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ON THE GLOBAL DIMENSION OF FIBRE PRODUCTS

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In this paper we will sharpen Wiseman's upper bound on the global dimension of a fibre product [Theorem 2] and use our bound to compute the global dimension of some examples. Our upper bound is used to prove a new change of rings theorem [Corollary 4]. Lower bounds on the global dimension of a fibre product seem more difficult; we obtain a result [Proposition 12] which allows us to compute lower bounds in some special cases.

A commutative square of rings and ring homomorphisms

$$\begin{array}{ccc} R & \xrightarrow{i_1} & R_1 \\ {\scriptstyle i_2}\downarrow & & \downarrow{\scriptstyle j_1} \\ R_2 & \xrightarrow[j_2]{} & R' \end{array}$$

is said to be a *Cartesian* square if given $r_1 \in R_1$, $r_2 \in R_2$ with $j_1(r_1) = j_2(r_2)$ there exists a unique element $r \in R$ such that $i_1(r) = r_1$ and $i_2(r) = r_2$. We will assume that j_2 is a surjection so that results of Milnor [**M**] apply. The ring R is called a *fibre product* (or pullback) of R_1 and R_2 over R'.

The homological properties of a fibre product R have been studied previously. Milnor [**M**, Chapter 2] has characterized projective modules over such a ring R. Facchini and Vamos [**FV**] have obtained analogues of Milnor's theorems for injective and flat modules. Wiseman [**W**] has used Milnor's results to obtain an upper bound on $\operatorname{lgldim} R$; in particular, Wiseman's results show that R has finite left global dimension whenever the rings R_i have finite left global dimension and $\operatorname{fd}(R_i)_R$ are both finite, where $\operatorname{fd}(R_i)_R$ represents the flat dimension of R_i as a right R-module. Vasconcelos [**V**, Chapters 3 and 4] and Greenberg [**G1** and **G2**] have studied commutative rings of finite global dimension which are fibre products and have used their results to classify commutative rings of global dimension 2. Osofsky's example of a commutative local ring of finite global dimension having zero divisors can be described as a fibre product (see [**V**, p. 29–30]). Fibre products have been used to construct noncommutative Noetherian rings of finite global dimension by Robson [**R2**, §2], by Stafford [**St**] and by the authors [**KK2**].

We begin by noting that a fibre product R can be thought of as the standard pullback $R = \{(r_1, r_2)\colon j_1(r_1) = j_2(r_2)\}$, a subring of $R_1 \oplus R_2$, with the maps $i_j\colon R \to R_j$ given by $i_j(r_1, r_2) = r_j$, $j = 1, 2$. Moreover, if A is a subring of a ring B and Q is an ideal of B, $Q \leq A$, then the diagram

$$\begin{array}{ccc} A & \longrightarrow & A/Q \\ \downarrow & & \downarrow \\ B & \longrightarrow & B/Q \end{array}$$

with the obvious maps, is a Cartesian square. Greenberg [**G1** and **G2**] has studied the case where B is a commutative, flat epimorphic image of A, and Q is A-flat (including the "$D+M$ construction", see Dobbs [**D**]). Two important examples of rings of finite global dimension can thus be regarded as fibre products: the trivial extension (see [**PR**]) $A = R \ltimes M$ (which can be regarded as a subring of the triangular matrix ring $B = \left(\begin{smallmatrix} R & M \\ 0 & R \end{smallmatrix}\right)$ with common ideal $Q = \left(\begin{smallmatrix} 0 & M \\ 0 & 0 \end{smallmatrix}\right)$) and the subidealizer R in S at Q (see [**R2**]) (where R can be regarded as a subring of $B = \mathrm{II}(Q)$, sharing the ideal Q).

We begin by stating Wiseman's upper bound and our generalization of it.

