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K-THEORY FOR GRADED BANACH ALGEBRAS II

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Let A be a real or complex Banach algebra and assume that A is equipped with a continuous automorphism α such that α^2 is the identity. In "K-theory for graded Banach algebras I" we have associated a group $K(A)$ to such a pair (A, α). In this paper we prove that this group $K(A)$ is isomorphic with $K(SA\hat{\otimes}C)$ where SA is the algebra of continuous functions $f : [0, 1] \to A$ with $f(0) = f(1) = 0$ and equipped with pointwise operations and where $SA\hat{\otimes}C$ denotes the graded tensor product of SA with the Clifford algebra $C = C^{0,1}$. The periodicity of Clifford algebras is used to show that $K(S^8A) = K(A)$ in general and $K(S^2A) = K(A)$ in the complex case. All this gives rise to an important periodic exact sequence associated to an algebra A and an invariant closed ideal I with

$$K(I) \to K(A) \to K(A/I) \to K(I\hat{\otimes}C) \to K(A\hat{\otimes}C) \to K(A/I\hat{\otimes}C)$$

as its typical part. The usual 6-term periodic exact sequence with K_0 and K_1 is a special case of this sequence.

1. Introduction. In a previous paper we have defined an abelian group $K(A)$ for any real or complex Banach algebra A equipped with a $\mathbf{Z}_2$-grading [4]. For convenience we work with an involutive automorphism α that determines the grading in the sense that $\deg a = 0$ if $\alpha(a) = a$ and $\deg a = 1$ if $\alpha(a) = -a$. The main result in [4] is the usual exact sequence

$$K(SI) \to K(SA) \to K(S(A/I)) \to K(I) \to K(A) \to K(A/I)$$

for any invariant closed two-sided ideal I of A. As usual SA is the algebra of continuous functions $f : [0, 1] \to A$ such that $f(0) = f(1) = 0$ with pointwise operations and supremum norm.

In this paper we will obtain another important exact sequence. It is related to the following lifting problem. As above let I be an invariant closed two-sided ideal of A. Denote by π the quotient map and also use the symbol α for the induced involution on A/I. Assume that A has an identity and take an element $x \in A/I$ such that $x^2 = 1$ and $\alpha(x) = -x$. Of course there is an element $a \in A$ such that $\pi(a) = x$ and by taking $\frac{1}{2}(a - \alpha(a))$ we may even assume that also $\alpha(a) = -a$. In general however it will not be possible to find a lifting a such that also $a^2 = 1$.

Consider the following two motivating examples. First let α be the flip automorphism on $A \oplus A$. Then any element $a \in A \oplus A$ with $a^2 = 1$ and $\alpha(a) = -a$ has the form $a = (b, -b)$ with $b = 2p - 1$ and p a projection (i.e. $p^2 = p$) in A. It is well known that projections in a quotient cannot always be lifted. Secondly let α be the automorphism on $M_2(A)$, the algebra of 2×2 matrices over A, defined by

$$\alpha\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = \begin{pmatrix} a & -b \\ -c & d \end{pmatrix}.$$

Then any element $a \in M_2(A)$ with $a^2 = 1$ and $\alpha(a) = -a$ has the form

$$a = \begin{pmatrix} 0 & b \\ b^{-1} & 0 \end{pmatrix}$$

with b invertible in A. Again it is well known that invertible elements cannot always be lifted to invertible elements.

So the general question arises to "measure" the obstruction for an element $x \in A/I$ with $x^2 = 1$ and $\alpha(x) = -x$ to have a lifting a such that also $a^2 = 1$ and $\alpha(a) = -a$. The idea is the following. First construct a good lifting in a somewhat bigger algebra and then "measure" the distance of this lifting to the original algebra. This is in fact also the basic idea of the connecting map $K(S(A/I)) \to K(I)$ in **[4]**. The bigger algebra there is CA, the algebra of continuous functions $f : [0, 1] \to A$ with $f(0) = 0$. The distance of an element $f \in CA$ to SA can be measured simply by $f(1)$.

Here we consider the crossed product of A by the action α of $\mathbf{Z}_2$ as the larger algebra. We will denote this by $A \hat{\otimes} C$ (for reasons which will be made clear in the text) and use the common realisation $\{a + bf | a, b \in A\}$ with $f^2 = 1$ and $fa = \alpha(a)f$ for all $a \in A$. The automorphism on $A \hat{\otimes} C$ will not be the dual action from crossed product theory but the automorphism $\tilde{\alpha}$ defined by $\tilde{\alpha}(a + bf) = \alpha(a) - \alpha(b)f$.

Then it is not hard to show that if $a \in A$ and $\alpha(a) = -a$ then $y = \sin \frac{1}{2}\pi a + f \cos \frac{1}{2}\pi a$ satisfies $y^2 = 1$ and $\tilde{\alpha}(y) = -y$. Moreover if $\pi(a) = x$ and $x^2 = 1$ then $\pi(y) = x$ simply because $\sin \frac{1}{2}\pi x = x \sin \frac{\pi}{2} = x$ and $\cos \frac{1}{2}\pi x = \cos \frac{\pi}{2} = 0$. So y would be a good lifting in $A \hat{\otimes} C$.

If y would be in A itself then $-yfy = -f\alpha(y)y = fy^2 = f$. So we could use $-yfy$ as a sort of measure for the obstruction. This element $z = -yfy$ has the property that $z^2 = 1$ and $\tilde{\alpha}(z) = -z$ as well. Moreover $\pi(z) = -xfx = f$ so that in fact $z \in I^+ \hat{\otimes} C$ where I^+ is the ideal I added with the identity of A.

