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POINT SEPARATION BY BOUNDED ANALYTIC FUNCTIONS OF A COVERING RIEMANN SURFACE

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Results of both positive and negative directions on the point separation by bounded analytic functions of two-sheeted covering Riemann surfaces are given when the points of base Riemann surfaces are separated by bounded analytic functions.

1. Introduction. Let R be a Riemann surface and $H^\infty(R)$ the set of bounded analytic functions on R. In the study of bounded analytic functions on Riemann surfaces, one of the basic problems is to determine when the points of R are separated by $H^\infty(R)$. Here we say that $H^\infty(R)$ separates the points of R (or $H^\infty(R)$ is *separating*) if for any pair of distinct points a and b of R there exists an f in $H^\infty(R)$ with $f(a) \neq f(b)$. Although we do not have any satisfactory answer to the problem, there is a very general result on the point separation by an algebra of analytic functions by Royden [**5**]. Applied to the present case the Royden result amounts to saying that if a Riemann surface $\tilde{R}$ admits a nonconstant bounded analytic function, then there is a quotient Riemann sueface R of $\tilde{R}$ with a quotient map ψ of $\tilde{R}$ onto R such that $H^\infty(R)$ is isomorphic to $H^\infty(\tilde{R})$ via the correspondence $f \to f \circ \psi$ and such that $H^\infty(R)$ is *weakly separating*, by which we mean that $H^\infty(R)$ separates the points of R except for a countable subset of R. At present, the gap between this general result of Royden and our knowledge on concrete examples is wide. For this reason it might be natural to consider the problem in the following special case as an experimental study.

Suppose there is given a Riemann surface R such that $H^\infty(R)$ is separating. For a Riemann surface $\tilde{R}$ with a holomorphic proper mapping ψ of $\tilde{R}$ onto R (i.e., $\tilde{R}$ is a ramified finitely sheeted unlimited covering surface of R), we ask when $H^\infty(\tilde{R})$ is separating.

The problem has been considered by Selberg [**8**], and later by Stanton [**9**] when the base domain R is the open unit disk, and then by Segawa [7] when R is a Riemann surface of Parreau-Widom type. In this note, we continue to study the problem in the case when the base

Riemann surface is rather more general. In this context we also cite here a related work of Forelli [1]. The contents of our paper are as follows.

In §2, we consider as $\tilde{R}$ a subdomain of the famous Myrberg surface (cf. e.g. [6]) which is obtained by deleting a sequence of disjoint disks around ramification points. We shall prove that $H^\infty(\tilde{R})$ for this *unramified* $\tilde{R}$ is not separating if the radii of disks are chosen to decrease rapidly enough, and also that $H^\infty(\tilde{R})$ is separating if they are chosen to decrease slowly enough. In spite of the fact that these results are likely to happen, our proof of the "not separating" case, in particular, will need some elaborate arguments.

In §3, we examine the following conjecture.

Conjecture 1.1. Suppose that $H^\infty(R)$ is separating. Consider two copies of $R\backslash J$, where J is the union of a finite number of mutually disjoint compact slits on R. Let $\tilde{R}$ be the Riemann surface obtained by joining one copy to the other along the two sides of the corresponding slits crosswise. Then, $H^\infty(\tilde{R})$ is separating.

We shall show that the conjecture is true if, in addition, the slits are chosen in a certain open subset of R, and that the conjecture fails, surprisingly, in general. In fact, an example of a Riemann surface R constructed in [4] will show that the conjecture is true if the slits sit in a certain place of R, but it fails if the slits sit in some other place of the same R.

In this note, we only consider the *two-sheeted* covering case. Thus, our results are by no means complete, and there are more problems left than answers given in this paper.

2. We denote by $\Delta(x, r)$ ($\overline{\Delta}(x, r)$, resp.) an open (closed, resp.) disk in the complex plane $\mathbf{C}$ with center x and radius r. Let $\Delta_0 = \Delta(0, 1)\backslash\{0\}$ and $\Delta_k = \overline{\Delta}(2^{-k}, r_k)$, $k = 1, 2, 3, \ldots$. We assume that the closed disks Δ_k are mutually disjoint and included in Δ_0, that is, $2^{-k-1} + r_{k+1} < 2^{-k} - r_k$. Put

$$R = \Delta_0 \backslash \bigcup_{k=1}^{\infty} \Delta_k.$$

Let $\tilde{\Delta}_0$ be the two-sheeted unlimited covering surface of Δ_0 whose ramification points are those over $z = 2^{-k}$ for $k = 1, 2, 3, \ldots$, and let ψ denote the covering map of $\tilde{\Delta}_0$ onto Δ_0. Define $\tilde{R} = \psi^{-1}(R)$. Then, the Riemann surface $\tilde{R}$ is a two-sheeted *smooth* covering of the domain R. Now we show the following.