THEOREM 1. [**W**, *Theorem* 3.1]. *If R is a fibre product of R_1, R_2 over R' then* $\operatorname{lgldim} R \leq \max_i\{\operatorname{lgldim}(R_i)\} + \max_i\{\operatorname{fd}(R_i)_R\}$. □

THEOREM 2. *If R is a fibre product of R_1, R_2 over R' then* $\operatorname{lgldim} R \leq \max_i\{\operatorname{lgldim}(R_i) + \operatorname{fd}(R_i)_R\}$. □

Theorem 2 is an immediate consequence of the following proposition.

PROPOSITION 3. *Let M be a left R-module such that* $\operatorname{Tor}^R_{n_i+m}(R_i, M) = 0$ *for $m \geq 1$, $i = 1, 2$. Then*

$$\operatorname{pd}_R M \leq \max_i\{n_i + \operatorname{pd}({}_{R_i}(R_i \otimes_R \operatorname{Im} f_{n_i}))\}$$

where

$$(*) \qquad \cdots \to P_{k+1} \xrightarrow{f_{k+1}} P_k \xrightarrow{f_k} \cdots \xrightarrow{f_2} P_1 \xrightarrow{f_1} P_0 \xrightarrow{f_0} M \to 0$$

is a projective resolution of M.

Proof. The projective resolution $(*)$ of M gives rise to a sequence of short exact sequences:

$$0 \to \operatorname{Im} f_1 \to P_0 \to M \to 0$$

and

$$0 \to \operatorname{Im} f_{k+1} \to P_k \to \operatorname{Im} f_k \to 0, \qquad k \geq 1.$$

From this we conclude that $\operatorname{Tor}^R_{m+k}(R_i, M) \cong \operatorname{Tor}^R_m(R_i, \operatorname{Im} f_k)$.

Let $n = \max\{n_i + \operatorname{pd}_{R_i}(R_i \otimes_R \operatorname{Im} f_{n_i})\}$, and consider the resolution

$$0 \to L \xrightarrow{f_n} P_{n-1} \xrightarrow{f_{n-1}} \cdots \xrightarrow{f_2} P_1 \xrightarrow{f_1} P_0 \xrightarrow{f_0} M \to 0$$

obtained from $(*)$ by letting $L = \operatorname{Im} f_n$. The isomorphism noted above gives $\operatorname{Tor}^R_m(R_i, \operatorname{Im} f_{n_i}) = 0$ for $m \geq 1$. Hence if we tensor the exact sequence

$$0 \to L \to P_{n-1} \to \cdots \to P_{n_i} \to \operatorname{Im} f_{n_i} \to 0$$

over R with R_i, we obtain an exact sequence

$$0 \to R_i \otimes_R L \to R_i \otimes_R P_{n-1} \to \cdots \to R_i \otimes_R P_{n_i} \to R_i \otimes_R \operatorname{Im} f_{n_i} \to 0.$$

Each $R_i \otimes_R P_k$ is R_i-projective, hence since $n \geq n_i + \operatorname{pd}_{R_i}(R_i \otimes_R \operatorname{Im} f_{n_i})$, $R_i \otimes_R L$ is R_i-projective. By [**W**, Theorem 2.3], L is R-projective and the result holds. □

We state Theorem 2 in the "shared ideal" case, where it can be regarded as a change of rings theorem; it bounds the global dimension of A by the maximum of two quantities: one involving a homomorphic image of A and the other involving an overring of A. Both quantities are similar to those in other change of rings theorems: the quantity involving the homomorphic image of A is the same as that in Small's change of rings theorem [**S1**], and the quantity involving the overring B can be compared to the McConnell-Roos Theorem [see **Rot**, Theorem 9.39, p. 250].