All this results in a map $K(A/I) \to K(I\hat{\otimes}C)$ and it should not be surprising that the sequence

$$K(I) \to K(A) \to K(A/I) \to K(I\hat{\otimes}C) \to K(A\hat{\otimes}C) \to K(A/I\hat{\otimes}C)$$

is exact. This sequence is obtained in §3 of this paper. If it is applied to the algebra CA with the ideal SA it yields an isomorphism $K(A) \to K(SA\hat{\otimes}C)$. In fact, in our paper this isomorphism is proved first (in §2) and it is combined with the main exact sequence of **[4]** (see above) to obtain the new sequence.

The notation $A\hat{\otimes}C$ stands for the graded tensor product of A with the Clifford algebra $C = C^{0,1}$. The periodicity of Clifford algebras leads to the periodicity of the groups $K(S^nA)$ and implies that our long exact sequences are in fact periodic. We have $K(S^8A) \cong K(A)$ in general and $K(S^2A) \cong K(A)$ if A is a complex Banach algebra.

Throughout the paper we will freely use notations and properties from the first paper **[4]**. We will also use the same point of view w.r.t. notations. So often we will use the same symbol to denote different but very similar things. Finally we wish to thank P. de la Harpe for his hospitality at the University of Genève and for discussions we had on this subject. We also express our gratitude to C. Hu for stimulating conversations on this material.

2. The isomorphism $K(A) \to K(SA\hat{\otimes}C)$. Let A and B be Banach algebras over the field $\mathbf{K}$ of real or complex numbers. Assume that α and β are continuous involutive automorphisms on A and B respectively. Let $A \otimes B$ denote the algebraic tensor product of the vector spaces A and B. We equip $A \otimes B$ with the graded product defined by

$$\begin{aligned} (a \otimes b)(c \otimes d) &= ac \otimes bd \quad &&\text{if } \alpha(c) = c \text{ or } \beta(b) = b \\ (a \otimes b)(c \otimes d) &= -ac \otimes bd \quad &&\text{if } \alpha(c) = -c \text{ and } \beta(b) = -b. \end{aligned}$$

It can be checked that this product is well defined on $A \otimes B$ and that it makes $A \otimes B$ into an associative algebra (see e.g. **[2]**, page 129 and **[3]**, page 522). We also equip $A\otimes B$ with the involutive automorphism $\alpha \otimes \beta$. The algebra thus obtained, together with the involution, will be denoted by $A\hat{\otimes}B$.

We will only use this notation when B is finite dimensional. In that case it doesn't matter for our purpose which norm is considered on $A\hat{\otimes}B$ as long as it is a Banach algebra norm which is compatible with the norms on A and B. Such a norm exists in all the cases considered in this paper.

Another important remark is that such a graded tensor product is an associative operation in the sense that $A\hat{\otimes}(B\hat{\otimes}C) = (A\hat{\otimes}B)\hat{\otimes}C$ for algebras A, B and C.

In this section we will only consider a very special case for B.

2.1. Notation. Let C denote the Clifford algebra $C^{0,1} \cong \mathbf{R} \oplus \mathbf{R}$ with the usual involution given by $(a, b) \to (b, a)$ where now $a, b \in \mathbf{R}$.

Throughout this section we will work with the algebra $A\hat{\otimes}C$ where A is a given Banach algebra with involution. The involution on $A\hat{\otimes}C$ will be denoted by $\tilde{\alpha}$.

In order to work with this algebra $A\hat{\otimes}C$ it is important to have a nice and easy description of it. We first realise it as a subalgebra of $M_2(A)$. That is not surprising since

$$A\hat{\otimes}C = A\hat{\otimes}C^{0,1} \subseteq A\hat{\otimes}C^{0,1} \otimes C^{1,0} \cong A \otimes C^{1,1} \cong A \otimes M_2(\mathbf{R}) = M_2(A).$$

Remark that different notations are used for the Clifford algebras in [2] and [3]. We will use the conventions of [2] and we also refer to [2] for all results on Clifford algebras.

2.2. Lemma. *The mapping*

$$a \otimes (1, 1) + b \otimes (1, -1) \to \begin{pmatrix} a & b \\ \alpha(b) & \alpha(a) \end{pmatrix}$$

defines an injective homomorphism of $A\hat{\otimes}C$ into $M_2(A)$.

It is a matter of straightforward checking that this result is true. Moreover, it is easily seen that the involution on $A\hat{\otimes}C$ is carried to the restriction of the involution γ on $M_2(A)$ which we defined before (see [4]) by

$$\gamma : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \to \begin{pmatrix} \alpha(a) & -\alpha(b) \\ -\alpha(c) & \alpha(d) \end{pmatrix}.$$

It is also interesting to notice that the algebra $A\hat{\otimes}C$ is isomorphic with the crossed product $A \otimes_\alpha \mathbf{Z}_2$ of the algebra A by the action of $\mathbf{Z}_2$ given by (see e.g. [5], page 11). However, the action γ does not correspond to the dual action $\hat{\alpha}$ which would be given by

$$\hat{\alpha} : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \to \begin{pmatrix} a & -b \\ -c & d \end{pmatrix}.$$

For notational convenience we will still work with another description of $A\hat{\otimes}C$. For this we suppose first that A has an identity. Then denote $f = 1 \otimes (1, -1)$ in $A\hat{\otimes}C$. It is clear that

$$a \otimes (1, -1) = (a \otimes (1, 1))f.$$

Moreover $a \to a \otimes (1, 1)$ is an injective homomorphism of A in $A\hat{\otimes}C$. If we identify A with its image in $A\hat{\otimes}C$ we find the following.

2.3. LEMMA. *If A has an identity then*

$$A\hat{\otimes}C = \{a + bf | a, b \in A\}$$

where $f^2 = 1$ and $af = f\alpha(a)$ for all a. The involution is given by $\tilde{\alpha}(a + bf) = \alpha(a) - \alpha(b)f$.

