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IRREDUCIBILITY OF UNITARY PRINCIPAL SERIES FOR COVERING GROUPS OF SL$(2, k)$

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This paper establishes the irreducibility of certain unitary principal series representations of covering groups of SL$(2, k)$, where k is a p-adic field, with p odd.

0.1. The theory of automorphic forms on covering groups of reductive groups over number fields has been shown to have important arithmetical applications [5], [3]. It is thus natural to study the representation theory of covering groups over p-adic fields. The representation-theoretic results which seem to be most applicable to automorphic forms are those concerning the reducibility of non-unitary principal series. The main results concern GL(n) and have been established by Kazhdan and Patterson [3]. In this paper we undertake the study of the unitary principal series by establishing complete reducibility results for n-sheeted covering groups of SL$(2, k)$, where k is a p-adic field containing the nth roots of unity. For ease of exposition, we assume p is odd. The proof uses a detailed analysis in the Fourier transform realization. This procedure is well known, but carrying out the details in the general case is rather involved. In particular, a careful study of matrix-valued Bessel functions is necessary.

The main result of the paper states that when n is even, all unitary principal series are irreducible, and that when n is odd, the only reducible ones are those induced from non-trivial characters of order 2 of $k^\times$. The reducibility results in the case of n odd follow from [6]; the proofs here deal with the irreducibility. These results can easily be applied to establish the reducibility of certain unitary principal series of covering groups of p-adic Chevalley groups. A more complete study, however, requires a completeness theorem like that proved by Harish-Chandra for reductive p-adic groups.

1.1. Let k be a p-adic field. Let n be a positive integer and assume k contains the nth roots of unity. Let $(\; , \;)$ be the norm residue symbol of degree n. Let $G = \mathrm{SL}(2, k)$. There is a covering group $\tilde{G}$ defined as follows [4]: if $\sigma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in G$, put

$$x(\sigma) = \begin{cases} c & \text{if } c \neq 0, \\ d & \text{if } c = 0. \end{cases}$$

For $\sigma, \tau \in G$, put $\beta(\sigma, \tau) = (x(\sigma), x(\tau))(-x(\sigma)^{-1}x(\tau), x(\sigma\tau))$. $\tilde{G}$ is the set $\{(\sigma, \gamma) \mid \sigma \in G, \gamma \in Z/nZ\}$, with multiplication defined by $(\sigma_1, \gamma_1)(\sigma_2, \gamma_2) = (\sigma_1\sigma_2, \gamma_1\gamma_2\beta(\sigma_1, \sigma_2))$.

We will assume in this paper that $p \neq 2$ and that p does not divide n. Let $\mathscr{O}$ be the ring of integers in k, P the prime ideal, and U the units in $\mathscr{O}$. Let $U^m = \{u^m \mid u \in U\}$. Let q be the order of the residue class field, τ a prime element of k, and ε a $(q-1)$st root of unity in k. Let χ be a character of k^+ with conductor $\mathscr{O}$. We take $\{1, \varepsilon, \dots, \varepsilon^{n-1}\}$ to be representatives for U/U^n. Let $\zeta = (\varepsilon, \tau)$, $|x|$ the absolute value on k, and ν the additive valuation. Once we fix n, we will let $(\ ,\ )_m$ be the norm residue symbol of degree m, where $m \neq n$, whenever the symbol is defined.

Let $N = \{\left(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right) \mid x \in k\}$, $A = \{\left(\begin{smallmatrix} a & 0 \\ 0 & a^{-1} \end{smallmatrix}\right) \mid a \in k^\times\}$, and $B = NA$. Let $\tilde{N}$, $\tilde{A}$, $\tilde{B}$ be the inverse images of N, A, B in $\tilde{G}$ with respect to the canonical surjection $\tilde{G} \to G$.

1.2. Let μ be a character of $k^\times$, and let θ be a character of Z/nZ of order n. We will write $\theta(\gamma) = \gamma^t$, with t and n relatively prime. Let $\tilde{A}_0 = \{(\left(\begin{smallmatrix} a & 0 \\ 0 & a^{-1} \end{smallmatrix}\right), \gamma) \in \tilde{A} \mid \nu(a) \equiv O(n)\}$. Put $k_0^\times = \{x \in k^\times \mid \nu(x) \equiv O(n)\}$. Then $\tilde{A}_0 \cong k_0^\times \times Z/nZ$. Characters of $\tilde{A}_0$ are thus of the form $\tilde{\mu}_0(\left(\begin{smallmatrix} a & 0 \\ 0 & a^{-1} \end{smallmatrix}\right), \gamma) = \theta(\gamma)\mu(a)$.

Suppose first that n is odd. Then the induced representations $\tilde{\mu} = \mathrm{Ind}_{\tilde{A}_0}^{\tilde{A}} \tilde{\mu}_0$ are irreducible n-dimensional representations. We will use the explicit matrix realization of $\tilde{\mu}$ obtained by choosing as representatives for $\tilde{A}/\tilde{A}_0$ the set $\{1, r^{-1}, \dots, r^{-(n-1)}\}$, where $r = (\left(\begin{smallmatrix} \tau & 0 \\ 0 & \tau^{-1} \end{smallmatrix}\right), 1)$. If $\tilde{x} = (\left(\begin{smallmatrix} x & 0 \\ 0 & x^{-1} \end{smallmatrix}\right), \gamma) \in \tilde{A}$, with $\nu(x) \equiv j(n), j \in \{0, 1, \dots, n-1\}$, the matrix $\tilde{\mu}(\tilde{x})$ is of the form $\left[\begin{smallmatrix} 0 & C \\ D & 0 \end{smallmatrix}\right]$, where C and D are respectively $j \times j$ and $(n-j) \times (n-j)$ diagonal matrices. In the (i, k)th place of $\tilde{\mu}(\tilde{x})$, where $i - k = -j$ or $n - j$, we have $\tilde{\mu}_0(r^{i-1}\tilde{x}r^{-k+1})$.

Now assume that n is even. Let $\tilde{A}^1 = \{(\left(\begin{smallmatrix} a & 0 \\ 0 & a^{-1} \end{smallmatrix}\right), \gamma) \in \tilde{A} \mid \nu(a) \equiv O(n/2)\}$. Each character of $\tilde{A}_0$ can be extended to $\tilde{A}^1$ in two ways. Choose $\tilde{x} = (\left(\begin{smallmatrix} x & 0 \\ 0 & x^{-1} \end{smallmatrix}\right), \gamma) \in \tilde{A}^1$. The two extensions of a character of $\tilde{A}_0$ defined by θ and μ are:

$$\tilde{\mu}^1(\tilde{x}) = \begin{cases} \theta(\gamma)\mu(x) & \text{if } \tilde{x} \in \tilde{A}_0, \\ \theta(\gamma)\theta((x, \tau))^{n/2}\theta((\tau, \tau))^{n/2}\alpha\mu(x) & \text{if } x \in \tilde{A}^1 - \tilde{A}_0, \end{cases}$$

where $\alpha^2 = \theta((\tau, \tau))^{n^2/4}$.

We obtain irreducible representations $\tilde{\mu} = \mathrm{Ind}_{\tilde{A}^1}^{\tilde{A}} \tilde{\mu}^1$ of dimension $n/2$, and we will use the matrix realization corresponding to the representatives $\{1, r^{-1}, \ldots, r^{-(n/2-1)}\}$ of $\tilde{A}^1$ in $\tilde{A}$.

