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FUNCTIONS IN $R^2(E)$ AND POINTS OF THE FINE INTERIOR

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Let $E \subset \mathbb{C}$ be a set that is compact in the usual topology. Let m denote 2-dimensional Lebesgue measure. We denote by $R_0(E)$ the algebra of rational functions with poles off E. For $p \geq 1$, let $L^p(E) = L^p(E, dm)$. The closure of $R_0(E)$ in $L^p(E)$ will be denoted by $R^p(E)$.

In this paper we study the behavior of functions in $R^2(E)$ at points of the fine interior of E. We prove that if $U \subset E$ is a finely open set of bounded point evaluations for $R^2(E)$, then there is a finely open set $V \subset U$ such that each $x \in V$ is a bounded point derivation of all orders for $R^2(E)$. We also prove that if $R^2(E) \neq L^2(E)$, there is a subset $S \subset E$ having positive measure such that if $x \in S$ each function in $\bigcup_{p>2} R^p(E)$ is approximately continuous at x. Moreover, this approximate continuity is uniform on the unit ball of a normed linear space that contains $\bigcup_{p>2} R^p(E)$.

1. Introduction. Let $E \subset \mathbb{C}$ be a set that is compact in the usual topology. Let m denote 2-dimensional Lebesgue measure. We denote by $R_0(E)$ the algebra of rational functions with poles off E. For $p \geq 1$, let $L^p(E) = L^p(E, dm)$. The closure of $R_0(E)$ in $L^p(E)$ will be denoted by $R^p(E)$.

In **[16]** we studied the smoothness properties of functions in $R^p(E)$, $p > 2$, at bounded point evaluations. The case $p = 2$ is different. Fernström has shown in [7] that $R^2(E)$ can be unequal to $L^2(E)$ without there being any bounded point evaluations for $R^2(E)$. In this paper we use the fine topology introduced by Cartan to study the behavior of functions in $R^2(E)$ at points of the fine interior of E. We prove that if $U \subset E$ is a finely open set of bounded point evaluations for $R^2(E)$, then there is a finely open set $V \subset U$ such that each $x \in V$ is a bounded point derivation of all orders for $R^2(E)$. Finely open sets of this kind are contained in certain "Swiss cheese sets". We also prove that if $R^2(E) \neq L^2(E)$, there is a set $S \subset E$ having positive measure such that if $x \in S$ each function in $\bigcup_{p>2} R^p(E)$ is approximately continuous at x. Moreover, this approximate continuity is uniform on the unit ball of a normed linear space that contains $\bigcup_{p>2} R^p(E)$.

2. Functions in $R^2(E)$ defined on finely open sets. When $R^2(E) \neq L^2(E)$, the fine interior is non-empty. This follows from a theorem of Havin [**10**] that we shall now state. Let $\Delta(x, r)$ denote the open disk of radius r centered at x. Let C_2 denote the Wiener capacity as defined in [**11**].

THEOREM 2.1 (*Havin*). *Let $E \subset \mathbb{C}$ be a compact set without interior in the usual topology. Then $R^2(E) \neq L^2(E)$ if and only if there is a set $S \subset E$ having positive measure such that for $x \in S$,*

$$\limsup_{r \to 0} \frac{C_2(\Delta(x, r) \backslash E)}{r^2} = 0.$$

One way to relate this theorem to fine interior points is to use Wiener's criterion. Let

$$A_n(x) = \left\{ z : \frac{1}{2^{n+1}} \leq |z - x| \leq \frac{1}{2^n} \right\}.$$

Then x is a fine interior point of E if and only if

$$\sum_{n=1}^{\infty} n C_2(A_n(x) \backslash E) < \infty.$$

For a proof see [**11**, p. 220]. It follows from Wiener's criterion and Theorem 2.1 that if $R^2(E) \neq L^2(E)$, the fine interior has positive measure.

Each point of the fine interior has a system of fine neighborhoods that are compact in the usual topology (see [**2**]). Debiard and Gaveau observed in [**5**] that if the fine interior of E is nonempty, it satisfies the Baire property: The intersection of a countable number of open dense sets in E is always dense in E. We give the following proof.