Again this is a matter of simple checking. Remark that the structure of $A\hat{\otimes}C$ is completely determined by the algebraic relations in the formulation of the lemma.

It turned out to be most convenient to use this description of $A\hat{\otimes}C$. If A has no identity we can consider A^+ and we can still use the same description. In that case however f is no longer in $A\hat{\otimes}C$.

In this section we will construct an isomorphism from $K(A)$ to $K(SA\hat{\otimes}C)$ where as before SA is the algebra of continuous functions $f : [0, 1] \to A$ with $f(0) = f(1)$ and pointwise operations. This result will have very important consequences such as periodicity. We will discuss this in the following section. The proof of this result is mainly a consequence of a theorem of Wood **[6]** which we will formulate in due time. Unfortunately there are some technical difficulties to overcome before we can get to the stage where this theorem can be used. Many of these have to do with the problems of defining $K(A)$ when A is a general Banach algebra. We refer to §3 of **[4]**.

Recall the definition of $K(A)$. First we considered A^+ and $M_2(A^+)$ with the involution γ defined by

$$\gamma : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \to \begin{pmatrix} \alpha(a) & -\alpha(b) \\ -\alpha(c) & \alpha(d) \end{pmatrix}.$$

In connection with this step in the procedure we have the following lemma.

2.4. LEMMA. *There is a natural isomorphism from $M_2(A^+)\hat{\otimes}C$ to $M_2(A^+\hat{\otimes}C)$ compatible with the natural involutions on the two algebras.*

Proof. Let $A^+\hat{\otimes}C = \{a + bf | a, b \in A^+\}$ with $f^2 = 1$ and $af = f\alpha(a)$ as in Lemma 2.3. Let $f_1 = \left(\begin{smallmatrix} f & 0 \\ 0 & -f \end{smallmatrix}\right)$. Then $f_1^2 = 1$ and $af_1 = f_1\gamma(a)$ for any $a \in M_2(A^+)$. So we have

$$M_2(A^+)\hat{\otimes}C \cong \{a + bf_1 | a, b \in M_2(A^+)\} = M_2(A^+\hat{\otimes}C).$$

The involution considered on $M_2(A^+)\hat{\otimes}C$ is $\tilde{\gamma}$ and given by $a+bf_1 \to \gamma(a)-\gamma(b)f_1$. It is straightforward to verify that this coincides with the involution

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \to \begin{pmatrix} \tilde{\alpha}(a) & -\tilde{\alpha}(b) \\ -\tilde{\alpha}(c) & \tilde{\alpha}(d) \end{pmatrix}$$

where now $a, b, c, d \in A^+\hat{\otimes}C$. □

We will in what follows work further with the set $\{a + bf_1 | a, b \in M_2(A^+)\}$ as a description of $M_2(A^+)\hat{\otimes}C$. Also we will use $\tilde{\gamma}$ to denote the involution on this algebra. Remark that loosely speaking the second statement of the lemma says that going from α to γ first and then to $\tilde{\gamma}$ is the same as first taking $\tilde{\alpha}$ and then associating $\tilde{\gamma}$ in the same way as γ is associated to α.

The next step is to consider $M_4(A^+) = M_2(M_2(A^+))$ with pointwise application of the involution γ. For this reason we here have to use the element $k = f_1 \oplus f_1$ to get a natural description

$$M_4(A^+)\hat{\otimes}C = \{a + bk | a, b \in M_4(A^+)\}$$

in such a way that still $k^2 = 1, ka = \gamma(a)k$, and involution given by $\tilde{\gamma}(a + bk) = \gamma(a) - \gamma(b)k$ when $a, b \in M_4(A^+)$.

2.5. Notation. In what follows we will use the notation $B = M_4(A^+)$.

Recall that now any element in $K(A)$ is of the form $[x]$ where $x \in M_n(B)$ with $x^2 = 1$ and $\gamma(x) = -x$. As before $[x]$ denotes the class of x with respect to a twofold equivalence relation. On the one hand there is homotopy equivalence while on the other hand we have the equivalence due to the inductive limit associated with the imbeddings $x \to x \oplus e$ where $e \in M_4(\mathbf{K})$ is still given by

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{bmatrix}.$$

We will use a number of steps. Each step will be in an obvious way compatible with the quotient map $\varphi : A^+ \to \mathbf{K}$. This is important since $K(A)$ is the kernel of the map $\varphi_x : K(A^+) \to K(\mathbf{K})$.

2.6. Notation. Let $D = \{y \in B\hat{\otimes}C | \tilde{\gamma}(y) = y\}$. Denote by $\hat{\gamma}$ the automorphism on D defined by $\hat{\gamma}(a+bk) = a - bk$. It is easy to check that D is indeed invariant under $\hat{\gamma}$.

Remark that here indeed $\hat{\gamma}$ is the dual action in crossed product theory associated to γ (cf. remark after Lemma 2.2.).

Now consider $x \in B$ and assume that $x^2 = 1$ and $\gamma(x) = -x$. Put $y = xk$. Then $y \in B \hat{\otimes} C$, $\tilde{\gamma}(y) = -\gamma(x)k = xk = y$ so that in fact $y \in D$. Moreover $y^2 = xkxk = x\gamma(x)k^2 = -x^2k^2 = -1$ and $\hat{\gamma}(y) = -y$. So to any $x \in B$ with $x^2 = 1$ and $\gamma(x) = -x$ we associate an element $y \in D$ with $y^2 = -1$ and $\hat{\gamma}(y) = -y$. This is the basis for the following definition and proposition.

2.7. Definition. Consider homotopy classes of elements $y \in M_n(D)$ such that $y^2 = -1$ and $\hat{\gamma}(y) = -y$ (with the convention that again $\hat{\gamma}$ is used for the elementwise application of $\hat{\gamma}$ on $M_n(D)$). Consider the inductive limit of these classes with respect to the imbeddings $y \to y + ek$. Denote this set by H.