For each n, whether odd or even, we obtain in this way all finite dimensional representations of $\tilde{A}$. We extend these to $\tilde{B}$ and form the principal series $(T_{\tilde{\mu}}, H_{\tilde{\mu}}) = \mathrm{Ind}_{\tilde{B}}^{\tilde{G}} \tilde{\mu}$. $H_{\tilde{\mu}}$ consists of all locally constant functions $\phi\colon \tilde{G} \to C^{\dim \tilde{\mu}}$ satisfying $\phi(\tilde{n}\tilde{x}\tilde{g}) = |x|\tilde{\mu}(\tilde{x})\phi(\tilde{g})$, where $\tilde{n} \in \tilde{N}$ and $\tilde{x} = ((\begin{smallmatrix} x & 0 \\ 0 & x^{-1} \end{smallmatrix}), \gamma) \in \tilde{A}$. Each function ϕ in $H_{\tilde{\mu}}$ is determined by the function $x \to \phi((\begin{smallmatrix} 1 & 0 \\ x & 1 \end{smallmatrix}), 1)$. These are functions on k, so we take Fourier transforms and obtain a realization of $T_{\tilde{\mu}}$ in a space of functions we denote by $\hat{k}_{\tilde{\mu}}$ (for details see [**6**]). The action $\hat{T}_{\tilde{\mu}}$ of $\tilde{G}$ on $\hat{k}_{\tilde{\mu}}$ is given by:

$$\hat{T}_{\tilde{\mu}}\left(\left(\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}, \gamma\right)\right) f(t) = |a|^{-1}\tilde{\mu}\left(\left(\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}, \gamma\right)\right) f(a^{-2}t),$$

$$\hat{T}_{\tilde{\mu}}\left(\left(\begin{pmatrix} 1 & 0 \\ v & 1 \end{pmatrix}, 1\right)\right) f(t) = \chi(-vt) f(t),$$

$$\hat{T}_{\tilde{\mu}}\left(\left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, 1\right)\right) f(t) = \int\int \tilde{\mu}\left(\left(\begin{pmatrix} x & 0 \\ 0 & x^{-1} \end{pmatrix}, 1\right)\right) \chi\left(ux + \frac{t}{x}\right) f(u)\, du \frac{dx}{|x|}.$$

1.3. In this paper we will study only the principal series $T_{\tilde{\mu}}$ coming from unitary characters μ of $k^{\times}$. We will determine which of these are irreducible. The element $\tilde{w} = ((\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}), 1) = (w, 1)$ of $\tilde{G}$ acts on representations $\tilde{\mu}$ of $\tilde{A}$ by $\tilde{\mu}^{\tilde{w}}(\tilde{x}) = \tilde{\mu}(\tilde{w}\tilde{x}\tilde{w}^{-1})$. An application of Bruhat theory [**1**] shows that if $\tilde{\mu}$ and $\tilde{\mu}^{\tilde{w}}$ are not equivalent, then $T_{\tilde{\mu}}$ is irreducible. We will now determine which $\tilde{\mu}$ satisfy $\tilde{\mu}^{\tilde{w}} \approx \tilde{\mu}$.

Suppose first that n is odd and θ is fixed. Then

$$\operatorname{trace} \tilde{\mu}\left(\left(\begin{pmatrix} x & 0 \\ 0 & x^{-1} \end{pmatrix}, \gamma\right)\right) = \begin{cases} 0 & \text{if } x \notin (k^{\times})^n, \\ \frac{n}{2}\theta(\gamma)\mu(x) & \text{if } x \in (k^{\times})^n. \end{cases}$$

Therefore, $\tilde{u}_1 \approx \tilde{u}_2 \Leftrightarrow \mu_1(x) = \mu_2(x)$ for all x in $(k^{\times})^n$. Also,

$$\operatorname{trace} \tilde{\mu}^{w}\left(\left(\begin{pmatrix} x & 0 \\ 0 & x^{-1} \end{pmatrix}, \gamma\right)\right) = \operatorname{trace} \tilde{\mu}\left(\left(\begin{pmatrix} x^{-1} & 0 \\ 0 & x \end{pmatrix}, \gamma\right)\right),$$

so $\tilde{\mu} \approx \tilde{\mu}^w \Leftrightarrow \mu^2(x) = 1$ for all x in $(k^\times)^n$. The characters μ which satisfy this property are those of the form $\mu(x) = (\alpha, x)_2(\varepsilon^i\tau^j, x)_{2n}$ for $i, j \in \{0, 1, \dots, n-1\}$. But there are only four inequivalent $\tilde{\mu}$ coming from these characters. They are the ones coming from the characters $\mu(x) = (\alpha, x)_2$, for $\alpha \in \{1, \varepsilon, \tau, \varepsilon\tau\}$. It thus suffices to consider these four characters.

Suppose now that n is even and θ is fixed. Then

$$\operatorname{trace}\tilde{\mu}\left(\begin{pmatrix} x & 0 \\ 0 & x^{-1} \end{pmatrix}, \gamma\right) = \begin{cases} 0, & \text{if } x \notin (k^\times)^{n/2}, \\ \frac{n}{2}\alpha\theta(\gamma)\theta((x,\tau))^{n/2}(\tau,\tau)^{n/2}\mu(x) & \text{if } x \in (k^\times)^{n/2}. \end{cases}$$

Therefore, $\tilde{\mu}_1 \approx \tilde{\mu}_2 \Leftrightarrow \mu_1(x) = \mu_2(x)$ for all x in $(k^\times)^{n/2}$, and $\tilde{\mu}^w \approx \tilde{\mu} \Leftrightarrow \mu^2(x) = 1$ for all $x \in (k^\times)^{n/2}$. The characters μ for which this is true are those of the form $\mu(x) = (\varepsilon^i\tau^j, x)$ for $i, j \in \{0, 1, \dots, n-1\}$. If $\mu_1(x) = (\varepsilon^k\tau^l, x)$ is another of these, then $\tilde{\mu} \approx \tilde{\mu}_1 \Leftrightarrow i \equiv k \pmod 2$ and $j \equiv l \pmod 2$. It thus suffices to consider the four characters $\mu(x) = (\alpha, x)$ for $\alpha \in \{1, \varepsilon, \tau^t, \varepsilon\tau^t\}$. It will prove more convenient to consider (τ^t, x) than (τ, x).

We can now state the main result of this paper.

THEOREM 1. *Let $\mu(x) = (\alpha, x)_2$ if n is odd, and let $\mu(x) = (\alpha, x)$ if n is even, where $\alpha \in \{1, \varepsilon, \tau^t, \varepsilon\tau^t\}$. Then*

(a) *If n is odd, $T_{\tilde{\mu}}$ is irreducible if $\alpha = 1$.*

(b) *If n is even, $T_{\tilde{\mu}}$ is irreducible for each α.*

REMARKS. (a) It is also true that if n is odd and $\mu(x) = (\alpha, x)_2$ for $\alpha \in \{\varepsilon, \tau^t, \varepsilon\tau^t\}$, then $T_{\tilde{\mu}}$ splits into a direct sum of two irreducible representations. This follows from the results of **[6]**.

(b) Since the result is well known when $n = 1$ or 2, we assume in the rest of this paper that $n > 2$.

1.4. We will assume in the rest of this paper that if n is odd, $\mu = 1$ and if n is even, $\mu(x) = (\alpha, x)$, $\alpha \in \{1, \varepsilon, \tau^t, \varepsilon\tau^t\}$.

Suppose that I is an intertwining operator for $\hat{T}_{\tilde{\mu}}$. Since I commutes with all the operators $\hat{T}_{\tilde{\mu}}(\left(\begin{smallmatrix} 1 & 0 \\ v & 1 \end{smallmatrix}\right), 1)$, I is given by an End $(C^{\dim \tilde{\mu}})$-valued function $a(x)$ on $k^\times$. Since I commutes with all $\hat{T}_{\tilde{\mu}}\left(\left(\begin{smallmatrix} x & 0 \\ 0 & x^{-1}\end{smallmatrix}\right), 1\right)$, we have

$$\left(I\hat{T}_{\tilde{\mu}}\left(\begin{pmatrix} x & 0 \\ 0 & x^{-1}\end{pmatrix}, 1\right) f\right)(t) = \left(\hat{T}_{\tilde{\mu}}\left(\begin{pmatrix} x & 0 \\ 0 & x^{-1}\end{pmatrix}, 1\right) If\right)(t)$$
$$\Rightarrow a(t)|x|^{-1}\tilde{\mu}\left(\begin{pmatrix} x & 0 \\ 0 & x^{-1}\end{pmatrix}, 1\right) f(x^{-2}t)$$
$$= |x|^{-1}\tilde{\mu}\left(\begin{pmatrix} x & 0 \\ 0 & x^{-1}\end{pmatrix}, 1\right) a(x^{-2}t)f(x^{-2}t)$$
$$\Rightarrow a(x^{-2}t) = \tilde{\mu}\left(\begin{pmatrix} x & 0 \\ 0 & x^{-1}\end{pmatrix}, 1\right)^{-1} a(t)\tilde{\mu}\left(\begin{pmatrix} x & 0 \\ 0 & x^{-1}\end{pmatrix}, 1\right).$$