COROLLARY 4. *Let A be a subring of B with Q an ideal of B, $Q \leq A$. Then*

$$\operatorname{lgldim} A \leq \max\{\operatorname{lgldim}(A/Q) + \operatorname{fd}(A/Q)_A, \operatorname{lgldim} B + \operatorname{fd}(B_A)\}.$$

EXAMPLE 5. Let

$$A = \begin{pmatrix} k[x] + tk[x, x^{-1}, t] & tk[x, x^{-1}, t] \\ k[x, x^{-1}, t] & k[x, x^{-1}, t] \end{pmatrix}$$

where k is a field and x and t are commuting indeterminates. (This affine PI ring is considered in [**S2**; p. 32]). We claim $\operatorname{lgldim} A = 2$. Let

$$B = \begin{pmatrix} k[x, x^{-1}, t] & tk[x, x^{-1}, t] \\ k[x, x^{-1}, t] & k[x, x^{-1}, t] \end{pmatrix}$$

and

$$Q = \begin{pmatrix} tk[x, x^{-1}, t] & tk[x, x^{-1}, t] \\ k[x, x^{-1}, t] & k[x, x^{-1}, t] \end{pmatrix}.$$

As B is a central localization of A, $\operatorname{lgldim} B \leq \operatorname{lgldim} A$, and $\operatorname{lgldim} B = 2$ by [**J**, Theorem 3.5]. Since $A/Q \cong k[x]$, $\operatorname{fd}(A/Q)_A = 1$, and $\operatorname{fd}(B_A) = 0$, Corollary 4 gives $\operatorname{lgldim} A \leq \max\{1 + 1, 2 + 0\}$, so that $\operatorname{lgldim} A = 2$ (and similarly $\operatorname{rgldim} A = 2$).

More generally, let $S = k[x_1, \dots, x_n, x_1^{-1}, \dots, x_n^{-1}, t_1, \dots, t_m]$, $R = k[x_1, \dots, x_n] + (t_1, \dots, t_m)S$, $I = (t_1, \dots, t_m)S$, $A = \left(\begin{smallmatrix} R & I \\ S & S \end{smallmatrix}\right)$, $B = \left(\begin{smallmatrix} S & I \\ S & S \end{smallmatrix}\right)$ and $Q = \left(\begin{smallmatrix} I & I \\ S & S \end{smallmatrix}\right)$. Similar arguments show that $\operatorname{rgldim} A = \operatorname{lgldim} A = n + m$ (note that the upper bound given by Theorem 1 is $\operatorname{lgldim} A \leq n + 2m$ since $\operatorname{fd}(A/Q)_A = m$; we know no other way of computing the global dimension of A). □

It is not hard to produce an example to show that the bound in Corollary 4 is not always an equality. Let

$$A = \begin{pmatrix} k & 0 \\ A_1(k) & A_1(k) \end{pmatrix}$$

where k is a field of characteristic 0 and $A_1(k)$ is the first Weyl algebra. Then A has $\operatorname{rgldim} A = \operatorname{lgldim} A = 1$ by [**PR**, Corollary 4′]. Take

$$B = \begin{pmatrix} A_1(k) & 0 \\ A_1(k) & A_1(k) \end{pmatrix} \quad \text{and} \quad Q = \begin{pmatrix} 0 & 0 \\ A_1(k) & A_1(k) \end{pmatrix};$$

since $\operatorname{gldim} B = 2$, the bound of Corollary 4 exceeds $\operatorname{gldim} A$.

To show the utility of Corollary 4 we provide a further example in which it can be applied.

EXAMPLE 6. Let R be an arbitrary ring; consider the ring

$$A' = \begin{pmatrix} R[x] & R[x] & R[x] \\ xR[x] & R[x] & R[x] \\ x^2R[x] & xR[x] & R[x] \end{pmatrix}$$

(which is a generalization of an example of Tarsy [**T**, Theorem 10]). Taking

$$B = \begin{pmatrix} R[x] & R[x] & R[x] \\ xR[x] & R[x] & R[x] \\ xR[x] & xR[x] & R[x] \end{pmatrix}$$

and

$$Q = \begin{pmatrix} xR[x] & xR[x] & xR[x] \\ xR[x] & xR[x] & xR[x] \\ x^2R[x] & xR[x] & xR[x] \end{pmatrix},$$