THEOREM 2.1. *Suppose, with the above notation, that the origin is an irregular boundary point for the domain Δ_0 in the sense of the potential theory, i.e.,*

$$\sum_{k=1}^{\infty} \frac{k}{\log(1/r_k)} < \infty. \tag{2.1}$$

Then, $H^\infty(\tilde{R})$ does not separate the points of any fibre $\psi^{-1}(z)$ for any $z \in R$. Thus, $H^\infty(\tilde{R}) = H^\infty(R) \circ \psi$.

We need the following lemma.

LEMMA 2.2. *Let $u(z)$ be a bounded real-valued harmonic function defined in an annulus $\{z\colon a < |z| < b\}$. Then,*

$$\sup_{\theta,\phi\in[0,2\pi]} |u(\sqrt{ab}e^{i\theta}) - u(\sqrt{ab}e^{i\phi})| \le \frac{24\sqrt{a/b}}{(1-\sqrt{a/b})^2}\|u\|_\infty.$$

Proof. Since the estimate is invariant under $z \to z/b$, we may assume that $b = 1$. We may also assume that u is continuous on $a \le |z| \le b$. First, we let $-1 \le u(\zeta) \le 1$ for $|\zeta| = 1$ and $u(\zeta) = 0$ for $|\zeta| = a$. We consider the Poisson integral

$$v(\mathrm{re}^{i\phi}) = \int_0^{2\pi} \frac{1-r^2}{1-2r\cos(\theta-\phi)+r^2} u(e^{i\theta})\, d\theta/2\pi$$

on the unit disk. Then,

$$|v(\mathrm{re}^{i\phi}) - v(0)| \le \int_0^{2\pi} \frac{2r+2r^2}{1-2r+r^2}\, d\theta/2\pi = \frac{4r}{(1-r)^2}. \tag{2.2}$$

Put $w = u - v$. Then, $w(\zeta) = 0$ for $|\zeta| = 1$, and $w(\zeta) = -v(\zeta)$ for $|\zeta| = a$. Hence,

$$|w(\zeta) + v(0)| \le 4a/(1-a)^2$$

for $|\zeta| = a$. By the maximum principle, it follows that

$$\left| w(z) + v(0)\frac{\log|z|}{\log a} \right| \le \frac{4a\log|z|}{(1-a)^2\log a}.$$

From this inequality, we see

$$|w(\sqrt{a}e^{i\theta}) - w(\sqrt{a}e^{i\phi})| \le 4a/(1-a)^2.$$

Also, by (2.2), we have

$$|v(\sqrt{a}e^{i\theta}) - v(\sqrt{a}e^{i\phi})| \le 8\sqrt{a}/(1-\sqrt{a})^2.$$

The last two inequalities imply

$$|u(\sqrt{a}e^{i\theta}) - u(\sqrt{a}e^{i\phi})| \leq 12\sqrt{a}/(1-\sqrt{a})^2.$$

For a general u, we write $u = u_1 + u_2$, where u_1 and u_2 are harmonic with boundary values $u_1(e^{i\theta}) = u(e^{i\theta})$ and $u_2(ae^{i\theta}) = u(ae^{i\theta})$. Applications of the above argument to $u_1(z)$ and $u_2(a/z)$ imply the required estimate.

Proof of Theorem 2.1. Let k be a positive integer. By (2.1), $k/\log_2(1/r_k) = o(1)$, i.e., if $r_k = 2^{-n(k)k}$, then $n(k) \to +\infty$ as $k \to \infty$. In this proof, we admit a conventional use of the symbol $n(k)$ to represent any sequence of real numbers with the property $n(k) \to +\infty$ as $k \to \infty$, that is, $n(k)$ will not be a fixed sequence. For every positive integer n, we have

$$\sum_{k=1}^{\infty} 2^{nk}\sqrt{r_k} < \infty; \quad \text{and} \tag{2.3}$$

$$\sum_{k=1}^{\infty} 2^{nk} r_k < \infty. \tag{2.4}$$

Let $f \in H^\infty(R)$. By (2.4), we see that

$$f^{(n)}(0) = \sum_{k=0}^{\infty} \frac{n!}{2\pi i}\int_{\partial\Delta_k} \frac{f(\zeta)}{\zeta^{n+1}}\, d\zeta \tag{2.5}$$