Since I commutes with $\hat{T}_{\tilde{\mu}}(w, 1)$, we have

$$(I\hat{T}_{\tilde{\mu}}(w, 1)f)(t) = (\hat{T}_{\tilde{\mu}}(w, 1)If)(t)$$
$$\Rightarrow a(t)\int\int \tilde{\mu}\left(\begin{pmatrix} x & 0 \\ 0 & x^{-1}\end{pmatrix}, 1\right)\chi\left(ux + \frac{t}{x}\right) f(u)\, du\frac{du}{|x|}$$
$$= \int\int \tilde{\mu}\left(\begin{pmatrix} x & 0 \\ 0 & x^{-1}\end{pmatrix}, 1\right)\chi\left(ux + \frac{t}{x}\right) a(u)f(u)\, du\frac{dx}{|x|}$$
$$\Rightarrow J_{\tilde{\mu}}(u, v)a(u) = a(v)J_{\tilde{\mu}}(u, v) \quad \text{for all } u, v \in k^\times,$$

where

$$J_{\tilde{\mu}}(u, v) = \int \tilde{\mu}\left(\begin{pmatrix} x & 0 \\ 0 & x^{-1}\end{pmatrix}, 1\right)\chi\left(ux + \frac{v}{x}\right)\frac{dx}{|x|}.$$

1.5. We will now establish some results for later use. Any $\Pi \in \hat{k}^\times$ has associated to it a p-adic gamma function $\Gamma(\Pi)$ and a p-adic Bessel function $J_\Pi(u, v)$ [7]. For $y \in k^\times$, $\Gamma(y)$ will denote $\Gamma(\Pi)$, where $\Pi(x) = (y, x)$. If $y \in k^\times$ and $\mu \in \hat{k}^\times$, $J_y^\mu(u, v)$ will denote $J_\Pi(u, v)$, where $\Pi(x) = (y, x)\mu(x)$. If $\mu = 1$, we will simply write $J_y(u, v)$.

Lemma 2. *Let $U_s = (1/n)\sum_{k=0}^{n-1}\zeta^{-ks}J_{\varepsilon^k}(u\tau^m, v)$, where $u, v \in U$.*

(a) *If $m = -1$, $U_0 = U_1 = -q^{-1}$, $U_2 = \cdots = U_{n-1} = 0$.*

(b) *If $m = 0$, $U_0 = U_{n-1} = 1 - q^{-1}$, $U_2 = \cdots = U_{n-2} = 0$, $U_1 = -q^{-1}$.*

(c) *If $m \in \{1, 2, \ldots, n-3\}$, $U_1 = U_{n-m-1} = -q^{-1}$, $U_0 = U_{n-m} = U_{n-m+1} = \cdots = U_{n-1} = 1 - q^{-1}$, $U_2 = U_3 = \cdots = U_{n-m-2} = 0$.*

(d) *If $m = n-2$, $U_1 = -2q^{-1}$, $U_0 = U_2 = U_3 = \cdots = U_{n-1} = 1-q^{-1}$.*

(e) *If $m = n - 1$, $U_0 = U_1 = 1 - 2q^{-1}$, $U_2 = \cdots = U_{n-1} = 1 - q^{-1}$.*

Proof.

$$\begin{aligned}
&J_1(u\tau^m, v) + \sum_{k=1}^{n-1} \zeta^{-ks} J_{\varepsilon^k}(u\tau^m, v) \\
&\quad = (m+1) - q^{-1}(m+3) \\
&\qquad + \sum_{k=1}^{n-1} \zeta^{-ks}[(\varepsilon^k, v)\Gamma(\varepsilon^{-k}) + (\varepsilon^{-k}, u\tau^m)\Gamma(\varepsilon^k)] \quad [4, \text{p. } 69], \\
&\quad = (m+1) - q^{-1}(m+3) \\
&\qquad + \sum_{k=1}^{n-1} \zeta^{-ks}\left[\frac{1-q^{-1}\zeta^k}{1-\zeta^{-k}} + \zeta^{-mk}\frac{1-q^{-1}\zeta^{-k}}{1-\zeta^k}\right] \\
&\quad = (m+1) - q^{-1}(m+3) \\
&\qquad + \sum_{k=1}^{n-1}\left[\frac{\zeta^{ks}}{1-\zeta^k} - q^{-1}\frac{\zeta^{k(s-1)}}{1-\zeta^k} + \frac{\zeta^{k(-s-m)}}{1-\zeta^k} - q^{-1}\frac{\zeta^{k(-s-m-1)}}{1-\zeta^k}\right].
\end{aligned}$$

Applying the identity

$$\sum_{k=1}^{n-1} \frac{\zeta^{kj}}{1-\zeta^k} = \begin{cases} \dfrac{-(n-2j+1)}{2} & \text{if } 1 \le j \le n-1, \\[2ex] \dfrac{n-1}{2} & \text{if } j = 0, \end{cases}$$

we obtain the result.

Recall that each $\Pi \in \hat{k}^\times$ can be written $\Pi(x) = \Pi^*(x)|x|^\alpha$. If Π is ramified of degree $h \ge 1$, then $\Gamma(\Pi) = c_{\Pi^*} q^{h(\alpha-1/2)}$ [7]. Suppose $\mu(x) = (\varepsilon^i\tau^j, x)|x|^\alpha$ is a ramified character. Then

$$\Gamma(\mu) = \zeta^{-i} C(\tau^j) q^{\alpha-1/2},$$

where $C(\tau^j) = (\tau^j, \tau)c_{\mu^*}$.

LEMMA 3. *Let* $R_s = (1/n)\sum_{k=0}^{n-1} \zeta^{-ks} J_{\varepsilon^k\tau^j}(u\tau^m, v)$, *where* $u, v \in U$ *and* $j \not\equiv O(n)$.

(a) *If* $s = 1$ *and* $m+2 \not\equiv O(n)$, $R_s = C(\tau^{-j})q^{-1/2}$.

(b) *If* $s = 1$ *and* $m+2 \equiv O(n)$, $R_s = q^{-1/2}C(\tau^{-j}) + (\tau^{-j}, u\tau^m)C(\tau^j)$.

(c) *If* $s \ne 1$, *then* $R_s = 0$ *unless* $s + m + 1 \equiv O(n)$, *in which case it equals* $(\tau^{-j}, u\tau^m)C(\tau^j)q^{-1/2}$.

Proof.

$$\frac{1}{n}\sum_{k=0}^{n-1}\zeta^{-ks}J_{\varepsilon^k\tau^j}(u\tau^m,v)$$
$$=\frac{1}{n}\sum_{k=0}^{n-1}\zeta^{-ks}[(\varepsilon^k\tau^j,v)\Gamma(\varepsilon^{-k}\tau^{-j})+(\varepsilon^{-k}\tau^{-j},u\tau^m)\Gamma(\varepsilon^k\tau^j)]$$
$$=\frac{1}{n}\sum_{k=0}^{n-1}\zeta^{-ks}[\zeta^k(\tau^j,v)C(\tau^{-j})q^{-1/2}$$
$$+\zeta^{-km}(\tau^{-j},u\tau^m)\zeta^{-k}C(\tau^j)q^{-1/2}]$$
$$=(\tau^j,v)C(\tau^{-j})q^{-1/2}\frac{1}{n}\sum_{k=0}^{n-1}\zeta^{-k(s-1)}$$
$$+(\tau^{-j},u\tau^m)C(\tau^j)q^{-1/2}\frac{1}{n}\sum_{k=0}^{n-1}\zeta^{-k(s+m+1)}.$$

2.1. In Part 2 of this paper we assume that n is odd and $\mu=1$. We will prove that $T_{\tilde\mu}$ is irreducible. The first step is to construct the matrix $J_{\tilde\mu}(u,v)$, for $u,v\in U$. If $x\in k^x$ and $\nu(x)\equiv s(n)$, for $s\in\{0,1,\dots,n-1\}$, then $\tilde\mu((\begin{smallmatrix}x&0\\0&x^{-1}\end{smallmatrix}),1)$ is a matrix with non-zero entries only in places (i,j), where $i-j=-s$ or $i-j=n-s$. The (i,j)th entry is $(x,\tau)^{t(i+j-2)}$. We thus obtain

$$J_{\tilde\mu}(u,v)=\frac{1}{n}\sum_{s=0}^{n-1}\int\sum_{k=0}^{n-1}\zeta^{-ks}(\varepsilon^k,y)M_s(y)\chi\left(uy+\frac{v}{y}\right)\frac{dy}{|y|},$$

where $M_s(y)$ is the $n\times n$ matrix with $(y,\tau)^{t(i+j-2)}$ in place (i,j), for $i-j=-s$ or $n-s$, and zeros elsewhere. Given i, j, and the corresponding s, we thus obtain in the (i,j)th place of $J_{\tilde\mu}(u,v)$ the term

$$\frac{1}{n}\int\sum_{k=0}^{n-1}\zeta^{-ks}(\varepsilon^k,x)(x,\tau)^{t(i+j-2)}\chi\left(ux+\frac{v}{x}\right)\frac{dx}{|x|}$$
$$=\frac{1}{n}\sum_{k=0}^{n-1}\zeta^{-ks}J_{\varepsilon^k\tau^{t(2-i-j)}}(u,v)$$
$$=\frac{1}{n}\sum_{k=0}^{n-1}\zeta^{-ks}J_{\varepsilon^k\tau^{-t(2i+s-2)}}(u,v).$$