It is clear from the above considerations that also $(ek)^2 = -1$ and $\tilde{\gamma}(ek) = -ek$. Because e is homotopic with $-e$ also here ek will be homotopic with $-ek$. So in fact H is similar to the group introduced in §2 of **[4]**. The only difference is that here we have elements y with $y^2 = -1$ instead of $y^2 = 1$. In the complex case this makes no difference. In the real case we get something else. But clearly this new situation is completely analogous.

We refer here to the Remark 2.14 of **[4]**. In that remark we have implicitly replaced B by D just by defining another product on B.

2.8. Proposition. *The map $x \to xk_n$ where $x \in \mathscr{F}_n(B)$ and k_n is the direct sum of n copies of k induces an isomorphism from $K_k(B)$ to H.*

Proof. This is almost obvious. We have seen already that $x^2 = 1$ and $\tilde{\gamma}(x) = -x$ implies $xk_n \in M_n(D)$ and $(xk_n)^2 = -1$ and $\hat{\gamma}(xk_n) = -xk_n$. Moreover e is sent to ek and the inductive limit in $K_e(B)$ and H are precisely taken with respect to these points. □

We are now able to use the theorem of Wood **[6]**. This will be our next step. Also here we will first motivate it. So let $y \in D$ such that $y^2 = -1$ and $\hat{\gamma}(y) = -y$. Define the continuous function $v : [0, 1] \to D$ by

$$v(t) = (\cos \tfrac{1}{2}\pi t + y \sin \tfrac{1}{2}\pi t)(\cos \tfrac{1}{2}\pi t - ek \sin \tfrac{1}{2}\pi t).$$

Then $v(t)$ is invertible in D, the inverse being

$$v(t)^{-1} = (\cos \tfrac{1}{2}\pi t + ek \sin \tfrac{1}{2}\pi t)(\cos \tfrac{1}{2}\pi t - y \sin \tfrac{1}{2}\pi t).$$

Furthermore $v(0) = 1$ and $v(1) = -yek$. Now $\hat{\gamma}(-yek) = -\hat{\gamma}(y)\hat{\gamma}(ek) = -(-y)(-ek) = -yek$. So $\hat{\gamma}(v(1)) = v(1)$. This is the basis for the following definition and the formulation of Wood's theorem.

2.9. Definition. Consider homotopy classes of continuous functions $v : [0, 1] \to M_n(D)$ such that $v(t)$ is invertible for all t, $v(0) = 1$ and $\hat{\gamma}(v(1)) = v(1)$. Let G be the inductive limit of these classes with respect to the imbeddings $v \to v \oplus 1$ where here 1 denotes the constant function with value $1 \in D$.

In the complex case the group G is nothing else but the group K_1 of the algebra of continuous functions $f : [0, 1] \to D$ where $f(0) \in \mathbf{C}1$ and $\hat{\gamma}(f(1)) = f(1)$. In the real case we only have "one half" of the K_1 since we have the restriction $v(0) = 1$.

2.10. Theorem (Wood). *Let $y \in M_n(D)$ be such that $y^2 = -1$ and $\hat{\gamma}(y) = -y$. Associate the function $v : [0, 1] \to M_n(D)$ defined by*

$$v(t) = (\cos \tfrac{1}{2}\pi t + y \sin \tfrac{1}{2}\pi t)(\cos \tfrac{1}{2}\pi t - e_n k_n \sin \tfrac{1}{2}\pi t).$$

Then the map $y \to v$ induces an isomorphism of the groups H and G.

Let us remark here that the map described above is clearly compatible with direct sum which induces the group composition law in the two cases. Further if $y = ek$ then v is defined precisely in such a way that $v = 1$. This assures that the map is compatible with the inductive limit structures.

The use of Wood's theorem is the crucial step. The following however is also important to bring us closer to the group $K(SA\hat{\otimes}C)$. So let $v : [0, 1] \to D$ where v is continuous, $v(t)$ is invertible, $v(0) = 1$ and $\hat{\gamma}(v(1)) = v(1)$.

Define $z : [0, 1] \to B\hat{\otimes}C$ by $z(t) = v(t)kv(t)^{-1}$. Because $k^2 = 1$ we also have $(z(t))^2 = 1$. Because $v(t) \in D$ we have $\tilde{\gamma}(v(t)) = v(t)$ for all t and since $\tilde{\gamma}(k) = -k$ we also have $\tilde{\gamma}(z(t)) = -z(t)$. Clearly $z(0) = k$ since $v(0) = 1$. We claim that also $z(1) = k$. Indeed, because $v(1) \in B\hat{\otimes}C$ and $\hat{\gamma}(v(1)) = v(1)$ we must have $v(1) \in B$. But as also $v(1) \in D$ we have $\tilde{\gamma}(v(1)) = v(1)$. On B however $\tilde{\gamma}$ coincides with γ. So all together we get

$$z(1) = v(1)kv(1)^{-1} = k\gamma(v(1))v(1)^{-1} = kv(1)v(1)^{-1} = k.$$

So we have associated to v a function $z : [0, 1] \to B\hat{\otimes}C$ such that $z(t)^2 = 1$, $\tilde{\gamma}(z(t)) = -z(t)$ for all t and $z(0) = z(1) = k$. This leads to the following.