defines a bounded linear functional on $H^\infty(R)$. Here, the directions of integral paths are taken counter-clockwise for $\partial\Delta_0$, and clockwise for $\partial\Delta_k$, $k = 1, 2, \ldots$, so that this formula is obviously true in the classical sense if f is analytic in a neighborhood of the origin. Also, we have

$$f(z) = \sum_{k=0}^{\infty} \frac{1}{2\pi i}\int_{\partial\Delta_k} \frac{f(\zeta)}{\zeta - z}\, d\zeta \tag{2.6}$$

for $z \in R$.

The theorem follows from the following assertion.

Assertion 2.3. Let $f \in H^\infty(R)$.

(i) If $f^{(n)}(0) = 0$ for all $n = 0, 1, 2, \ldots$, then $f = 0$.

(ii) If there are a sequence $\{\rho_k\}$ of points in R and positive constants A, A_n such that $\rho_k \to 0$,

$$|f(\rho_k)| \leq A2^{-n(k)k} \quad \text{and} \quad |\rho_{k+i} - \rho_{k+j}| \geq A_n 2^{-k} \quad \text{for all } k \text{ and } i \neq j,\ 0 \leq i, j \leq n, \tag{2.7}$$

then $f^{(n)}(0) = 0$ for all $n = 0, 1, 2, \ldots$.

In fact, let $g \in H^\infty(\tilde{R})$. For each point z in R, we write $\psi^{-1}(z) = \{z_+, z_-\}$ and define

$$f(z) = (g(z_+) - g(z_-))^2.$$

As is well-known, $f(z)$ is a single-valued bounded analytic function on R. In order to see the conditions in the hypothesis of the assertion (ii), we consider the annulus $\Delta_k^* = \{z \colon r_k < |z - 2^{-k}| < 2^{-k}/4\}$. Note that Δ_k^* is included in R for every large k and that $\psi^{-1}(\Delta_k^*)$ is conformally equivalent to the annulus $\{w \colon \sqrt{r_k} < |w| < 2^{-k/2-1}\}$. The conformal radius of $\psi^{-1}(\Delta_k^*)$ is estimated as follows:

$$\sqrt{r_k}/2^{-k/2-1} = (r_k 2^{k+2})^{1/2} \leq 2^{-n(k)k}$$

when k is large. Now, put $\rho_k = 2^{-k} + \sqrt{r_k}2^{-k/2-1}$. Then, $|\rho_{m+i} - \rho_{m+j}| \geq 2^{-m}/2^{n+1}$ and, by Lemma 2.2,

$$|f(\rho_k)| < 2^{-n(k)k}\|g\|_\infty^2.$$

Hence, we have $f = 0$ by the assertion. It remains to prove the assertion.

Proof of Assertion 2.3. (i) Set $\Delta_k' = \overline{\Delta}(2^{-k}, \sqrt{r_k})$. By (2.3), disks Δ_k' are mutually disjoint for large k. For the simplicity of the argument, we may assume that all Δ_k' are mutually disjoint, and we define

$$Y = \Delta_0 \setminus \bigcup_{k=1}^{\infty} \Delta_k'.$$

Suppose $f^{(n)}(0) = 0$ for all $n = 0, 1, 2, \ldots$. Assume on the contrary that $f \neq 0$. Multiplying a nonzero constant to f, if necessary, we may assume that $|f(z)| < 1$. From (2.5) and (2.6), an induction on n yields the identity

$$f(z) = z^n \sum_{k=0}^{\infty} \frac{1}{2\pi i} \int_{\partial \Delta_k} \frac{f(\zeta)}{\zeta^n(\zeta - z)} d\zeta. \tag{2.8}$$

It follows from (2.3) that

$$C_n = \sup_{z \in Y} \sum_{k=0}^{\infty} \frac{1}{2\pi} \int_{\partial \Delta_k} \frac{1}{|\zeta^n(\zeta - z)|} |d\zeta| < +\infty.$$