and noting that $\mathrm{fd}({}_{A'}Q) = 0$, $\mathrm{fd}({}_{A'}B) \leq 1$, $\mathrm{rgldim}(A'/Q) = \mathrm{rgldim}\, R + 1$, and $\mathrm{rgldim}\, B = \mathrm{rgldim}\, R + 1$ **[KK1]**, we get $\mathrm{rgldim}\, A' \leq \mathrm{rgldim}\, R + 2$ (when R is a field, $\mathrm{rgldim}\, A' = 2$). Now take

$$A = \begin{pmatrix} (R[x])^* & R[x] & R[x] \\ xR[x] & R[x] & R[x] \\ x^2R[x] & xR[x] & (R[x])^* \end{pmatrix}$$

where * entries agree modulo x (this example is a generalization of an example of Fields [**F1**, p. 129]), $B = A'$,

$$Q = \begin{pmatrix} xR[x] & R[x] & R[x] \\ xR[x] & xR[x] & R[x] \\ x^2R[x] & xR[x] & xR[x] \end{pmatrix};$$

since $\mathrm{fd}({}_AQ) \leq 1$, $\mathrm{fd}({}_AB) \leq 1$, we get that $\mathrm{rgldim}\, A \leq \mathrm{rgldim}\, R + 3$ (when R is a field, $\mathrm{rgldim}\, A = 2$; so the bound is not sharp in this case). □

In using Corollary 4 to show that the ring A has finite global dimension, it is necessary to compute two flat dimensions. The following corollary shows that often it is, in fact, necessary to compute only one.

COROLLARY 7. *If A is a subring of a ring B of finite left global dimension with Q an ideal of B, $Q \leq A$,* $\mathrm{fd}(Q_A) < \infty$, $\mathrm{rgldim}(A/Q) < \infty$ *and* $\mathrm{lgldim}(A/Q) < \infty$ *then* $\mathrm{lgldim}\, A < \infty$.

Proof. By Corollary 4 (or Theorem 1) it suffices to show that $\mathrm{fd}(B_A) < \infty$. Consider the exact sequences of right A-modules $0 \to Q \to B \to B/Q \to 0$. Since $\mathrm{fd}(B/Q)_A \leq \mathrm{fd}(B/Q)_{(A/Q)} + \mathrm{fd}(A/Q)_A$ by [**McR**, Proposition 2.2], $\mathrm{fd}(B/Q)_{A/Q} < \infty$ so $\mathrm{fd}(B_A) < \infty$. □

We note that we have constructed a ring R of finite global dimension which is a fibre product of two rings of infinite global dimension, so that the conditions of Corollary 7 (or Theorems 1 or 2) are not necessary conditions for the ring R to have finite global dimension. The problem of determining the global dimension of R from homological properties of the rings or modules in the commutative diagram seems difficult, except in some special cases. For example, when R_1,

R_2 are von Neumann regular, so is R, and it is not difficult to show that $\operatorname{rgldim} R = \max\{\operatorname{rgldim} R_i\}$. More generally we have the following proposition (which applies to examples of Robson **[R2**, §2] and Osofsky [**V**, p. 29–30]).

PROPOSITION 8. *Let R be the fibre product of R_1 and R_2 over $R_1/U_1 \cong R_2/U_2$. Suppose that both U_i are idempotent, and $(U_i)_{R_i}$ are flat. Then $U_1 \oplus U_2$ is a flat right R-module and*

$$\max_i\{\operatorname{lgldim} R_i\} \leq \operatorname{lgldim} R \leq \max_i\{\operatorname{lgldim} R_i\} + 1.$$