Lemma 2 shows that for $2i+s-2 \equiv O(n)$, $(1/n)\sum_{k=0}^{n-1}\zeta^{-ks}J_{\varepsilon^k}(u,v)$ is non-zero only if $s=0,1$, or $n-1$. The contributions to $J_{\tilde{\mu}}(u,v)$ in this case are thus $1-q^{-1}$ in $(1,1)$, $-q^{-1}$ in $((n+1)/2,(n+3)/2)$, and $-q^{-1}$ in $((n+3)/2,(n+1)/2)$. Lemma 3 shows that if $2i+s-2 \not\equiv O(n)$,

$$\frac{1}{n}\sum_{k=0}^{n-1}\zeta^{-ks}J_{\varepsilon^k\tau^{-t(2i+s-2)}}(u,v)$$
$$=\begin{cases} \alpha_i=(\tau^{-t(2i-1)},v)C(\tau^{t(2i-1)})q^{-1/2} & \text{if } s=1, i\neq \frac{n+1}{2}, \\ \beta_i=(\tau^{t(2i-3)},u)C(\tau^{-t(2i-3)})q^{-1/2} & \text{if } s=n-1, \\ & \quad i\neq\frac{n+3}{2}, \\ 0 \quad \text{in all other cases.} \end{cases}$$

We set $\alpha_{(n+1)/2}=\beta_{(n+3)/2}=-q^{-1}$. We have thus shown

LEMMA 4. *For $u,v\in U$,*

$$J_{\tilde{\mu}}(u,v)=\begin{bmatrix} 1-q^{-1} & \alpha_1 & 0 & 0 & & & \beta_1 \\ \beta_1 & 0 & \alpha_2 & 0 & & & 0 \\ 0 & \beta_2 & 0 & \alpha_3 & & & 0 \\ 0 & 0 & \beta_3 & 0 & & & \\ & & & & & & \\ & & & & & 0 & \alpha_n-1 \\ \alpha_n & & & & & \beta_n & 0 \end{bmatrix}$$

with α_i and β_i given above.

2.2. In this section we begin the proof that if n is odd and $\mu=1$, any intertwining operator of T_μ is scalar.

PROPOSITION 5. *$a(1)$ is scalar.*

Proof. Using the relations established in §1.4, we have that

$$a(\varepsilon^{-2k})=\tilde{\mu}\left(\begin{pmatrix}\varepsilon^k & 0\\ 0 & \varepsilon^{-k}\end{pmatrix},1\right)^{-1}a(1)\tilde{\mu}\left(\begin{pmatrix}\varepsilon^k & 0\\ 0 & \varepsilon^{-k}\end{pmatrix},1\right)$$

and that

$$J_{\tilde{\mu}}\left(\varepsilon^{-2k},1\right)a(\varepsilon^{-2k})=a(1)J_{\tilde{\mu}}(\varepsilon^{-2k},1).$$

Combining these equations, we see that $a(1)$ commutes with $J_{\bar{\mu}}(\varepsilon^{-2k}, 1)\tilde{\mu}((\begin{smallmatrix}\varepsilon^k & 0 \\ 0 & \varepsilon^{-k}\end{smallmatrix}), 1)^{-1}$ for $k = 0, 1, \ldots, n-1$. $a(1)$ thus commutes with

$$M = \sum_{k=0}^{n-1} J_{\bar{\mu}}(\varepsilon^{-2k}, 1)\tilde{\mu}\left(\begin{pmatrix}\varepsilon^k & 0 \\ 0 & \varepsilon^{-k}\end{pmatrix}, 1\right)^{-1}.$$

Using the formulas for $J_{\bar{\mu}}(\varepsilon^{-2k}, 1)$ we derived above, a calculation shows that the matrix M has only three non-zero entries. They are: $M_{11} = 1 - q^{-1}$, $M_{1n} = C(\tau^t)q^{-1/2}$, and $M_{n1} = C(\tau^{-t})q^{-1/2}$.

Writing $a(1) = (a_{ij})$, the equation $Ma(1) = a(1)M$ implies that for $2 \le i \le n-1$, we have $a_{1i}a_{ni} = a_{in} = a_{i1} = 0$.

We now use the equation $a(1)J_{\bar{\mu}}(1, 1) = J_{\bar{\mu}}(1, 1)a(1)$. Notice that $\beta_i = \bar{\alpha}_{i-1}$ for $2 \le i \le n$ and that $\beta_1 = \bar{\alpha}_n$. Also, $\bar{\alpha}_i = \alpha_{n-i+1}$ and $\alpha_i\alpha_{n-i+1} = q^{-1}$ for $1 \le i \le n$.

Equating the first rows gives:

$$a_{11} = a_{22}, \tag{1}$$

$$\bar{\alpha}_n a_{n1} = a_{1n}\alpha_n, \tag{2}$$

$$a_{2j} = 0 \quad \text{for } 3 \le j \le n-2, \tag{3}$$

$$\alpha_1 a_{2,n-1} = \bar{\alpha}_{n-1}a_{1n}, \tag{4}$$

$$(1 - q^{-1})a_{1n} + \bar{\alpha}_n a_{nn} = a_{11}\bar{\alpha}_n. \tag{5}$$

Equating the ith rows, for $2 \le i \le (n-3)/2$ gives:

$$a_{ii} = a_{i+1,i+1}, \tag{6}$$

$$\alpha_i a_{i+1,n-i} = a_{i,n-i}\bar{\alpha}_{n-i}, \tag{7}$$

$$a_{i+1,j} = 0 \quad \text{for } j \ne i+1, n-i. \tag{8}$$

An inductive step is necessary here.

Equating the $(n-1)/2$st rows gives:

$$a_{(n+1)/2,j} = 0 \quad \text{for } j \ne \frac{n+1}{2}. \tag{9}$$

Equating the $(n+1)/2$st rows gives:

$$\bar{\alpha}_n a_{(n-1)/2,(n-1)/2} + \alpha_{(n+1)/2}a_{(n+3)/2,(n-1)/2} = a_{(n+1)/2,(n+1)/2}\bar{\alpha}_{(n-1)/2}, \tag{10}$$

$$\bar{\alpha}_{(n-1)/2}a_{(n-1)/2,(n+3)/2} + \alpha_{(n+1)/2}a_{(n+3)/2,(n+3)/2} = a_{(n+1)/2,(n+1)/2}\alpha_{(n+1)/2}, \tag{11}$$

$$a_{(n+3)/2,j} = 0 \quad \text{if } j \ne \frac{n+1}{2}, \frac{n+3}{2}. \tag{12}$$

Now we start at the bottom row and proceed upwards. Equating the nth rows gives:

$$a_{nn} = a_{n-1,n-1}, \tag{13}$$

$$a_{n-1,j} = 0 \quad \text{for } j \neq 2,\ n-1, \tag{14}$$

$$\bar{\alpha}_{n-1} a_{n-1,2} = a_{n1}\alpha_1. \tag{15}$$

Equating the ith rows, for $n-1 \geq i \geq (n+5)/2$ gives:

$$a_{i-1,j} = 0 \quad \text{for } j \neq i-1,\ n-i+2, \tag{16}$$

$$\bar{\alpha}_{i-1} a_{i-1,n-i+2} = \alpha_{n-i+1} a_{i,n-i+1}, \tag{17}$$

$$a_{ii} = a_{i-1,i-1}. \tag{18}$$

An inductive step is also necessary here.