By (2.8), $|f(z)| \leq C_n |z|^n$ $(n = 0, 1, 2, \ldots)$ for $z \in Y$, or equivalently, $\log|1/f(z)| \geq n \log|1/z| - \log C_n$ $(n = 0, 1, 2, \ldots)$. Let u be the lower envelope of the family $\{s\}$ of nonnegative superharmonic functions s on Y such that the inferior limit of $s(z)$ as $z \in Y$ approaches to ζ is not less than $\log|1/\zeta|$ for every $\zeta \in \partial Y \backslash \{0\}$. Then, $u(z)$ is harmonic on Y with boundary values $\log|1/\zeta|$ on $\partial Y \backslash \{0\}$. By the condition (2.1), $z = 0$ is the irregular boundary point of Y in the sense of potential theory. Hence, $v(z) = \log|1/z| - u(z) > 0$ on Y, which can be seen, for instance, by an application of Fatou's lemma to the formula [**10**; Theorem III.41]. Since $u(z) > 0$, $\log|1/z| > v(z)$ and we have

$$\log \frac{1}{|f(z)|} \geq nv(z) - \log C_n \qquad (n = 0, 1, 2, \ldots) \tag{2.9}$$

on $\overline{Y} \backslash \{0\}$. Fix an arbitrary positive number ε and consider the superharmonic function

$$s_{n,\varepsilon}(z) = \log \frac{1}{|f(z)|} + \varepsilon \log \frac{1}{|z|} - nv(z)$$

on $\overline{Y} \backslash \{0\}$. By (2.9) and the effect of the term $\varepsilon \log|1/z|$, we see that $\lim_{z \in Y, z \to 0} s_{n,\varepsilon}(z) = +\infty$. Since $|f(z)| < 1$ and $v(z) = 0$ on $\partial Y \backslash \{0\}$, we conclude that

$$\liminf_{z \in Y, z \to \zeta} s_{n,\varepsilon}(z) \geq 0$$

for every $\zeta \in \partial Y$. The maximum principle for superharmonic functions assures that $s_{n,\varepsilon}(z) \geq 0$ on Y. Letting $\varepsilon \downarrow 0$, we have

$$\log \frac{1}{|f(z)|} \geq nv(z) \qquad (n = 0, 1, 2, \ldots)$$

on Y, a contradiction.

(ii) Define

$$(R_n f)(z) = f(z) - \sum_{l=0}^{n} \frac{f^{(l)}(0)}{l!} z^l.$$

Then,

$$(R_n f)[\rho_{m+n}, \ldots, \rho_m] = f[\rho_{m+n}, \ldots, \rho_m] - f^{(n)}(0)/n!,$$

where

$$F[x_n, \ldots, x_1, x_0] = \sum_{k=0}^{n} \frac{F(x_k)}{(x_k - x_n) \cdots (x_k - x_{k+1})(x_k - x_{k-1}) \cdots (x_k - x_0)}.$$

It follows from (2.7) that $f[\rho_{m+n}, \ldots, \rho_m]$ tends to zero as $m \to \infty$. On the other hand, we estimate $(R_n f)[\rho_{m+n}, \ldots, \rho_m]$ as follows. By

(2.8),

$$(R_nf)(z) = z^{n+1}\sum_{k=0}^{\infty}\frac{1}{2\pi i}\int_{\partial\Delta_k}\frac{(R_nf)(\zeta)}{\zeta^{n+1}(\zeta - z)}\,d\zeta.$$

Now, if $\zeta \in \partial\Delta_k$, then

$$\left|\frac{(R_nf)(\zeta)}{\zeta^{n+1}(\zeta-\rho_k)}\right| \le \|R_nf\|_\infty/(2^{-k}-r_k)^{n+1}(\sqrt{r_k}2^{-k/2-1}-r_k)$$

and if $\zeta \in \partial\Delta_l$, $l \neq k$, then

$$\left|\frac{(R_nf)(\zeta)}{\zeta^{n+1}(\zeta-\rho_k)}\right| \le \|R_nf\|_\infty/(2^{-l}-r_l)^{n+1}2^{-l/4}.$$

Hence, by Lebesgue's dominated covergence theorem, we conclude

$$\lim_{m\to\infty}(R_nf)[\rho_{m+n},\ldots,\rho_m] = 0. \tag{2.10}$$

This shows that $f^{(n)}(0) = 0$ for all n, as was to be proved.

One can also prove (2.10) using an estimate in terms of analytic capacity [**3**: Estimate 2.7 (E-3′)].

THEOREM 2.4. *Suppose that, with the same notation as above, we have* $\inf_k r_k 2^k > 0$. *Then,* $H^\infty(\tilde{R})$ *separates the points of* $\tilde{R}$.