2.11. Proposition. *Let $v : [0, 1] \to M_n(D)$ be a continuous function such that $v(t)$ is invertible for all t, $v(0) = 1$ and $\hat{\gamma}(v(1)) = v(1)$. Define $z(t) = v(t)k_n v(t)^{-1}$ for all t. Then the map $v \to z$ defines an injective* *homomorphism of the group G into the group $K_k(\Omega B\hat{\otimes}C)$ where ΩB is the algebra of continuous functions $g : [0,1] \to B$ such that $g(0) = g(1)$ with pointwise operations.*

Proof. We have seen already that z is a continuous function from $[0,1]$ into $M_n(B\hat{\otimes}C)$ and that $z(t)^2 = 1$ and $\tilde{\gamma}(z(t)) = -z(t)$ for all t. We also showed that $z(0) = z(1) = k_n$. Moreover if v is the constant function 1 then z is the constant function with value k_n. Then it clearly follows that we obtain a map from the group G (where the inductive limit is taken w.r.t. to the imbeddings $v \to v \oplus 1$) to the group $K_k(\Omega B\hat{\otimes}C)$ (where the inductive limit is taken w.r.t. the imbedding $z \to z \oplus k$, where now k also denotes the constant function on $[0,1]$ with value k).

We will now show that this map is injective. Because we already have a group homomorphism it will be sufficient to prove the following. Consider a continuous function $v : [0,1] \to M_n(D)$ such that $v(t)$ is invertible, $v(0) = 1$ and $\hat{\gamma}(v(1)) = v(1)$. Assume that the associated function z given by $z(t) = v(t)k_n v(t)^{-1}$ is homotopic with k_n in $\mathscr{F}_n(\Omega B\hat{\otimes}C)$. If necessary we first may have to pass to trivial extensions. We then must show that v is homotopic with 1.

Because z is homotopic with k_n within $\mathscr{F}_n(\Omega B\hat{\otimes}C)$ by Proposition 2.3 of [**4**] we can find a continuous function $w : [0,1] \times [0,1] \to M_n(B\hat{\otimes}C)$ such that

(i) $\tilde{\gamma}(w(s,t)) = w(s,t)$ for all s, t,
(ii) $w(s,0) = w(s,1)$ for all s,
(iii) $w(0,t) = 1$ for all t,
(iv) $w(s,t)$ invertible for all s and t,
(v) $z(t) = w(1,t)k_n w(1,t)^{-1}$ for all t.

Let $a(t) = v(t)^{-1}w(1,t)$ for all t. Because $v(t)k_n v(t)^{-1} = z(t) = w(1,t)k_n w(1,t)^{-1}$ we have that $a(t)$ commutes with k_n. Because also $\tilde{\gamma}(a(t)) = a(t)$ we get that $a(t) \in M_n(B)$ for all t.

Define $u(s,t) = w(s,t)w(s,0)^{-1}$ for all s, t. We first show that $s \to u(s,\cdot)$ defines a homotopy between the functions 1 and $t \to v(t)a(t)a(0)^{-1}$. Indeed

(i) $\tilde{\gamma}(u(s,t)) = u(s,t)$ so that $u(s,t) \in M_n(D)$,
(ii) $u(s,0) = 1$ for all s,
(iii) $u(s,1) = 1$ for all s,
(iv) $u(0,t) = 1$ for all t,
(v) $u(1,t) = w(1,t)w(1,0)^{-1} = v(t)a(t)a(0)^{-1}$.

Next consider $(s,t) \to v(t)a(st)a(0)^{-1}$. We claim that this defines a homotopy between the functions v and $t \to v(t)a(t)a(0)^{-1}$. Indeed, for all s we have $v(1)a(s)a(0)^{-1} \in M_n(B)$ and if $s = 0$ we have $v(t)a(st)a(0)^{-1} = v(t)$ and if $s = 1$ we have $v(t)a(st)a(0)^{-1} = v(t)a(t)a(0)^{-1}$.

Combining the two results we see that v is homotopic with 1 within the set of continuous functions $g : [0,1] \to M_n(D)$ such that $g(0) = 1$, $g(t)$ invertible for all t and $\tilde{\gamma}(g(1)) = g(1)$ (which is equivalent with $g(1) \in M_n(B)$). □

In the following proposition we find the range of this map.

2.12. PROPOSITION. *Let* $z : [0,1] \to M_n(B\hat{\otimes}C)$ *be a continuous mapping such that* $z(t)^2 = 1, \tilde{\gamma}(z(t)) = -z(t)$ *for all* t *and* $z(0) = z(1) = k_n$. *Then there exists a continuous function* $v : [0,1] \to M_n(D)$ *such that* $v(t)$ *is invertible for all* $t, v(0) = 1$ *and* $\tilde{\gamma}(v(1)) = v(1)$ *and* $z(t) = v(t)k_n v(t)^{-1}$.

Proof. It is clear that z is homotopic with the constant function with value k_n by means of functions with value k_n at 0. By Proposition 2.3 of [**4**] there exists a function $v : [0,1] \to M_n(B\hat{\otimes}C)$ such that $v(t)$ is invertible and $\tilde{\gamma}(v(t)) = v(t)$ for all t and such that $v(0) = 1$ and $z(t) = v(t)k_n v(t)^{-1}$. Because $\tilde{\gamma}(v(t)) = v(t)$ we have $v(t) \in M_n(D)$ and because $z(1) = k_n$ we have that $v(1)$ commutes with k_n so that $v(1) \in M_n(B)$ and $\tilde{\gamma}(v(1)) = v(1)$. □

The group $K(SA\hat{\otimes}C)$ is a subgroup of the group $K_e(M_4((SA\hat{\otimes}C)^+))$. Clearly $M_4((SA\hat{\otimes}C)^+) \subseteq \Omega B\hat{\otimes}C$ because $B = M_4(A^+)$. The mapping $v \to z$ in Proposition 2.10 brings us to the group $K_k(\Omega B\hat{\otimes}C)$. This group is isomorphic with $K_e(\Omega B\hat{\otimes}C)$ as we have seen in [**4**]. In this case it is easy to describe this isomorphism.