Proof. We will show that $(U_1, 0)$ is right R-flat. Let I be a left ideal of R; we need to show that $(U_1, 0) \otimes_R I \to (U_1, 0)I$ is one-to-one. Since $(U_1, 0)^2 = (U_1, 0)$, $(U_1, 0) \otimes_R I = (U_1, 0) \otimes_R (U_1, 0)I$ and hence, without loss of generality, we may assume $I = (J, 0)$ for $J \leq R_1$. Now $(U_1, 0) \otimes_R (J, 0) \cong U_1 \otimes_{R_1} J$ because $R/(0, U_2) \cong R_1$, $(J, 0)(0, U_2) = 0 = (U_1, 0)(0, U_2)$, and $(U_1, 0)(J, 0) = (U_1 J, 0)$. But $U_1 \otimes_{R_1} J \to U_1 J$ is one-to-one since $(U_1)_{R_1}$ is flat. Similarly $(0, U_2)$ is right R-flat. The upper bound then follows from Theorem 2, thinking of R as arising from the Cartesian square:

$$\begin{array}{ccc} R & \longrightarrow & R/(0, U_2) \cong R_1 \\ \downarrow & & \downarrow \\ R_2 \cong R/(U_1, 0) & \longrightarrow & R'. \end{array}$$

Since $R/(0, U_2) \cong R_1$, and since $(0, U_2)$ is a flat idempotent right ideal of R, it follows from Fields **[F2]** that $\operatorname{lgldim} R \geq \operatorname{lgldim} R_1$. Similarly $\operatorname{lgldim} R \geq \operatorname{lgldim} R_2$. □

As an example where Proposition 8 can be applied, we present the following:

EXAMPLE 9. Let

$$R = \left[\begin{pmatrix} Z & Z \\ 2Z & Z^* \end{pmatrix}, \begin{pmatrix} Z & 2Z \\ Z & Z^* \end{pmatrix}\right]$$

where Z is the integers and * entries agree modulo 2. Here

$$R_1 = \begin{pmatrix} Z & Z \\ 2Z & Z \end{pmatrix}, \quad R_2 = \begin{pmatrix} Z & 2Z \\ Z & Z \end{pmatrix},$$
$$U_1 = \begin{pmatrix} Z & Z \\ 2Z & 2Z \end{pmatrix}, \quad U_2 = \begin{pmatrix} Z & 2Z \\ Z & 2Z \end{pmatrix}.$$

As R is not hereditary, Proposition 8 shows that $\operatorname{gldim} R = 2$. We note that R is not a right or left subidealizer in $M_2(Z) \oplus M_2(Z)$, so the trick of thinking of R as a subidealizer used in **[R2]** and **[KK2]** cannot be used to show that R has finite global dimension. □

Proposition 8 does not extend to nilpotent ideals (or hence to eventually idempotent ideals) or to idempotent ideals of finite flat dimension.

EXAMPLES 10. (a) Let

$$R = \left[\begin{pmatrix} a & 0 \\ c & b \end{pmatrix}, \begin{pmatrix} a & d \\ 0 & b \end{pmatrix}\right]$$

where $a, b, c, d \in k$, a field. It is not hard to show that R has infinite global dimension, despite the fact that the R_i are hereditary and the U_i are projective, nilpotent ideals.

(b) Let

$$R_1 = R_2 = \begin{bmatrix} Z & 2Z & 4Z \\ Z & Z & 2Z \\ Z & Z & Z \end{bmatrix},$$

a ring of gldim $= 2$. Let

$$U_1 = U_2 = \begin{bmatrix} Z & 2Z & 4Z \\ Z & 2Z & 2Z \\ Z & Z & Z \end{bmatrix},$$

an idempotent ideal of flat dimension 1. Then

$$R = \left[\begin{pmatrix} Z & 2Z & 4Z \\ Z & Z^* & 2Z \\ Z & Z & Z \end{pmatrix}, \begin{pmatrix} Z & 2Z & 4Z \\ Z & Z^* & 2Z \\ Z & Z & Z \end{pmatrix}\right]$$

where the indicated entries agree modulo 2. Since the exact sequences below do not split, R has infinite right global dimension:

$$0 \to ([2Z, 2Z, 4Z], [0,0,0]) \longrightarrow \begin{matrix} ([Z, 2Z, 4Z], [0,0,0]) \\ \oplus \\ ([2Z, 2Z, 2Z], [0,0,0]) \end{matrix} \longrightarrow ([Z, 2Z, 2Z], [0,0,0]) \to 0$$

$$0 \to ([0,0,0], [Z, 2Z, 2Z] \to ([Z, Z^*, 2Z]), [Z, Z^*, 2Z]) \to ([Z, Z, 2Z], [0,0,0]) \to 0. \quad \square$$