PROPOSITION 2.1. *If E is a set having non-empty fine interior E', then E' satisfies the Baire property.*

Proof. Let $D_1, D_2, \ldots$ be a sequence of finely open dense sets in E. We must show that for each finely open set $U \subset E'$, $U \cap (\bigcap_1^\infty D_i) \neq \varnothing$. Now $U \cap D_1 \neq \varnothing$ because D_1 is dense. Pick $x_1 \in U \cap D_1$ and a fine neighborhood B_1 of x_1 such that B_1 is compact in the usual topology. Since D_2 is dense, there exists $x_2 \in B_1 \cap D_2$ and a fine neighborhood B_2 of x_2 compact in the usual topology such that $B_2 \subset B_1 \cap D_2$. Continuing in this way, we get a sequence $\{B_n\}$ of compact finely open sets such that $B_n \subset B_{n-1} \cap D_n$. Since B_1 is compact, the finite intersection property implies that $\bigcap_1^\infty B_n \neq \emptyset$. Hence $\bigcap_1^\infty D_n \neq \emptyset$, and E' satisfies the Baire property.

Each point of E is a point of full area density for E (see [**6**, p. 170]). Moreover, one can use results in [**1**, p. 43], due to Beurling to show that any finely open subset of $\mathbb{C}$ includes circles of arbitrarily small radii centered at each of its points. Next we define those points of the fine interior at which the functions in $R^2(E)$ may have smoothness properties.

Definition 2.1. A point $x \in E$ is a bounded point evaluation (BPE) for $R^2(E)$ if there exists a constant C such that

$$|f(x)| \leq C\|f\|_{L^2(E)}$$

for all $f \in R_0(E)$.

Definition 2.2. A point $x \in E$ is a bounded point derivation (BPD) of order s for $R^2(E)$ if there exists a constant C such that

$$|f^{(s)}(x)| \leq C\|f\|_{L^2(E)}$$

for all $f \in R_0(E)$.

If x is a BPE for $R^2(E)$, the map $f \mapsto f(x)$ extends from $R_0(E)$ to a bounded linear functional on $R^2(E)$. Let $N(x)$ equal the norm of this linear functional. We will need the following lemma and proposition.

Lemma 2.1. *The function N is lower semi-continuous on the set of BPE's for $R^2(E)$.*

For the proof see [**16**, p. 72].

The proof of the next statement is in [**15**, p. 148].

Proposition 2.2. *Let $f\colon X \mapsto \mathbb{R}$ be a lower semi-continuous function on a Baire space X. Every non-empty open set in X contains a non-empty open set on which f is uniformly bounded.*

If $X \subset \mathbb{C}$ is compact in the usual topology, we let $R(X)$ denote the closure of $R_0(X)$ in the sup norm on X.

Theorem 2.2. *Suppose that $U \subset E'$ is a finely open set such that every point of U is a BPE for $R^2(E)$. Then there is a compact set $X \subset U$ such that X has non-empty fine interior, and for each $f \in R^2(E)$, $f|_X \in R(X)$.*

Proof. Let $U \subset E'$ be a finely open set of BPE's for $R^2(E)$. By Proposition 2.2 there is a finely open set $V \subset U$ on which the $R^2(E)$

norm of "evaluation at x" is bounded. Let $X \subset V$ be a set that is compact in the usual topology and that contains a finely open set. Let $f \in R^2(E)$. Then there is a sequence $\{f_n\}$ in $R_0(E)$ such that $||f_n - f||_{L^2(E)} \to 0$. By the choice of X there is a constant C such that

$$\sup_{z \in X} |f_n(z) - f_m(z)| \leq C||f_n - f_m||_{L^2(E)}.$$

Thus the sequence obtained by restricting the f_n's to X converges in $R(X)$ to the restriction of f to X. We conclude that $f|_X \in R(X)$.

Let X be as in the above theorem.

COROLLARY 2.2. *Every point of X is a BPD of all orders for $R^2(E)$.*

Proof. Let $x \in X$, and let s be a positive integer. By **[4]**, x is a BPD of all orders for $R(X)$. Hence there is a constant C such that if $f \in R_0(E)$, $|f^{(s)}(x)| \leq C||f||_X$ where $|| \; ||_X$ denotes the sup norm on X. By the choice of X (see the proof of Theorem 2.2), there is another constant C' such that $||f||_X \leq C'||f||_{L^2(E)}$. Taken together these inequalities imply that x is a BPD of order s for $R^2(E)$.