Proof. Set $\delta = \inf_k r_k 2^k$, which is seen to be less than 1/3. If we replace r_k by $\delta 2^{-k}$, then $\tilde{R}$ is replaced by a larger subdomain of $\tilde{\Delta}_0$. Hence, we may assume that $r_k = \delta 2^{-k}$. Set $g_n(z) = z/(z-2^{-n})$ for $n = 1, 2, 3, \ldots$. The following inequalities are immediate:

$$|1-g_n(z)| \le 2^{-n+1}/|z|, \qquad |z| \ge 2^{-n+1}, \tag{2.11}$$

$$|g_n(z)| \le 2^{-n+1}/r_n = 2/\delta, \qquad |z-2^{-n}| \ge r_n, \tag{2.12}$$

$$|g_n(z)| \le 1, \qquad \operatorname{Re} z \le 2^{-n-1}, \tag{2.13}$$

$$|g_n(z)| \le 1 + 4\cdot 2^{k-n}, \qquad |z-2^{-k}| = r_k, \quad k \le n-2. \tag{2.14}$$

Here, (2.14) follows from (2.11). By estimate (2.11), the function

$$g = \prod_{n=1}^{\infty} g_n$$

converges almost uniformly on $\mathbf{C}\backslash\{0\}$. Regarding the function g as a product of functions $g_{2n-1}g_{2n}$, we see that the square root $f = \sqrt{g}$

defines a single-valued meromorphic function on the covering surface $\tilde{\Delta}_0$ of Δ_0. By (2.12), (2.13) and (2.14), we have

$$\left|\prod_{n=1}^{m} g_n\right| \leq \frac{2}{\delta} \prod_{n=k+2}^{m} (1 + 4/2^{n-k})$$

on $|z - 2^{-k}| = r_k$ for any integer $m \geq k + 2$. Hence,

$$\left|\prod_{n=1}^{m} g_n\right| \leq \frac{2}{\delta} \prod_{n=1}^{\infty} (1 + 4/2^{n+1})$$

on R for any positive integer m. This implies that the function g is bounded on R, and hence, f is a bounded analytic function on $\tilde{R}$. Clearly, the function f separates the fibre $\psi^{-1}(z)$ for every $z \in R$.

In the above notations, it is an interesting unsettled question to find a necessary and sufficient condition on radii r_k in order that $H^\infty(\tilde{R})$ is separating.

3. In this section an arbitrary open Riemann surface R will be considered. We note by $M^\infty(R)$ the set of meromorphic functions on R bounded off compact subsets of R and by $\mathscr{P}(R)$ the set of points p of R such that there exists an f in $M^\infty(R)$ for which p is a pole. The set $\mathscr{P}(R)$, referred to as the *pole set* of R, is seen to be open by considering $1/(f - \alpha)$ for f in $M^\infty(R)$ and α complex numbers with large absolute values. In the proof of the following theorem, a *Cauchy differential* $A(\zeta, z)\, d\zeta$ $((\zeta, z) \in \mathscr{P}(R) \times R)$ constructed in **[2]** under the assumption that $H^\infty(R)$ is weakly separating will be used essentially. Let V be an arbitrary parametric disk in $\mathscr{P}(R)$. The coefficients $A(\zeta, z)$ $((\zeta, z) \in V \times R)$ of the above differential enjoy the following properties: (α) $A(\zeta, z)$ is holomorphic on $V \times R$ except for the set $\zeta = z$, (β) $A(\zeta, z) = 1/(\zeta - z) +$ (holomorphic function) on $V \times V$, (γ) $A(\zeta, \cdot)$ is bounded analytic on $R \backslash V$ for any fixed ζ in V, and (δ) $A(\zeta_n, \cdot) \to A(\zeta, \cdot)$ uniformly on $R \backslash V$ as $\zeta_n \to \zeta$ in V.

We have the following.

THEOREM 3.1. *Let ψ be a two-sheeted unlimited covering map of a Riemann surface $\tilde{R}$ onto a Riemann surface R. Suppose that $H^\infty(R)$ is (weakly) separating and that there exists an open subset W of R with* *the following properties*:

(3.1) $H^\infty(\tilde{W})$ *is (weakly) separating, where* $\tilde{W} = \psi^{-1}(W)$;

(3.2) *the boundary* ∂W *of* W *is a compact subset of the pole set* $\mathscr{P}(R)$; *and*

(3.3) $\psi^{-1}(R\backslash W)$ *splits into two disjoint copies of* $R\backslash W$.

Then, $H^\infty(\tilde{R})$ *is (weakly) separating on* $\tilde{R}$.