We next calculate the global dimension of the particular rings $R_n = Z + (x_1, \ldots, x_n)Q[x_1, \ldots, x_n]$ where Z is the integers and Q is the rationals. Such rings were considered by Carrig **[C**, Example 1.8] and are mentioned by Greenberg **[G2]** for $n \geq 2$ as behaving differently than when the common ideal is flat; they are symmetric algebras $R_n = S(M)$ where $M = Q^{(n)} = Q \oplus \cdots \oplus Q$, over Z. Carrig was able to show that $\operatorname{gldim} R_n \leq n+1$ by showing that $\operatorname{wdim}(R_n) = n$ (where wdim stands for the weak or Tor dimension) and then using Jensen's lemma **[Je]** and the fact that R_n is countable to conclude that $\operatorname{gldim}(R_n) \leq n+1$. If $R_n = D + (x_1, \ldots, x_n)K[x_1, \ldots, x_n]$ for any Dedekind domain D (not necessarily countable) with quotient field K, Corollary 4 shows that $\operatorname{gldim}(R_n) \leq n+1$ taking $A = R_n$, $B = K[x_1, \ldots, x_n]$, $Q = (x_1, \ldots, x_n)K[x_1, \ldots, x_n]$, and $\operatorname{fd}(A/Q)_A = n$, $\operatorname{gldim}(A/Q) = 1$, $\operatorname{gldim} B = n$, and $\operatorname{fd}(B_A) = 0$. Using chain conditions, Carrig notes that $\operatorname{gldim} R_1 = 2$ (since R_1 is not Noetherian) and $\operatorname{gldim} R_2 = 3$ (since R_2 is not coherent); he conjectures that $\operatorname{gldim} R_n = n+1$, which we will prove using generalizations of two change of rings theorems. Our proofs follow those of Kaplansky **[K]**. The original theorems concern the change of rings from A to A/xA where x is a central regular element of A; our generalizations concern the change of rings from A to A/xB where xB is a shared ideal between A and a flat epimorphic image B.

LEMMA 11. (*Compare to* **[K**, *Theorem 8, p.* 176].) *Let A be a subring of B, x a regular element of B with $Bx = xB \leq A$ and ${}_AB$ flat. Let T be a submodule of a free A-module. Then* $\operatorname{pd}(T/T(xB))_{A^*} \leq \operatorname{pd}(T)_A$, *where $A^* = A/xB$.*

Proof. Since $Bx \cong B$, $\operatorname{fd}({}_AA^*) \leq 1$. Taking a projective A-resolution of T, $0 \to P_k \to \cdots \to P_1 \to P_0 \to T \to 0$ and tensoring over A with A^* we get $0 \to P_k \otimes_A A^* \to \cdots \to P_0 \otimes_A A^* \to T \otimes_A A^* \cong T/T(xB) \to 0$ since T is a submodule of a free R-module and $\operatorname{fd}({}_AA^*) \leq 1$. □

PROPOSITION 12. (*Compare with* **[K**, *Theorem* 3, *p.* 172].) *Let A be a subring of B, x a central regular element of B, $xB \leq A$, ${}_AB$ flat, and B an epimorphic image of A (i.e. $B \otimes_A B \cong B$); then for any right $B^* = B/xB$-module C, with* $\operatorname{pd} C_{A^*}$ *finite,* $\operatorname{pd} C_A \geq \operatorname{pd} C_{A^*} + 1$, *where $A^* = A/xB$.*