Using (1), (6), (13), and (18), we obtain

$$a_n = a_{22} = \cdots = a_{(n-1)/2,(n-1)/2} \quad \text{and} \quad a_{(n+3)/2,(n+3)/2} = \cdots = a_{nn}. \tag{19}$$

We also have

$$a_{ij} = 0 \quad \text{unless } j = i \text{ or } j = n-i+1. \tag{20}$$

Using (15) and (17) gives

$$a_{n1} = a_{(n+3)/2,(n-1)/2}\bar{\alpha}_{(n+3)/2}(\alpha_1)^{-1}. \tag{21}$$

Using (14) and (17) gives

$$a_{1n} = a_{(n-1)/2,(n+3)/2}\alpha_1(\bar{\alpha}_{(n+3)/2})^{-1}. \tag{22}$$

But (2) implies

$$a_{(n-1)/2,(n+3)/2} = \frac{a_{1n}\bar{\alpha}_{(n+3)/2}}{\alpha_1} = \frac{\bar{\alpha}_{(n+3)/2}\bar{\alpha}_n}{\alpha_1\alpha_n}a_{n1} = \frac{\bar{\alpha}_{(n+3)/2}}{\alpha_n}a_{n1} = \frac{(\bar{\alpha}_{(n+3)/2})^2}{\alpha_n\alpha_1}a_{(n+3)/2,(n-1)/2} = \frac{\bar{\alpha}_{(n+3)/2}}{\alpha_{(n+3)/2}}a_{(n+3)/2,(n-1)/2}. \tag{23}$$

Recalling that $a_{11} = a_{(n-1)/2,(n-1)/2}$, (10) implies that

$$a_{11} - a_{(n+1)/2,(n+1)/2} = -\frac{\alpha_{(n+1)/2}}{\bar{\alpha}_{(n-1)/2}}a_{(n+3)/2,(n-1)/2}. \tag{24}$$

Recalling that $a_{(n+3)/2,(n+3)/2} = a_{nn}$, (11) implies that

$$(25) \qquad a_{(n+1)/2,(n+1)/2} - a_{nn} = \frac{\bar{\alpha}_{(n-1)/2}}{\alpha_{(n+1)/2}} a_{(n-1)/2,(n+3)/2}.$$

Adding (24) and (25) and employing (23), we get

$$(26) \quad a_{11} - a_{nn} = \left[\frac{-\alpha_{(n+1)/2}}{\alpha_{(n-1)/2}} + \frac{\alpha_{(n-1)/2}\alpha_{(n+3)/2}}{\alpha_{(n+1)/2}\alpha_{(n+3)/2}}\right] a_{(n+3)/2,(n-1)/2}$$
$$= \left[\frac{q^{-1}(-q^{-1}) + q^{-1}}{(-q^{-1})\alpha_{(n+3)/2}}\right] a_{(n+3)/2,(n-1)/2}$$
$$= \frac{(q^{-1}-1)}{\alpha_{(n+3)/2}} a_{(n+3)/2,(n-1)/2}.$$

Using (5) we find that

$$(27) \; a_{11} - a_{nn} = (1-q^{-1})\frac{a_{1n}}{\bar{\alpha}_n} = \frac{(1-q^{-1})}{\bar{\alpha}_n}\frac{\alpha_1}{\bar{\alpha}_{(n+3)/2}} a_{(n-1)/2,(n+3)/2}$$
$$= \frac{1-q^{-1}}{\alpha_{(n+3)/2}} a_{(n+3)/2,(n-1)/2}.$$

Comparing (26) and (27), we see that $a_{(n+3)/2,(n-1)/2} = 0$; implying that

$$a_{n1} = a_{n-1,2} = \cdots = a_{(n+3)/2,(n-1)/2}$$
$$= a_{(n-1)/2,(n+3)/2} = \cdots = a_{2,n-1} = a_{1n} = 0.$$

This implies also that $a_{11} = a_{nn}$. Recalling (19) and (20), we see that $a(1)$ is scalar.

REMARK. For small values of n, of course, the above proof is not precisely true, but the same method applies to these special cases.

2.3. In this section we complete the proof that $T_{\tilde{\mu}}$ is irreducible. It suffices to show $a(x) = a(1)$ for all $x \in k^{x}$. Since $a(x^{-2}t) = \tilde{\mu}((\begin{smallmatrix} x & 0 \\ 0 & x^{-1}\end{smallmatrix}), 1)^{-1} a(t)\tilde{\mu}((\begin{smallmatrix} x & 0 \\ 0 & x^{-1}\end{smallmatrix}), 1)$, it suffices to show $a(\alpha) = a(1)$ for $\alpha \in \{\varepsilon, \tau^{-1}, \varepsilon\tau^{-1}\}$. If $J_{\tilde{\mu}}(\alpha, 1)$ is invertible, then we have $a(\alpha) = J_{\tilde{\mu}}(\alpha, 1)^{-1} a(1) J_{\tilde{\mu}}(\alpha, 1) = a(1)$. We therefore proceed to calculate the determinants of the $J_{\tilde{\mu}}(\alpha, 1)$.

Lemma 6. $\det J_{\tilde{\mu}}(\varepsilon, 1) \neq 0 \Leftrightarrow n \equiv 3(4)$ *or* $q \neq 3$.

Proof. We showed in Lemma 4 that

$$J_{\tilde{\mu}}(\varepsilon, 1) = \begin{bmatrix} 1-q^{-1} & \alpha_1 & 0 & & & \beta_1 \\ \beta_1 & 0 & \alpha_2 & & & \\ 0 & \beta_2 & 0 & & & \\ & & & & & \\ & & & & 0 & \alpha_n - 1 \\ \alpha_n & & & & \beta_n & 0 \end{bmatrix} \quad \text{where}$$

$\alpha_i = C(\tau^{t(2i-1)})q^{-1/2}$ if $i \neq (n+1)/2$, $\beta_i = C(\tau^{-t(2i-3)})q^{-1/2}\zeta^{-t(2i-3)}$ if $i \neq (n+3)/2$, and $\alpha_{(n+1)/2} = \beta_{(n+3)/2} = -q^{-1}$. An easy calculation shows that

$$\det J_{\tilde{\mu}}(\varepsilon, 1) = (-1)^{(n-1)/2}(1-q^{-1}) \prod_{i=1}^{(n-1)/2} \alpha_{2i}\beta_{2i+1} + \prod_{i=1}^{n} \alpha_i + \prod_{i=1}^{n} \beta_i.$$

Using the values for α_i and β_i, we obtain $\prod \alpha_i = \prod \beta_i = -q^{(-n-1)/2}$. Now consider the remaining term. $\alpha_{(n+1)/2}$ appears in this product if and only if $\beta_{(n+3)/2}$ appears, and this happens if and only if $(n+1)/2$ is even. If $(n+1)/2$ is even, we thus obtain $(-1)(1-q^{-1})q^{(-n-1)/2}$. If $(n+1)/2$ is odd, we get $q^{(-n+1)/2}$. Combining the three terms, we find that if $(n+1)/2$ is even,

$$\begin{aligned} \det J_{\tilde{\mu}}(\varepsilon, 1) &= (-1-q^{-1})q^{(-n-1)/2} - 2q^{(-1-n)/2} \\ &= q^{(-3-n)/2} - 3q^{(-1-n)/2} = q^{(-3-n)/2}(1-3q) \neq 0. \end{aligned}$$

If $(n+1)/2$ is odd,

$$\begin{aligned} \det J_{\tilde{\mu}}(\varepsilon, 1) &= (1-q^{-1})q^{(-n+1)/2} - 2q^{(-n-1)/2} \\ &= q^{(-n+1)/2} - 3q^{(-n-1)/2} = q^{(-n+1)/2}(1-3q^{-1}), \end{aligned}$$

which equals zero $\Leftrightarrow q = 3$. If $q = 3$, however, the field cannot contain an nth root of unity for any $n > 2$, and we are not concerned with the case $n = 2$.

We now construct the matrices $J_{\tilde{\mu}}(u\tau^{-1}, 1)$, for $u \in U$. We must consider the sums

$$\frac{1}{n}\sum_{k=0}^{n-1} \zeta^{-ks} J_{\varepsilon^k \tau^{-t(2i+s-2)}}(u\tau^{-1}, 1).$$

If $2i + s - 2 \equiv O(n)$, Lemma 2 shows that the sum is non-zero only if $s = 0$ or 1. The contributions to $J_{\tilde{\mu}}(u\tau^{-1}, 1)$ in this case are $-q^{-1}$

in (1,1) and $((n+1)/2, (n+3)/2)$. If $2i+s-2 \not\equiv O(n)$, Lemma 3 shows that the sum is non-zero only if $s=0$ or 1. If $s=0$, it equals $\beta_i = (\tau^{t(2i-2)}, u)C(\tau^{t(2i-2)})q^{-1/2}$ for $i \neq 1$. If $s=1$, it equals $\alpha_i = C(\tau^{t(2i-1)})q^{-1/2}$ for $i \neq (n+1)/2$. We set $\alpha_{(n+1)/2} = \beta_1 = -q^{-1}$. The α_i occur in places (i,j), for $i-j=-1$ or $n-1$, and the β_i occur in places (i,i). We have thus shown:

LEMMA 7. *For $u \in U$,*

$$J_{\tilde{\mu}}(u\tau^{-1}, 1) = \begin{bmatrix} \beta_1 & \alpha_1 & 0 & & & \\ 0 & \beta_2 & \alpha_2 & & & \\ 0 & 0 & \beta_3 & & & \\ & & & & & \\ & & & & \beta_{n-1} & \alpha_{n-1} \\ \alpha_n & & & & 0 & \beta_n \end{bmatrix}$$

with α_i and β_i given above.