Before proceeding to the proof of the above, we consider some examples. Let $J_1, J_2, \dots, J_n$ be a finite number of mutually disjoint compact analytic Jordan arcs contained in the set $\mathscr{P}(R)$. Let R_+ and R_- be two copies of $R\backslash(\bigcup_k J_k)$, and let $\tilde{R}$ be the Riemann surface obtained by joining R_+ with R_- along two sides of each J_k crosswise. Choosing a disjoint union of disks containing J_k as the open subset W, we see that conditions (3.1), (3.2) and (3.3) are satisfied. Therefore Conjecture 1.1 is true in this case. Furthermore, we can apply the theorem to the case when $\{J_k\}$ is an infinite sequence if J_k are suitably chosen (see remarks after the following example).

Example 3.2. Conjecture 1.1 fails in general. More precisely, let J be a compact slit in R and $\tilde{R}$ the two-sheeted covering Riemann surface obtained by joining two copies of $R\backslash J$ along J crosswise. Then, there exists a Riemann surface R such that $H^\infty(R)$ is separating, but $H^\infty(\tilde{R})$ may or may not be separating, depending upon the choice of the slit J in R.

Proof. We recall an example of a Riemann surface R constructed in **[4]**. Namely, let $I_1, I_2, \dots$ be disjoint closed intervals on the interval $[0, 1)$ such that I_k converges to $z = 1$. We consider a union I_k^* of a finite number of disjoint subintervals of I_k for each $k = 1, 2, \dots$. Let $R_0 = \Delta\backslash(\bigcup_{n=1}^\infty I_n^*)$ and $R_k = \Delta\backslash\{I_k^* \cup (\bigcup_{j=1}^{k-1} I_j)\}$, where Δ is the open unit disk. The Riemann surface R is now obtained by joining every R_k with R_0 along every subinterval contained in I_k^* crosswise. If one chooses I_k^* in such a fashion that it includes sufficiently many subintervals, then $f|R_k$ converges to $f|R_0$ almost uniformly for every bounded analytic function f on R. Also, we know that $H^\infty(R)$ is separating and that $\mathscr{P}(R) = \bigcup_{k=1}^\infty R_k$ (cf. **[4]** for details). If we choose the slit J in $\mathscr{P}(R)$, then $H^\infty(\tilde{R})$ is separating by the theorem. Next, we choose the slit J in the bottom sheet R_0. Let ψ be the covering map of $\tilde{R}$ onto R. Consider a closed Jordan curve C on R_0 enclosing J.

Choosing the union I_k^* of subintervals more carefully, if necessary, we can assume that $g|R_k$ converges uniformly to $g|(R_0\backslash J)$ on C for every bounded analytic function g on $R\backslash J$, so that g extends analytically to R. Now, let f be a bounded analytic function on $\tilde{R}$ and let R_+ be one of two connected components of $\psi^{-1}(R\backslash J)$. Since ψ is one-to-one on the domain R_+, $f \circ \psi^{-1}$ defines a function g on $R\backslash J$. Hence, g extends analytically to R. Since the analytic function $g \circ \psi$ on $\tilde{R}$ agrees with f on R_+, we have $g \circ \psi = f$ on $\tilde{R}$ throughout. This shows that $H^\infty(\tilde{R})$ does not separate any fibre $\psi^{-1}(p)$ for any $p \in R$.

Here are two remarks on the above example. First, suppose we construct a Riemann surface S by joining $R\backslash J$ with $\Delta\backslash J$ along J crosswise. Namely, we attach an open disk to the bottom sheet of R. In this case, applying a similar argument, we easily see not only the fact that $H^\infty(S)$ is not separating but also the fact that $H^\infty(S)$ turns back to the algebra $H^\infty(\Delta)$ on the unit disk Δ. That is, $H^\infty(S)$ does not separate any pair of distinct points in the fibre $\phi^{-1}(z)$ for every $z \in \Delta$, where ϕ is a covering map of S onto the open unit disk Δ.

Second, suppose we choose an infinite number of mutually disjoint slits J_k in a fixed sheet (or finite number of sheets) R_n ($n \geq 1$) of R such that $\sum(1-|a_k|) < \infty$, where a_{2k-1} and a_{2k} are the end points of the slit J_k. If we construct a two-sheeted covering $\tilde{R}$ of R as before, then Theorem 3.1 shows that $H^\infty(\tilde{R})$ is separating.

The following lemma will be needed in the proof of Theorem 3.1.