There do exist examples of compact nowhere dense sets E that contain finely open subsets of BPE's for $R^2(E)$.

3. The case of no BPE's **for** $R^2(E)$. In this section we show that whenever $R^2(E) \neq L^2(E)$, there is a subset of E on which functions in $R^2(E)$ that are not continuous may still have smoothness properties. To describe this set of points we begin by letting φ be a positive function defined on $(0, \infty)$ such that φ is decreasing and $\lim_{r \downarrow 0^+} \varphi(r) = \infty$.

DEFINITION 3.1. A point $x \in E$ is a BPE of type φ for $R^2(E)$ if there is a constant C such that

$$|f(x)| \leq C \left\{ \int_E |f(z)|^2 \varphi(|z - x|) \, dm(z) \right\}^{1/2}$$

for all $f \in R_0(E)$.

Fernström introduced BPE's of type $\varphi(r) = \log^\beta 1/r$ for $\beta > 1$ in **[7]**. The proof of the following theorem is similar to the proof of Theorem 3 in **[8]**.

THEOREM 3.1. *Let $E \subset \mathbb{C}$ be compact. Then x is a BPE of type φ for $R^2(E)$ if and only if*

$$\sum_{n=1}^{\infty} \varphi^{-1}(2^{-n}) 2^{2n} C_2(A_n(x) \backslash E) < \infty.$$

For certain φ's the above series will converge on a set of positive measure whenever $R^2(E) \neq L^2(E)$.

DEFINITION 3.2. A non-negative, real-valued function φ defined on $(0, \infty)$ is *nice* if it satisfies the following conditions:

(i) There is an $r_0 > 0$ such that φ is decreasing on $(0, r_0)$, and $\lim_{r\downarrow 0^+} \varphi(r) = +\infty$.

(ii) $\lim_{r\downarrow 0^+} r \cdot \varphi(r) = 0$, and there is an $s_0 > 0$ such that $1/(r \cdot \varphi(r))$ is decreasing on $(0, s_0)$; and

(iii) there is a $t_0 > 0$ such that $\int_0^{t_0} (1/(r \cdot \varphi(r)))\, dr < \infty$.

EXAMPLES.

$$\varphi(r) = \frac{1}{r^\alpha}, \qquad 0 < \alpha < 1. \tag{1}$$

$$\varphi(r) = \log^\beta \frac{1}{r}, \quad \beta > 1, \quad \text{for } 0 < r \leq 1, \quad \varphi(r) = 0 \quad \text{for } r > 1. \tag{2}$$

$$\varphi(r) = \left(\log \frac{1}{r}\right) \left[\log \left(\log \frac{1}{r}\right)\right]^\beta, \ \beta > 1, \quad \text{for } 0 < r \leq 1/2, \tag{3}$$
$$\varphi(r) = (\log 2) \cdot [\log(\log 2)]^\beta, \quad \text{for } r > 1/2.$$

Condition (iii) of Definition 3.2 combined with Theorem 2.1 and Theorem 3.1 imply the following:

THEOREM 3.2. *Let $E \subset \mathbb{C}$ be a compact set without interior in the usual topology. Let φ be nice. Then if $R^2(E) \neq L^2(E)$ the set of BPE's of type φ has positive measure.*

Let S denote the set of $x \in E$ such that $\limsup_{r\to 0} C_2(\Delta(x, r)\backslash E)/r^2 = 0$. Suppose that $x \in E$ is a BPE of type φ. We define a norm $\|\ \|_\varphi$ on functions in $L^2(E)$ as follows:

$$\|f\|_\varphi = \sup_{y\in S} \|f \cdot \varphi(|z - y|) \cdot \varphi(|z - x|)\|_{L^2(E)}$$

where f is a function of z. Let $R^\varphi(E)$ be the closure of $R_0(E)$ in this norm. For certain φ such as $\varphi(r) = \log^\beta 1/r$, $\beta > 0$, Hölder's inequality implies that $\bigcup_{p>2} R^p(E) \subset R^\varphi(E)$.