LEMMA 8. $\det J_{\tilde{\mu}(u\tau^{-1},1)} = -2q^{(-n-1)/2} \neq 0$.

Proof. A calculation shows $\det J_{\tilde{\mu}}(u\tau^{-1}, 1) = \prod_{i=1}^{n} \alpha_i + \prod_{i=1}^{n} \beta_i$. Substituting the values for α_i and β_i, we obtain the result.

Letting $u=1$ and ε, we see that $a(\tau^{-1}) = a(\varepsilon\tau^{-1}) = a(1)$. This completes the proof of the first part of our main result.

THEOREM 1(a). *If n is odd and $\mu = 1$, $T_{\tilde{\mu}}$ is irreducible.*

REMARK. Let $J_1(x,y)$ denote the Bessel function attached to the trivial character of the field. It seems likely that for $m \geq -1$ and $n \equiv 1\ (2m+4)$, $\det J_{\tilde{\mu}}(u\tau^m, 1) = q^{(-n+1)/2} J_1(u\tau^m, 1)$, where $u \in U$. Lemmas 6 and 8 show this to be true when $m=-1$ and 0. Additional calculations show this is so for $m=1$ and 2 and for some cases when $m=3$ and 4. The restriction on n is necessary, as the following results show.

(a) If $n \equiv 3\ (6)$ and $n > 3$,

$$\det J_{\tilde{\mu}}(u\tau, 1) = 2q^{(-n-3)/2}[q^2 + 1 - 6q - q(z+\bar{z})],$$

where

$$z = \prod_{l=0}^{n-3/3} [C(\tau^{t(6l+1)})C(\tau^{-t(6l+2)})(\tau^{t(6l+2)}, u)];$$

(b) If $n \equiv 5\ (6)$, $\det J_{\tilde{\mu}}(u\tau, 1) = -2q^{(-n-1)/2}$.

(c) If $n \equiv 5\ (8)$, $\det J_{\tilde{\mu}}(u\tau^2, 1) = q^{(-n+1)/2}(4 - 12q^{-1} + 7q^{-2} - q^{-3})$.

3.1. In Part 3 of this paper we assume that n is even and that $\mu(x) = (\alpha, x)$ for $\alpha \in \{1, \varepsilon, \tau^t, \varepsilon\tau^t\}$. We will show $T_{\tilde{\mu}}$ is irreducible for each μ. Since n is even, we write, letting $m = n/2$,

$$\begin{aligned} J_{\tilde{\mu}}(u, v) &= \int \tilde{\mu}\left(\begin{pmatrix} y & 0 \\ 0 & y^{-1} \end{pmatrix}, 1\right) \chi\left(uy + \frac{v}{y}\right) \frac{dy}{|y|} \\ &= \sum_{r=0}^{1} \sum_{s=0}^{m-1} \int_{A_{r,s}} \tilde{\mu}\left(\begin{pmatrix} y & 0 \\ 0 & y^{-1} \end{pmatrix}, 1\right) \chi\left(uy + \frac{v}{y}\right) \frac{dy}{|y|}, \end{aligned}$$

where $A_{r,s} = \{y \in k^\times \mid \nu(y) \equiv mr + s(n)\}$.

For $y \in A_{r,s}$, $\tilde{\mu}((\begin{smallmatrix} y & 0 \\ 0 & y^{-1} \end{smallmatrix}), 1)$ has non-zero entries only in places (i, j), where $i - j = -s$ if $1 \le i \le m - s$, and $i - j = m - s$ if $m - s + 1 \le i \le m$. The (i, j)th entry of $\tilde{\mu}((\begin{smallmatrix} y & 0 \\ 0 & y^{-1} \end{smallmatrix}), 1)$ is

$$\tilde{\mu}^1\left(\begin{pmatrix} \tau^{i-j}y & 0 \\ 0 & \tau^{j-i}y^{-1} \end{pmatrix}, (y, \tau)^{i+j-2}(\tau, \tau)^{j(1-i)}\right).$$

Using the formula for $\tilde{\mu}^1$ in §1.2, we obtain the following result.

LEMMA 9. *Given* $1 \le i, j \le m$, *choose the unique* $s \in \{0, 1, \ldots, m - 1\}$ *for which* $i - j = -s$ *or* $i - j = m - s$. *Then the* (i, j)*th entry of* $J_{\tilde{\mu}}(u, v)$ *is*

$$\frac{1}{n} \sum_{r=0}^{1} a(i, r, s) \sum_{k=0}^{n-1} \zeta^{-k(mr+s)} J^{\mu}_{\varepsilon^k \tau^{-t}}(2i + s - 2 + mr)(u, v),$$

where for $1 \le i \le m - s$, $a(i, r, s) = \mu(\tau^{-s})\theta((\tau, \tau))^{s(1-i)+rm(s+1)}$, *and for* $m - s + 1 \le i \le m$, $a(i, r, s) = \mu(\tau^{m-s})\theta((\tau, \tau))^{s(1-i)+m(i+1+rs)}$.

3.2. In this section we assume first that $n/2$ is even. In this case, we may take the value of α to be one in the definition of μ^1. We also take $\mu \equiv 1$. Consider the sum

$$\frac{1}{n} \sum_{k=0}^{n-1} \zeta^{-k(mr+s)} J_{\varepsilon^k \tau^{-t(2i+s-2+mr)}}(u, v), \qquad \text{for } u, v \in U.$$

If $r = 0$ and $2i+s-2 \equiv O(n)$, Lemma 2 shows that the sum is non-zero only if $s = 0$ or 1, when it equals $1 - q^{-1}$ and $-q^{-1}$ respectively. If $r = 1$ and $2i+s-2+m \equiv O(n)$, the sum is non-zero only if $s = m-1$, when it equals $-q^{-1}$. The only contribution to $J_{\tilde{\mu}}(u, v)$ in this case is $1-q^{-1}$ in (1,1), since $2i+s-2 \equiv O(n)$ cannot be solved for i if $s = 1$, and $2i+s-2+m \equiv O(n)$ cannot be solved for i if $s = m-1$. If $r = 0$ and $2i+s-2 \not\equiv O(n)$, Lemma 3 shows that the sum is non-zero only if $s = 1$, in which case it equals $a_i = (\tau^{-t(2i-1)}, v)C(\tau^{t(2i-1)})q^{-1/2}$. If $r = 1$ and $2i+s-2+m \not\equiv O(n)$, the sum in non-zero only if $s = m-1$, when it equals $b_i = (\tau^{t(2i-3)}, u)C(\tau^{-t(2i-3)})q^{-1/2}$. We have thus shown:

LEMMA 10. *For $\mu = 1$ and $u, v \in U$,*

$$J_{\tilde{\mu}}(u,v) = \begin{bmatrix} 1-q^{-1} & \alpha_1 & 0 & & & \beta_1 \\ \beta_2 & 0 & \alpha_2 & & & \\ & & & & & \\ & & & & & \\ & & & & 0 & \alpha_{m-1} \\ \alpha_m & & & & \beta_m & 0 \end{bmatrix},$$

where $\alpha_i = (\tau, \tau)^{i-1} a_i$, and $\beta_i = (\tau, \tau)^{i-1} b_i$.

Letting $a(x)$ denote the function on k^x determined by an intertwining operator of $T_{\tilde{\mu}}$, we have:

PROPOSITION 11. *$a(1)$ is scalar.*

Proof. As in the case of n odd, $a(1)$ commutes with

$$N = \sum_{k=0}^{n-1} J_{\tilde{\mu}}(\varepsilon^{-2k}, 1)\tilde{\mu}\left(\begin{pmatrix} \varepsilon^k & 0 \\ 0 & \varepsilon^{-k} \end{pmatrix}, 1\right)^{-1}.$$

The only non-zero entries of N are

$$N_{11} = 1 - q^{-1}, \qquad N_{1m} = C(\tau^t)q^{-1/2}(\tau, \tau)^m,$$

and

$$N_{m1} = C(\tau^{-t})q^{-1/2}(\tau, \tau).$$

This condition implies that for $2 \le i \le m-1$, we have $a_{1i} = a_{i1} = a_{mi} = a_{im}$.