2.13. PROPOSITION. *Let* $p = (1/\sqrt{2})(1+ek)$. *Then* $\tilde{\gamma}(p) = p, p$ *is invertible and* $pkp^{-1} = e$. *Hence the automorphism* $a \to pap^{-1}$ *of* $B\hat{\otimes}C$ *induces an isomorphism from* $K_k(\Omega B\hat{\otimes}C)$ *to* $K_e(\Omega B\hat{\otimes}C)$.

Proof. Because $\tilde{\gamma}(e) = -e$ and $\tilde{\gamma}(k) = -k$ we have $\tilde{\gamma}(p) = p$. Moreover because $e \in B$ and $\gamma(e) = -e$ we have $ek = k\gamma(e) = -ke$. Then simple calculations give that $p^{-1} = (1/\sqrt{2})(1+ke)$ and $pkp^{-1} = e$. All the rest is then obvious. □

If we put everything together we have now obtained an injective homomorphism from $K_e(B)$ into $K_e(\Omega B\hat{\otimes}C)$. The mapping is given by the following formula. If $x \in M_n(B)$ and $x^2 = 1$ and $\gamma(x) = -x$ then let

$$z(t) = \frac{1}{\sqrt{2}}(1 + e_n k_n)v(t)k_n v(t)^{-1}\frac{1}{\sqrt{2}}(1 + k_n e_n)$$

with

$$v(t) = (\cos\tfrac{1}{2}\pi t + xk_n \sin\tfrac{1}{2}\pi t)(\cos\tfrac{1}{2}\pi t - e_n k_n \sin\tfrac{1}{2}\pi t).$$

We are now ready to complete the argument and to prove our main result.

2.14. THEOREM. *The mapping described above defines an isomorphism from $K(A)$ to $K(SA\hat{\otimes}C)$.*

Proof. Let us first show that our mapping really goes from $K(A)$ to $K(SA\hat{\otimes}C)$. In [**4**, Proposition 3.7] we saw that any element in $K(A)$ is of the form $[x]$ where $x \in M_n(M_4(A^+))$ such that $x^2 = 1, \gamma(x) = -x$ and $\varphi(x) = e_n$.

If we apply our map to this x we have seen before that $z \in M_n(\Omega B\hat{\otimes}C)$ and $z(t)^2 = 1$ and $\tilde{\gamma}(z(t)) = -z(t)$ for all t and $z(0) = z(1) = e_n$. Now if we apply φ we get $\varphi(v(t)) = 1$ because $\varphi(x) = e_n$ and so $\varphi(z(t)) = e_n$ for all t. This means that in fact

$$z \in M_n(M_4((SA\hat{\otimes}C)^+)).$$

So $[z] \in K_e(M_4((SA\hat{\otimes}C)^+))$ and precisely because $\varphi(z(t)) = e_n$ for all t we have $[z] \in K(SA\hat{\otimes}C)$.

It is clear that all steps are compatible with direct sums and with the imbeddings (as we have remarked before) so that we get a homomorphism.

Let us show that it is injective. So assume that $[z] = 0$. This means that z is homotopic with e_n within $\mathscr{F}_n(M_4((SA\hat{\otimes}C)^+))$ (if necessary we first pass to a trivial extension). Clearly then also z is homotopic with e_n in the larger set $\mathscr{F}_n(\Omega B\hat{\otimes}C)$. But because of the injectivity at the different levels (essentially Theorem 2.10 and Proposition 2.11), it follows that also $[x] = 0$ in $K_e(B)$, hence in $K(A)$.

Finally we prove that it is surjective. Again by [**4**, Proposition 3.7.] any element in $K(SA\hat{\otimes}C)$ has the form $[z]$ where $\varphi(z) = e_n$ and $z \in \mathscr{F}_n(M_4((SA\hat{\otimes}C)^+))$. In particular $z : [0, 1] \to \mathscr{F}_n(B\hat{\otimes}C)$, z is continuous, $z(0) = z(1) = e_n$ and $\varphi(z(t)) = e_n$ for all t. By Proposition 2.11 we have that

$$z(t) = \frac{1}{\sqrt{2}}(1 + e_n k_n)v(t)k_n v(t)^{-1}\frac{1}{\sqrt{2}}(1 + k_n e_n)$$

where $v : [0,1] \to M_n(D)$ is a continuous function such that $v(t)$ is invertible, $v(0) = 1$ and $\hat{\gamma}(v(1)) = v(1)$. Now apply φ. Then it follows that $\varphi(v(t))$ commutes with k_n. If we then replace $v(t)$ by $v(t)\varphi(v(t)^{-1})$ we see that we could assume that $\varphi(v(t)) = 1$ for all t. By the theorem of Wood we know that there is an element $x \in \mathscr{F}_n(B)$, such that v is homotopic with the function v' given by

$$v'(t) = (\cos \tfrac{1}{2}\pi t + xk_n \sin \tfrac{1}{2}\pi t)(\cos \tfrac{1}{2}\pi t - e_n k_n \sin \tfrac{1}{2}\pi t)$$

(if necessary pass to trivial extensions). Now if we apply φ we get that 1 is homotopic with $\varphi(v')$. By the injectivity in the theorem we know that therefore also $\varphi(x)$ and e_n have to be homotopic within $\mathscr{F}_n(B)$. This means precisely that $[x] \in K(A)$. This completes the proof.

3. Periodicity and exact sequences. Consider the Clifford algebra $C^{0,n}$ generated by elements $f_1, f_2, \dots, f_n$ such that $f_k^2 = 1$ and $f_k f_j + f_j f_k = 0$ if $k \neq j$. Then it is well known that $C^{0,p} \hat{\otimes} C^{0,q} \cong C^{0,p+q}$. So if we iterate the result of the previous section we get an isomorphism between $K(S^n A \hat{\otimes} C^{0,n})$ and $K(A)$ for all n. Here of course $S^n A = S(S^{n-1}A)$ when $n \geq 2$. It is also known that $C^{0,2} \cong M_2(\mathbf{R})$ and $C^{0,8} \cong M_{16}(\mathbf{R})$. We will now draw several conclusions from these facts.