Proof. The result is clear when $\operatorname{pd}(C_{A^*}) = 0$. Suppose that $\operatorname{pd} C_{A^*} = n$ and $\operatorname{pd} C_A \leq n$. Let H be a free A-module mapping onto C

$$0 \to T \to H \to C \to 0 \qquad (\text{-**-})$$

so $\operatorname{pd} T_A \leq n-1$. We have $0 \to T/H(xB) \to H/H(xB) \to C \to 0$ exact, so $\operatorname{pd}(T/H(xB))_{A^*} \leq n-1$ (assuming $n \geq 1$). By Lemma 11 $\operatorname{pd}(T/TxB)_{A^*} \leq n-1$, so the exact sequence $0 \to HxB/TxB \to T/TxB \to T/HxB \to 0$ yields $\operatorname{pd}(HxB/TxB)_{A^*} \leq n-1$. But tensoring (-∗∗-) above over A with B gives

$$\begin{array}{ccccccc} 0 \to & T \otimes_A B & \to & H \otimes_A B & \to C \otimes_A B \to 0 \\ & \| \cong & & \| \cong & \\ & T \otimes_A Bx & \to & H \otimes_A Bx & \\ & \| \cong & & \| \cong & \\ & TBx & \to & HBx & \end{array}$$

Then $(HxB)/(TxB) \cong C \otimes_A B \cong C \otimes_B B$ since $B \otimes_A B \cong B$; but $C \otimes_B B \cong C \otimes_{B^*} B^* \cong C$ so $\operatorname{pd}(C_{A^*}) \leq n-1$, a contradiction. □

THEOREM 13. *For $R_n = D + (x_1, \dots, x_n)K[x_1, \dots, x_n]$ for D a Dedekind domain with quotient field K,* $\operatorname{gldim} R_n = n+1$.

Proof. By remarks above, it suffices to show $n+1 \leq \operatorname{gldim} R_n$, which will be shown inductively. We know that $\operatorname{gldim} R_1 = 2$, and it is not hard to show that $\operatorname{pd}(K[x_1]/\langle x_1 \rangle) = 2$; inductively assume $\operatorname{pd}(K[x_1, \dots, x_{n-1}]/\langle x_1, \dots, x_{n-1} \rangle)_{R_{n-1}} = n$. In Proposition 12, let $A = R_n$, $B = K[x_1, \dots, x_n]$, $C = K[x_1, \dots, x_n]/\langle x_1, \dots, x_n \rangle$ and $x = x_n$; then since $A^* = A/x_nB = R_{n-1}$, we have $\operatorname{pd} C_{R_n} \geq \operatorname{pd} C_{R_{n-1}} + 1 = n+1$. □

We conclude with the following example which illustrates how the preceding techniques can be used to calculate (or bound) the global dimensions of particular rings.

EXAMPLE 14. Let k be a field,

$$R = k[x_1, \dots, x_n] + (t_1, \dots, t_m)k(x_1, \dots, x_n)[t_1, \dots, t_m],$$

$$I = (t_1, \dots, t_m)k(x_1, \dots, x_n)[t_1, \dots, t_m], \quad S = k(x_1, \dots, x_n)[t_1, \dots, t_m],$$

$$A = \begin{bmatrix} R & I \\ S & S \end{bmatrix}, \quad Q = \begin{bmatrix} I & I \\ S & S \end{bmatrix}, \quad B = \begin{bmatrix} S & I \\ S & S \end{bmatrix}, \text{ and } C = \begin{bmatrix} S & S \\ S & S \end{bmatrix}.$$

CLAIM.

$$\begin{aligned} \operatorname{rgldim} A &= \max\{m, n, \operatorname{pd}(B/Q)_{A/Q} + 1\} \\ &= \max\{m, n, \operatorname{pd}_{k[x_1, \dots, x_n]} k(x_1, \dots, x_n) + 1\}. \end{aligned}$$

