l(\bar{t}, n, j, k) - l(\bar{p}, n, j, k) = l(\bar{t} - 2, n - 1, j, k) - l(\bar{p} - 2, n - 1, j, k)$, this diagram proves the theorem. □

COROLLARY 4.2. *Let $R_i^{\bar{m}}(\mathbf{P}_n)$ be the peripheral invariant of Goresky and Siegel for the middle perversity (cf.* **[GS**; §9]). *Then*

$$R_{2j}^{\bar{m}} = \mathbf{Z} \Big/ \prod_{p:\ \text{prime}} \prod_{k=1}^{j'} p^{n_p(n-k+1)},$$

where $j' = \min(j, n - j)$. Thus the middle groups of $\mathbf{P}_n(b)$ satisfy Poincaré duality over the integers if and only if $\mathbf{P}_n(b)$ is isomorphic to the ordinary projective space. □

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