3.1. PROPOSITION. *For any real or complex graded Banach algebra A we have that $K(A \hat{\otimes} C^{0,8}) \cong K(A)$. If A is complex, then already $K(A \hat{\otimes} C^{0,2}) \cong K(A)$.*

Proof. Consider the algebra $C^{0,2}$ and let $v = f_1 f_2$. Then $v^2 = -1$ and $v f_j v^{-1} = -f_j$ for $j = 1, 2$. So v implements the natural involutive automorphism η on $C^{0,2}$. Next consider the algebra $C^{0,8}$ and let $v = f_1 f_2 \cdots f_8$. Then $v^2 = 1$ and also $v f_j v^{-1} = -f_j$ for all j. So also here v implements the natural involutive automorphism η on $C^{0,8}$.

Now write B for $C^{0,2}$ or $C^{0,8}$. Define a mapping $\psi : A \hat{\otimes} B \to A \otimes B$ given by $\psi(a \otimes b) = a \otimes b$ if $\alpha(a) = a$ and $\psi(a \otimes b) = a \otimes vb$ if $\alpha(a) = -a$. It is straightforward to verify that ψ gives an isomorphism from the graded tensor product $A \hat{\otimes} B$ to the usual tensor product $A \otimes B$. The involution on $A \hat{\otimes} B$ is given by $\alpha \otimes \eta$ and is not affected by this isomorphism.

It follows that $A \hat{\otimes} C^{0,2} \cong A \otimes M_2(\mathbf{R}) \cong M_2(A)$ and $A \hat{\otimes} C^{0,8} = A \otimes M_{16}(\mathbf{R}) \cong M_{16}(A)$. In the two cases the involution becomes $\alpha \otimes \eta$. The main difference is that in the first case η is implemented by an element v such that $v^2 = -1$ while in the second case $v^2 = 1$.

Consider $K(A\hat{\otimes}C^{0,8}) \cong K(M_{16}(A))$. In defining this last group, one of the steps is to consider $M_2(M_{16}(A)) \cong A \otimes M_{16}(\mathbf{R}) \otimes M_2(\mathbf{R})$ with the automorphism $\alpha \otimes \eta \otimes \delta$ where δ is implemented by $w = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$. Because $w^2 = 1$ and $v^2 = 1$ we can find an isomorphism $M_{16}(\mathbf{R}) \otimes M_2(\mathbf{R}) \to M_2(\mathbf{R}) \otimes M_{16}(\mathbf{R})$ carrying $\eta \otimes \delta$ to $\delta \otimes 1$. This means that we are working with $M_2(A)$ equipped with γ and further $M_{16}(M_2(A))$ where γ is applied elementwise. This clearly will result in an isomorphism from $K(A\hat{\otimes}C^{0,8})$ to $K(A)$.

Finally, suppose that A is a complex algebra. Then $A \otimes M_2(\mathbf{R}) \cong A \otimes M_2(\mathbf{C})$ where we have a tensor product over the complex numbers. The automorphism is $\alpha \otimes \eta$ where η now can be implemented by iv. In this case $(iv)^2 = 1$. Again the same argument as above will give us that $K(A \otimes C^{0,2})$ is isomorphic with $K(A)$. □

Now we can combine this result in the obvious way with the result from the previous section to obtain the following periodicity theorem.

3.2. THEOREM. *For any real or complex graded Banach algebra A we have $K(S^8A) \cong K(A)$. If the algebra is complex then $K(S^2A) \cong K(A)$.*

This periodicity theorem is one main application of the isomorphism of §2. We will now use this result to transform our exact sequence in **[4]** to a new one. Recall the exact sequence which we obtained there (**[4**, Theorem 4.6]):

$$K(SI) \to K(SA) \to K(S(A/I)) \to K(I) \to K(A) \to K(A/I).$$

3.3. THEOREM. *For any invariant closed ideal there exists an exact sequence*

$$K(I) \to K(A) \to K(A/I) \to K(I\hat{\otimes}C) \to K(A\hat{\otimes}C) \to K(A/I\hat{\otimes}C).$$

The sequence can be extended in the obvious way to a periodic sequence with 6 *terms in the case of a complex algebra and* 24 *terms in general.*

Proof. It is clear that for any invariant ideal I of A also $I\hat{\otimes}C$ is an invariant ideal of $A\hat{\otimes}C$ with quotient $A/I\hat{\otimes}C$. Then the left and right part of the sequence are nothing else but the short exact sequences associated to an ideal (**[4]**).

By the result of §2 we have a diagram

$$\begin{array}{ccccc} K(I) & \longrightarrow & K(A) & \longrightarrow & K(A/I) \\ \downarrow & & \downarrow & & \downarrow \\ K(SI\hat{\otimes}C) & \longrightarrow & K(SA\hat{\otimes}C) & \longrightarrow & K(S(A/I)\hat{\otimes}C). \end{array}$$

And it is not hard to see that this diagram is commutative. Furthermore it is clear that the long exact sequence of **[4]** applied to the algebra $A\hat{\otimes}C$ and ideal $I\hat{\otimes}C$ gives the desired result. □

We have proved our theorem by using the exact sequence of **[4]** and the isomorphism $K(A) \to K(SA\hat{\otimes}C)$ obtained in the previous section. On the other hand it is easy to see that Theorem 3.3 in turn will imply both the exact sequence of **[4]** and the isomorphism of §2. For this the theorem is applied to the algebra CA of continuous functions $f : [0,1] \to A$ with $f(0) = 0$ and the ideal SA. Because CA is contractible we get $K(CA) = \{0\}$ and $K(CA\hat{\otimes}C) = \{0\}$. This yields the desired isomorphism as A is a quotient of CA over SA. Again as before the isomorphism transforms the sequence of Theorem 3.3 into Theorem 4.6 of **[4]**. Moreover one can check that the same homomorphisms are found again.