LEMMA 3.3. *Let Γ be a union of a finite number of disjoint closed Jordan curves contained in $\mathscr{P}(R)$, and let g be an analytic function defined on a neighborhood of Γ. If g has no zeros on Γ, then there exists a function h in $M^\infty(R)$ such that* $\log(hg)$ *has a single-valued analytic branch on a neighborhood of Γ.*

Proof. Let $\Gamma = \Gamma_1 \cup \cdots \cup \Gamma_p$, where Γ_j are disjoint closed Jordan curves. For each Γ_j, we can choose an annular neighborhood V_j of it and a conformal analtyic map h_j of V_j onto an annulus $\{w\colon \delta < |w| < 1/\delta\}$ $(0 < \delta < 1)$. We may further assume that V_j's are mutually disjoint. Note that the period of $\log g$ along the curve Γ_j is an integer multiple of $2\pi i$, say $2\pi i m_j$. Shrinking V_j, if necessary, we see that $\log g/h_j^{m_j}$ has a single-valued analytic branch on V_j. Since V_j's are mutually disjoint, by using the Cauchy differential $A(\zeta, z)$ introduced in the beginning of this section, a Runge type approximation gives us a meromorphic function h in $M^\infty(R)$ such that $|1/h_j^{m_j} - h| < \delta^{m_j}$ on Γ_j.

Then, $|1 - h_j^{m_j} h| < 1$ on Γ_j, and hence, $\log(h_j^{m_j} h)$ has a single-valued analytic branch on a neighborhood of Γ_j. Consequently,

$$\log(hg) = \log(h_j^{m_j} h) + \log(g/h_j^{m_j})$$

has a single-valued analytic branch on a neighborhood of Γ_j. This proves the lemma.

Proof of Theorem 3.1. Set $\Gamma = \partial W$ and consider a neighborhood V of Γ such that $\overline{V} \subset \mathscr{P}(R)$ and such that the boundary ∂V of V consists of a finite number of disjoint closed Jordan curves. By condition (3.3), there are no ramification points over Γ. Hence, we may further assume that $\tilde{V} = \psi^{-1}(V)$ consists of two disjoint homeomorphic copies V^+ and V^- of V. Now we see that all the hypotheses remain valid even when one replaces the open set W by a slightly smaller one. Therefore, we may assume, from the beginning, that Γ consists of a finite number of disjoint closed Jordan curves, and also, that the neighborhood V is chosen as a disjoint union of annular neighborhoods which include each one of the components of Γ. Let $\{R_j\}_{j=1}^p$ be the connected components of $R \backslash W$ and set $\Gamma_j = \partial R_j$, where each Γ_j may consist of a finite number of disjoint closed Jordan curves. Denote by V_j the union of the annular components of V containing a component of Γ_j. Set

$$W_0 = W \cup V \quad \text{and} \quad W_j = R_j \cup V_j \quad \text{for } j = 1, \dots, p.$$

Shrinking W and V, if necessary, we may assume by condition (3.1) that $H^\infty(\tilde{W}_0)$ is (weakly) separating, where $\tilde{W}_0 = \psi^{-1}(W_0)$.

Now, if $z \in R$ is not a ramification point, then the point z has two pre-images, i.e., $\psi^{-1}(z) = \{z^+, z^-\}$. Define $\tau(z^+) = z^-$ and $\tau(z^-) = z^+$ for every such z. As is well-known, τ extends to a conformal mapping of $\tilde{R}$ onto itself and $\tau \circ \tau = \text{identity}$. Note that $f \circ \psi \in H^\infty(\tilde{R})$ for every $f \in H^\infty(R)$. Hence, in order to see that $H^\infty(\tilde{R})$ is weakly separating, it suffices to find a function F in $H^\infty(\tilde{R})$ with $F \circ \tau \neq F$. Since $H^\infty(\tilde{W}_0)$ is (weakly) separating, there is a function G in $H^\infty(\tilde{W}_0)$ such that $G \circ \tau \neq G$. Replacing G by $G - G \circ \tau$, we assume that

$$G \circ \tau = -G \tag{3.4}$$

and G is not identically zero. Also, we may assume that $G(\tilde{z})$ has no zeros on $\tilde{V} = \psi^{-1}(V)$, for one may modify, if necessary, the boundary Γ of W slightly, and replace the neighborhood V of Γ by a smaller one. Let us regard the restriction of G to V^+ as a function on V, which we denote by g. By Lemma 3.3, there is a function h in $M^\infty(R)$ such that $\log(gh)$ is single-valued and analytic in V, where we replace V by a smaller one if necessary, again. Set $u = \log(gh)$. Let $A(\zeta, z)\,d\zeta$ be the Cauchy differential on $\mathscr{P}(R) \times R$ introduced in the beginning of this section. For $z \in V_j$, it follows that