Now suppose that x is a BPE of type φ. Let $L^2(E, \varphi\, dm)$ be the space of all complex measurable functions f defined on E such that

$\{\int_E |f^2(z)| \cdot \varphi(|z-x|)\, dm(z)\}^{1/2} < \infty$. By a well known theorem **[14]**, there is a function $g \in L^2(E, \varphi\, dm)$ such that

$$f(x) = \int_E f \cdot g \cdot \varphi(|z-x|)\, dm(z)$$

for all $f \in R_0(E)$. We have the following theorem.

THEOREM 3.3. *Let φ be a nice function such that $\int_0^1 \varphi^3(r) r\, dr < \infty$. Suppose that $x \in E$ is a BPE of type φ. Let $\varepsilon > 0$. Then there is a set $A \subset E$ having full area density at x such that if $y \in A$ and $f \in R_0(E)$,*

$$|f(y) - f(x)| \leq \varepsilon \|f\|_\varphi.$$

We will give an outline of the proof. For more details see **[16]**.

Outline of Proof of Theorem 3.3. Let $\varepsilon > 0$. Let $g \in L^2(E, \varphi\, dm)$ be the representing function for x as defined above. Then if

$$c(y) = \int_E \frac{z-x}{z-y} g(z) \cdot \varphi(|z-x|)\, dm(z)$$

is defined and $\neq 0$,

$$\frac{1}{c(y)} \frac{z-x}{z-y} g(z) \cdot \varphi(|z-x|)$$

is a representing function for y. Among the points where $c(y)$ is defined are those in the set A_1 of the following lemma:

LEMMA 3.1. *For each $\delta > 0$, the sets*

$$A_1 = \left\{ y \in \mathbb{C} \colon |y-x| \int_E \frac{|g(z)| \cdot \varphi(|z-x|)}{|z-y|}\, dm(z) < \delta \right\} \quad \textit{and}$$

$$A_2 = \left\{ y \in \mathbb{C} \colon |y-x| \left[\int_E \frac{|g(z)|^2 \cdot \varphi(|z-x|)}{|z-y|^2 \cdot \varphi^2(|z-y|)}\, dm(z) \right]^{1/2} < \delta \right\}$$

have full area density at x.

The proof uses the properties of the nice function φ and is similar to that of Lemma 3.3 in **[16]**. Now if $c(y)$ is defined and $\neq 0$, and if

$f \in R_0(E)$, we have

$$\begin{aligned} f(y) - f(x) &= \frac{1}{c(y)} \int_E \frac{[f(z) - f(x)] \cdot (z - x)}{(z - y)} g(z) \cdot \varphi(|z - x|)\, dm(z) \\ &= \frac{1}{c(y)} \int_E [f(z) - f(x)] \left[1 + \frac{y - x}{z - y}\right] g(z) \cdot \varphi(|z - x|)\, dm(z) \\ &= \frac{y - x}{c(y)} \int_E \left[\frac{f(z) - f(x)}{z - y}\right] g(z) \cdot \varphi(|z - x|)\, dm(z) \\ &= \frac{y - x}{c(y)} \int_E \frac{f(z) - f(x)}{z - y} \frac{\varphi(|z - y|)}{\varphi(|z - y|)} g(z) \cdot \varphi(|z - x|)\, dm(z). \end{aligned}$$

From Hölder's inequality, the assumption that x is a BPE of type φ, and the assumption that $\int_0^1 \varphi^3(r) r\, dr < \infty$, it follows that

$$|f(y) - f(x)| \le \frac{C|y - x|}{c(y)} \|f\|_\varphi \left\{ \int_E \frac{|g(z)|^2 \cdot \varphi(|z - x|)}{|z - y|^2 \cdot \varphi^2(|z - y|)}\, dm(z) \right\}^{1/2}$$

where C is independent of f.

Choose $\delta > 0$ so small that if $y \in A_1 \cap A_2$ (see Lemma 3.1), then

$$\frac{C}{c(y)} |y - x| \left\{ \int_E \frac{|g(z)|^2 \cdot \varphi(|z - x|)}{|z - y|^2 \cdot \varphi^2(|z - y|)}\, dm(z) \right\}^{1/2} < \varepsilon.$$

Lemma 3.1 implies that the set $A = A_1 \cap A_2$ has full area density at x. Moreover, if $y \in A$ and $f \in R_0(E)$,

$$|f(y) - f(x)| \le \varepsilon \|f\|_\varphi.$$

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