It is both important and interesting to have the following description of the connecting map $K(A/I) \to K(I\hat{\otimes}C)$ of Theorem 3.3.

3.4. PROPOSITION. *Let $[x] \in K(A/I)$ and assume that $\varphi(x) = e_n$ and $x \in \mathscr{F}_n(M_4((A/I)^+))$. Define*

$$y = -\frac{1}{\sqrt{2}}(1 + e_n k_n)(\sin \pi a + k_n \cos \pi a)\frac{1}{\sqrt{2}}(1 + k_n e_n)$$

where a is any element in $M_n(M_4(A^+))$ such that $\pi(a) = x$ and $\gamma(a) = -a$. Then $[y]$ is the image in $K(I\hat{\otimes}C)$ of $[x]$ under the connecting map.

Proof. By the result of §2 the image of $[x]$ in $K(S(A/I)\hat{\otimes}C)$ is $[z]$ where

$$z(t) = \frac{1}{\sqrt{2}}(1 + e_n k_n)v(t)k_n v(t)^{-1}\frac{1}{\sqrt{2}}(1 + k_n e_n)$$

and

$$v(t) = (\cos \tfrac{1}{2}\pi t + x k_n \sin \tfrac{1}{2}\pi t)(\cos \tfrac{1}{2}\pi t - e_n k_n \sin \tfrac{1}{2}\pi t).$$

To find the image of $[z]$ in $K(I\hat{\otimes}C)$ we first have to lift this function z to a good function y (see **[4]**, Proposition 4.8). We do this by choosing any element $a \in M_n(M_4(A^+))$ such that $\pi(a) = x$ and still $\gamma(a) = -a$.

(The point is that in general we won't be able to choose a such that also $a^2 = 1$, cf. the introduction.) Now define

$$w(t) = (\cos\tfrac{1}{2}\pi ta - k_n \sin\tfrac{1}{2}\pi ta)(\cos\tfrac{1}{2}\pi t - e_n k_n \sin\tfrac{1}{2}\pi t).$$

Because $\gamma(a) = -a$ and the facts that cos is an even function and sin is an odd function we will still have that $\tilde{\gamma}(w(t)) = w(t)$ for all t. Moreover

$$(\cos\tfrac{1}{2}\pi ta + k_n \sin\tfrac{1}{2}\pi ta)(\cos\tfrac{1}{2}\pi ta - k_n \sin\tfrac{1}{2}\pi ta) = 1$$

because k_n commutes with $\cos\frac{1}{2}\pi ta$ and anti-commutes with $\sin\frac{1}{2}\pi ta$ as it anti-commutes with a. So $w(t)$ is invertible. Because $x^2 = 1$ we have that $\cos\frac{1}{2}\pi tx = \cos\frac{1}{2}\pi t$ and $\sin\frac{1}{2}\pi tx = x\sin\frac{1}{2}\pi t$. So we get from $\pi(a) = x$ that $\pi(w(t)) = v(t)$ for all t. Let

$$y(t) = \frac{1}{\sqrt{2}}(1 + e_n k_n) w(t) k_n w(t)^{-1} \frac{1}{\sqrt{2}}(1 + k_n e_n)$$

then clearly $(y(t))^2 = 1$ and $\tilde{\gamma}(y(t)) = -y(t)$ for all t, $y(0) = e_n$ and $\pi(y(t)) = z(t)$ for all t. So by Proposition 4.8 of [4] we have that $[y(1)]$ is the image of $[x]$ in $K(I\hat{\otimes}C)$. Now

$$w(1) = (\cos\tfrac{1}{2}\pi a - k_n \sin\tfrac{1}{2}\pi a)(-e_n k_n)$$

and

$$\begin{aligned} y(1) &= \frac{1}{\sqrt{2}}(1 + e_n k_n)(\cos\tfrac{1}{2}\pi a - k_n \sin\tfrac{1}{2}\pi a)(e_n k_n e_n) \\ &\qquad \times (\cos\tfrac{1}{2}\pi a + k_n \sin\tfrac{1}{2}\pi a)\frac{1}{\sqrt{2}}(1 + k_n e_n) \\ &= -\frac{1}{\sqrt{2}}(1 + e_n k_n)(\sin \pi a + k_n \cos \pi a)\frac{1}{\sqrt{2}}(1 + k_n e_n). \end{aligned}$$

(To check that this last element really belongs to $\mathscr{F}_n(M_4((I\hat{\otimes}C)^+))$ simply apply π and indeed $\pi(y(1)) = e_n$ because $\sin \pi x = x \sin \pi = 0$ and $\cos \pi x = \cos \pi = -1$.)

As we mentioned already in [4] the group $K(A)$ is in fact $K_1(A)$ if A is given the trivial automorphism α, and that $K_0(A) \cong K(A \oplus A)$ when the flip automorphism on $A \oplus A$ is considered. In fact $A \oplus A \cong A\hat{\otimes}C$ when α is trivial on A. So $K(A\hat{\otimes}C) \cong K_0(A)$.

It can be shown that our exact sequence in Theorem 3.3 gives the sequence

$$K_1(I) \to K_1(A) \to K_1(A/I) \to K_0(I) \to K_0(A) \to K_0(A/I),$$

if it is applied to an algebra A with the trivial automorphism. If $A \oplus A$ is considered with the flip and if A is a complex algebra then our sequence becomes

$$K_0(I) \to K_0(A) \to K_0(A/I) \to K_1(I) \to K_1(A) \to K_1(A/I).$$

The exponential map can be recognised in terms like $\cos \pi a + k \sin \pi a$. For details we refer to a paper by Hu Chuanpu [**1**].

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