We next use the relation $a(1)J_{\tilde{\mu}}(1,1) = J_{\tilde{\mu}}(1,1)a(1)$. Equating the first rows gives:

$$a_{11} = a_{22}, \tag{28}$$

$$a_{2j} = 0 \quad \text{for } 3 \le j \le m-2, \tag{29}$$

$$a_{1m}\beta_m = \alpha_1 a_{2,m-1}, \tag{30}$$

$$a_{11}\beta_1 = (1-q^{-1})a_{1m} + \beta_1 a_{mm}. \tag{31}$$

Equating the second rows gives

$$a_{22} = a_{33}, \tag{32}$$

$$a_{3j} = 0 \quad \text{for } j \ne 3, m-2, \tag{33}$$

$$a_{2,m-1}\beta_{m-1} = \alpha_2 a_{3,m-2}. \tag{34}$$

Equating the ith rows for $3 \le i \le n/4 - 1$ and using an inductive step gives:

$$a_{ii} = a_{i+1,i+1}, \tag{35}$$

$$a_{i,m-i+1}\beta_{m-i+1} = \alpha_i a_{i+1,m-i}, \tag{36}$$

$$a_{i+1,j} = 0 \quad \text{for } j \ne i+1, m-i. \tag{37}$$

Equating the $n/4$th rows gives:

$$a_{n/4,(n/4)+1}\beta_{(n/4)+1} = \alpha_{n/4} a_{(n/4)+1,n/4}, \tag{38}$$

$$a_{n/4,n/4} = a_{(n/4)+1,(n/4)+1}, \tag{39}$$

$$a_{(n/4)+1,j} = 0 \quad \text{for } j \ne \frac{n}{4}, \frac{n}{4}+1. \tag{40}$$

Equating the ith rows for $n/4 + 1 \le i \le m-2$ and using an inductive step also gives (35), (36), and (37) for these values of i.

Equating the $(m-1)$st rows gives

$$a_{m-1,2}\beta_2 = \alpha_{m-1} a_{m1} \tag{41}$$

$$a_{m-1,m-1} = a_{mm}. \tag{42}$$

We now have $a_{11} = a_{22} = \cdots = a_{mm}$. By (31), $a_{1m} = 0$. Using (30), (34), (36), (38), and (41), we see that each of the elements $a_{2,m-1}, a_{3,m-2}, \cdots, a_{m1}$ is a non-zero constant times a_{1m} and is thus zero. We conclude that $a(1)$ is scalar.

We now proceed to show $a(x) = a(1)$ for each $x \in k^x$. As in the case of n odd, we will calculate $\det J_{\tilde{\mu}}(\alpha, 1)$, for $\alpha \in \{\varepsilon, \tau^{-1}, \varepsilon\tau^{-1}\}$.

LEMMA 12. $\det J_{\tilde{\mu}}(\varepsilon, 1) \neq 0$.

Proof. Lemma 10 gives the matrix $J_{\tilde{\mu}}(\varepsilon, 1)$. A calculation shows that

$$\det J_{\tilde{\mu}}(\varepsilon, 1) = (-1)^{m/2}\left[\prod_{i=1}^{m/2} \alpha_{2i-1}\beta_{2i} + \prod_{i=1}^{m/2} \beta_{2i-1}\alpha_{2i}\right] - \prod_{i=1}^{m} \alpha_i - \prod_{i=1}^{m} \beta_i.$$

Substituting the values for α_i and β_i given in Lemma 10, with $u = \varepsilon$ and $v = 1$, we obtain $\det J_{\tilde{\mu}}(\varepsilon, 1) = -2q^{-m/2}(\tau, \tau)^{m/2} \neq 0$.

LEMMA 13. *For* $u \in U$, $\det J_{\tilde{\mu}}(u\tau^{-1}, 1) \neq 0$.

Proof. The usual calculations give

$$J_{\tilde{\mu}}(u\tau^{-1}, 1) = \begin{bmatrix} \beta_1 & \alpha_1 & 0 & & & \\ 0 & \beta_2 & \alpha_2 & & & \\ 0 & 0 & \beta_3 & & & \\ & & & & & \\ & & & & & \\ & & & & \beta_{m-1} & \alpha_{m-1} \\ \alpha_m & & & & 0 & \beta_m \end{bmatrix},$$

with $\beta_1 = -q^{-1}$, $\beta_i = C(\tau^{-t(2i-2)})q^{-1/2}(\tau^{t(2i-2)}, u)$ for $i \neq 1$, and $\alpha_i = q^{-1/2}(\tau, \tau)^{i-1}C(\tau^{t(2i-1)})$. We then obtain

$$\det J_{\tilde{\mu}}(u\tau^{-1}, 1) = \prod_{i=1}^{m} \beta_i + - \prod_{i=1}^{m} \alpha_i = q^{-n/4}[1 + q^{-1/2}(\tau^t, u)^{n/2}] \neq 0.$$

Letting $u = 1$ and ε, we see that $a(\tau^{-1}) = a(\varepsilon\tau^{-1}) = a(1)$. This completes the proof that $T_{\tilde{\mu}}$ is irreducible if $\mu = 1$ and $n/2$ is even.

Assume now that $\mu(x) = (\varepsilon, x)$ and $n/2$ is even: the proof that $T_{\tilde{\mu}}$ is irreducible is virtually identical to the case when $\mu = 1$, so we omit the details.

3.3. In this section we assume first that $\mu(x) = (\tau^t, x)$ and $n/2$ is even. We first consider, for $u \in U$, the sums

$$\frac{1}{n}\sum_{k=0}^{n-1} \zeta^{-k(mr+s)} J^{\mu}_{\varepsilon^k \tau^{-t(2t+s-2+mr)}}(u, 1) = \frac{1}{n}\sum_{k=0}^{n-1} \zeta^{-k(mr+s)} J_{\varepsilon^k \tau^{-t(2t+s-3+mr)}}(u, 1).$$

If $r = 0$ and $2i + s - 3 \equiv O(m)$, Lemma 2 shows that the sum is non-zero only if $s = 0$, when it equals $1 - q^{-1}$, and if $s = 1$, when it equals $-q^{-1}$. If $r = 1$ and $2i + s - 3 + m \equiv O(n)$, the sum is non-zero only if $s = m - 1$, when it equals $-q^{-1}$. If $r = s = 0$, there is no contribution to $J_{\tilde{\mu}}(u, 1)$, since $2i - 3 \equiv O(n)$ is not solvable. The only contributions in these cases are therefore $r = 0$, $s = 1$, which gives $-q^{-1}$ in $(1, 2)$, and $r = 1$, $s = m - 1$, which gives $-q^{-1}$ in (2,1). For the cases when $2i + s - 3 + mr \equiv O(n)$, the sum is non-zero only when $r = 0$, $s = 1$, in which case it equals $a_i = C(\tau^{t(2i-2)})q^{-1/2}$, for $i \neq 1$; or when $r = 1$, $s = m - 1$, in which case it equals $b_i = (\tau^{t(2i-4)}, u)C(\tau^{-t(2i-4)})q^{-1/2}$, for $i \neq 2$. We have thus shown:

Lemma 14. *For $u \in U$,*

$$J_{\tilde{\mu}}(u, 1) = \begin{bmatrix} 0 & \alpha_1 & 0 & & & \beta_1 \\ \beta_2 & 0 & \alpha_2 & & & \\ 0 & \beta_3 & 0 & & & \\ & & & & & \\ & & & & & \\ & & & & 0 & \alpha_{m-1} \\ \alpha_m & & & & \beta_m & 0 \end{bmatrix},$$

where $\alpha_1 = -(\tau, \tau)q^{-1}$, $\alpha_i = (\tau, \tau)^i a_i$ for $i \neq 1$, $\beta_2 = -q^{-1}$, and $\beta_i = (\tau, \tau)^i b_i$ for $i \neq 2$.