$$u(z) = \frac{1}{2\pi i}\int_{\Gamma_j''} u(\zeta)A(\zeta, z)\,d\zeta - \frac{1}{2\pi i}\int_{\Gamma_j'} u(\zeta)A(\zeta, z)\,d\zeta,$$

where $\Gamma_j' = W \cap \partial V_j$ and $\Gamma_j'' = R_j \cap \partial V_j$. Define

$$u_0(z) = \frac{1}{2\pi i}\sum_j \int_{\Gamma_j''} u(\zeta)A(\zeta, z)\,d\zeta, \qquad z \in W_0,$$

$$u_j(z) = \frac{1}{2\pi i}\sum_{k \neq j} \int_{\Gamma_k''} u(\zeta)A(\zeta, z)\,d\zeta + \frac{1}{2\pi i}\int_{\Gamma_j'} u(\zeta)A(\zeta, z)\,d\zeta, \qquad z \in W_j$$

for $j = 1, \ldots, p$. Note that

(3.5) u_0 is a bounded analytic function on $W_0 \backslash V$ and u_j is a bounded analytic function on $W_j \backslash V$.

Furthermore, we have

$$u_j = u_0 - u, \qquad z \in V_j. \tag{3.6}$$

By condition (3.3), $\psi^{-1}(W_j)$ consists of two homeomorphic copies W_j^+ and W_j^- of W_j, where W_j^+ and W_j^- intersect with V^+ and V^-, respectively. We now define a meromorphic function F on $\tilde{R}$ by

$$F(\tilde{z}) = \begin{cases} \exp(-u_j \circ \psi(\tilde{z})), & \tilde{z} \in W_j^+, \\ -\exp(-u_j \circ \psi(\tilde{z})), & \tilde{z} \in W_j^-, \\ G(\tilde{z})(h \circ \psi)(\tilde{z})\exp(-u_0 \circ \psi(\tilde{z})), & \tilde{z} \in \tilde{W}_0. \end{cases}$$

By (3.4) and (3.6), $F(\tilde{z})$ is well-defined. It follows that $F \in M^\infty(\tilde{R})$ and that $F \circ \tau = -F$. By [2], we can find a nonzero bounded analytic function f on R such that $fh \in H^\infty(R)$. Hence, $(f \circ \psi)F \in H^\infty(\tilde{R})$. Therefore, $H^\infty(\tilde{R})$ is weakly separating. If $H^\infty(\tilde{W})$ is separating, we can choose in the foregoing discussion a function G and a function f for each $z \in W$ such that $G(z^+) = -G(z^-) \neq 0$ and $(fh)(z) \neq 0$. Hence, the function of the form $(f \circ \psi)F$ separates the points z^+ and z^- for each $z \in W$. The same is true for $z \in R \backslash W$, because F does not vanish on $\tilde{R} \backslash \tilde{W}$. This proves that $H^\infty(\tilde{R})$ is separating.

REFERENCES

[1] F. Forelli, *A note on divisibility in $H^\infty(X)$*, Canad. J. Math., **36** (1984), 458–469.

[2] T. W. Gamelin and M. Hayashi, *The algebra of the bounded analytic functions on a Riemann surface*, J. Reine Angew. Math., **382** (1987), 49–73.

[3] M. Hayashi, *Smoothness of analytic functions at boundary points*, Pacific J. Math., **67** (1976), 171–202.

[4] ———, *The maximal ideal space of the bounded analytic functions on a Riemann surface*, J. Math. Soc. Japan, **39** (1987), 337–344.

[5] H. L. Royden, *Algebras of bounded analytic functions on Riemann surfaces*, Acta Math., **114** (1965), 113–142.

[6] L. Sario and M. Nakai, *Classification Theory of Riemann Surfaces*, Springer-Verlag, 1970.

[7] S. Segawa, *Bounded analytic functions on unbounded covering surfaces*, Pacific J. Math., **79** (1978), 183–187.

[8] H. L. Selberg, *Ein Satz über beschränkte endlichvieldeutige analytische Funktionen*, Comm. Math. Helv., **9** (1937), 104–108.

[9] C. M. Stanton, *Bounded analytic functions on a class of Riemann surfaces*, Pacific J. Math., **59** (1975), 557–565.

[10] M. Tsuji, *Potential Theory in Modern Function Theory*. 2nd. ed., Chelsea, New York, 1975.

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