Since B is a flat epimorphic image of A we have $m \leq \operatorname{rgldim}(A)$; since Q is an idempotent, projective left A-module, $n \leq \operatorname{rgldim}(A)$ by [**F2**]. As in [**G2**, Proposition 3.11], note that B is isomorphic to a right ideal of A, and hence by [**W**, Proposition 3.3]

$$\begin{aligned}\operatorname{pd} B_A &= \max\{\operatorname{pd}(B \otimes_A B)_B, \operatorname{pd}(B \otimes_A (A/Q))_{(A/Q)}\}\\ &= \max\{\operatorname{pd} B_B, \operatorname{pd}(B/Q)_{(A/Q)}\}\\ &= \operatorname{pd}_{k[x_1,\dots,x_n]} k(x_1,\dots,x_n);\end{aligned}$$

therefore $\operatorname{rgldim} A \geq \operatorname{pd}_{k[x_1,\dots,x_n]} k(x_1,\dots,x_n) + 1$.

To show equality, let I be a right ideal of A. As in [**G2**, Lemma 2.3], $I \leq F_A \leq F_B$ where F_A is a free right A-module and F_B is a free right B-module. Then $IQ \leq I \leq IB$, so that $I/IQ \leq IB/IQ$, a module over B/Q, a field. Hence I/IQ is contained in a free B/Q-module, and we have the exact sequence $0 \to I/IQ \to \bigoplus B/Q \to \text{cokernel} \to 0$. If $\operatorname{pd}(B/Q)_{(A/Q)} \lneqq n$, then $\operatorname{pd}(I/IQ) \lneqq n$; if $\operatorname{pd}(B/Q)_{(A/Q)} = n$, then $\operatorname{pd}(I/IQ) \leq n$. By [**W**, Proposition 3.3]

$$\begin{aligned}\operatorname{pd}(I_A) &= \max\{\operatorname{pd}(I \otimes_A B)_B,\ \operatorname{pd}(I \otimes_A (A/Q))\}\\ &= \max\{\operatorname{pd}(IB)_B,\ \max\{\operatorname{pd}(B/Q)_{(A/Q)},\ n-1\}\}\\ &\leq \max\{m-1,\ \operatorname{pd}(B/Q)_{(A/Q)},\ n-1\}\end{aligned}$$

so $\operatorname{rgldim} A \leq \max\{m, \operatorname{pd}(B/Q)_{(A/Q)} + 1, n\}$.

CLAIM. $\max\{\operatorname{pd}(B/Q)_{(A/Q)} + m, n\} \leq \operatorname{lgldim} A \leq n + m$.

Since a projective resolution of Q over B gives a flat resolution of Q over A, $\operatorname{fd}_A(A/Q) \leq m$, and the upper bound follows from Theorem 2.

To obtain the lower bound, consider first the case in which $m = 1$. Let $u = \left[\begin{smallmatrix} t_1 & 0 \\ 0 & 1\end{smallmatrix}\right]$; then $uAu^{-1} = \left[\begin{smallmatrix} R & S \\ I & S\end{smallmatrix}\right]$ so that $\operatorname{lgldim} A = \operatorname{rgldim} A = \max\{\operatorname{pd}_{A/Q}(B/Q) + 1, n\}$. For an arbitrary m, let

$$Q' = \begin{bmatrix} t_1 S & t_1 S \\ t_1 S & t_1 S \end{bmatrix} = \begin{bmatrix} t_1 & 0 \\ 0 & t_1 \end{bmatrix} C \leq A$$

and $Q' \leq C$. Note that A/Q' is isomorphic to a similar ring A with one fewer t_j. Both A and B are subidealizers in C, so by [**R1**, Lemma 2.1] $C \otimes_B C \cong C \cong C \otimes_A C$. Furthermore, C is left and right projective over B and C is right projective and left flat over A.

By Proposition 12, $\operatorname{pd}_A(C/Q') \geq \operatorname{pd}_{(A/Q')}(C/Q') + 1$, so inductively $\operatorname{lgldim} A \geq \operatorname{pd}_{k[x_1,\dots,x_n]} k(x_1, \dots, x_n) + m$. As in the case of the right global dimension of A, **[F2]** implies that $\operatorname{lgldim} A \geq n$. □

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