Similar calculations yield the following two lemmas:

Lemma 15. *For $u \in U$,*

$$J_{\tilde{u}}(u^{-2}\tau^2, 1) = \begin{bmatrix} 0 & \alpha_1 & 0 & \beta_1 & 0 & 0 \\ 1 - q^{-1} & 0 & \alpha_2 & & \beta_2 & 0 \\ 0 & 0 & 0 & & & \beta_3 \\ \beta_4 & 0 & 0 & & & \\ 0 & \beta_5 & 0 & & & \\ 0 & 0 & \beta_6 & & & \\ & & & & & \\ & & & & & \alpha_{m-1} \\ \alpha_m & & & & & 0 \end{bmatrix},$$

where $\alpha_1 = \beta_3 = -(\tau, \tau)q^{-1}, \alpha_i = (\tau, \tau)^i C(\tau^{t(2i-2)})q^{-1/2}$ *for* $i > 1$, *and* $\beta_i = (\tau, \tau)^i(\tau^{t(2i-6)}, u^{-2})C(\tau^{-t(2i-6)})q^{-1/2}$ *for* $i \neq 3$.

Lemma 16. *For* $u \in U$,

$$J_{\tilde{u}}(u^{-2}\tau^6, 1) = \left[\begin{array}{ccccc|ccccccc}
0 & \alpha_1 & 0 & & & \beta_1 & 0 & 0 & 0 & 0 & 0 & 0 \\
\gamma & 0 & \alpha_2 & & & & \beta_2 & 0 & 0 & 0 & 0 & 0 \\
 & & & & & & & \beta_3 & 0 & 0 & 0 & \gamma \\
 & & & & & & & & \beta_4 & 0 & \gamma & 0 \\
 & & & & & & & & & \beta_5 & 0 & 0 \\
 & & & & & & & & & & \beta_6 & 0 \\
 & & & & & & & & & & & \beta_7 \\
\hline
\beta_8 & & & & & & & & & & & \\
 & \cdot & & & & & & & & & & \\
 & & \cdot & & & & & & & & & \\
 & & & \cdot & & & & & & & & \\
 & & & & \beta_m & & & & & & & \alpha_{m-1} \\
 & & & & & & & & & & & 0
\end{array}\right]$$

where $\gamma = 1 - q^{-1}, \alpha_1 = \beta_5 = {}^-(\tau, \tau)q^{-1}, \alpha_i = C(\tau^{t(2i-2)})q^{-1/2}$ *if* $i > 1$, *and* $\beta_i = (\tau^{t(2i-10)}, u^{-2})C(\tau^{-t(2i-10)})q^{-1/2}$ *if* $i \neq 5$.

Proposition 17. $a(1)$ *is scalar.*

Proof. Using Lemma 15, a calculation shows that

$$A_1 = \sum_{k=0}^{m-1} J_{\tilde{\mu}}(\varepsilon^{-2k}\tau^2, 1)\tilde{\mu}\left(\begin{pmatrix} \varepsilon^k\tau^{-1} & 0 \\ 0 & \varepsilon^{-k}\tau \end{pmatrix}, 1\right)^{-1}$$

is a matrix with all zero entries except for places (2,2), $(2, m)$, and $(m, 2)$, which are occupied by distinct non-zero constants. The relation a(1) $A_1 = A_1\, a(1)$ implies that for $i \neq 2, m$, we have $a_{i2} = a_{im} = a_{2i} = a_{mi} = 0$.

Using Lemma 16, we see that

$$A_2 = \sum_{k=0}^{m-1} J_{\tilde{\mu}}(\varepsilon^{-2k}\tau^6, 1)\mu\left(\begin{pmatrix} \varepsilon^k\tau^{-3} & 0 \\ 0 & \varepsilon^{-k}\tau^3 \end{pmatrix}, 1\right)^{-1}$$

is a matrix with all zero entries except for places $(3, 3)$, $(3, m-1)$, and $(m-1, 3)$, which are occupied by distinct non-zero constants.

The relation a(1) $A_2 = A_2\, a(1)$ implies that for $i \neq 3, m-1$, we have $a_{3i} = a_{i3} = a_{m-1,i} = a_{i,m-1} = 0$.

Now we use the relation $a(1)\, J_{\tilde{\mu}}(1,1) = J_{\tilde{\mu}}(1,1) a(1)$. Equating the first rows gives:

$$a_{11}\alpha_1 = \alpha_1 a_{22} + \beta_1 a_{m2}, \tag{43}$$

$$a_{1j} = 0 \quad \text{for } j = 4, 5, \ldots, m-2, \tag{44}$$

$$a_{11}\beta_1 = \beta_1 a_{mm} + \alpha_1 a_{2m}. \tag{45}$$

Equating the second rows gives:

$$a_{11} = a_{22} = a_{33}, \tag{46}$$

$$a_{2m} = 0. \tag{47}$$

Note that (46) and (43) imply that $a_{m2} = 0$. Equating the ith rows, for $3 \leq i \leq m-2$, gives:

$$a_{i+1,j} = 0 \quad \text{for } j \neq i+1, \tag{48}$$

$$a_{ii} = a_{i+1,i+1}. \tag{49}$$

Equating the $(m-1)$st rows gives:

$$a_{m-1,m-1} = a_{mm}, \tag{50}$$

$$a_{m-1,3} = 0. \tag{51}$$

Employing all these identities yields the result that $a(1)$ is scalar.

The next step is to prove that $J_{\tilde{\mu}}(\alpha, 1)$ is invertible for $\alpha \in \{\varepsilon, \tau^{-1}, \varepsilon\tau^{-1}\}$.

Lemma 18. $\det J_{\tilde{\mu}}(\varepsilon, 1) \neq 0$.

Proof. Lemma 14 gives the form of $J_{\tilde{\mu}}(\varepsilon, 1)$. A calculation shows that

$$\det J_{\tilde{\mu}}(\varepsilon, 1) = (-1)^{m/2}\left[\prod_{i=1}^{m/2} \alpha_{2i-1}\beta_{2i} + \prod_{i=1}^{m/2} \beta_{2i-1}\alpha_{2i}\right] - \prod_{i=1}^{m} \alpha_i - \prod_{i=1}^{m} \beta_i.$$

Using the values of α_i and β_i from Lemma 14, we obtain $\det J_{\tilde{\mu}}(\varepsilon, 1) = (\tau, \tau)^{m/2} q^{-m/2}[1 - q^{-1}] \neq 0$.

Lemma 19. $\det J_{\tilde{\mu}}(u\tau^{-1}, 1) \neq 0$ *for* $u \in U$.

Proof. The usual calculations show that

$$J_{\tilde{\mu}}(u\tau^{-1},1)=\begin{bmatrix}\beta_1 & \alpha_1 & 0 & & & \\ 0 & \beta_2 & \alpha_2 & & & \\ & & & \cdot & & \\ & & & & \cdot & \\ & & & & & \cdot \\ & & & & & \alpha_{m-1} \\ \alpha_m & & & & & \beta_m\end{bmatrix}, \quad \text{where}$$

$\alpha_1 = -(\tau,\tau)q^{-1}$, $\alpha_i = (\tau,\tau)^i C(\tau^{t(2i-2)})q^{-1/2}$ for $i > 1$, and $\beta_i = (\tau,\tau)(\tau^{t(2i-3)},u)C(\tau^{-t(2i-3)})q^{-1/2}$. We therefore have

$$\det J_{\tilde{\mu}}(u\tau^{-1},1)=\prod_{i=1}^{m}\alpha_i+\prod_{i=1}^{m}\beta_i=q^{-m/2}(\tau,\tau)^{m/2}[1-q^{-1/2}]\neq 0.$$

Letting $u = 1$ and ε, we see that $a(\tau^{-1}) = a(\varepsilon\tau^{-1}) = a(1)$. This completes the proof that $T_{\tilde{\mu}}$ is irreducible if $\mu(x) = (\tau^t, x)$ and $n/2$ is even.

Assume now that $\mu(x) = (\varepsilon\tau^t, x)$ and $n/2$ is even. The proof that $T_{\tilde{\mu}}$ is irreducible is virtually identical to the case when $\mu(x) = (\tau^t, x)$, so we omit the details.

3.4. In §§3.2 and 3.3, the proofs are not precisely as given if n is small, but the same methods apply and the results still hold, so we omit the details.

As for the cases when n is even and $n/2$ is odd, there is nothing new here. If $\mu(x) = (\alpha, x)$, where $\alpha = 1$ or ε, the proof is generally the same as for n odd, $\mu = 1$. If $\mu(x) = (\alpha, x)$, where $\alpha = \tau^t$ or $\varepsilon\tau^t$, we proceed as we did for n divisible by four.

Combining these remarks with the results of §§3.2 and 3.3, we obtain:

THEOREM 1(b). *If n is even and $\mu(x) = (\alpha, x)$ for $\alpha \in \{1, \varepsilon, \tau^t, \varepsilon\tau^t\}$, $T_{\tilde{\mu}}$ is irreducible.*

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