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AMENABILITY AND KUNZE-STEIN PROPERTY FOR GROUPS ACTING ON A TREE

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We characterize the amenable groups acting on a locally finite tree. In particular if the tree is homogeneous and the group G acts transitively on the vertices then we prove that G is amenable iff G fixes one point of the boundary of the tree. Moreover we prove that a group G which acts transitively on the vertices and on an open subset of the boundary is either amenable or a Kunze-Stein group.

1. Introduction and notations. Let X be a locally finite tree, that is, a connected graph without circuits such that every vertex belongs to a finite set of edges. Let V be the set of vertices and E the set of edges. If v_1 and v_2 are in V, let $[v_1, v_2]$ be the unique geodesic connecting v_1 to v_2; the distance $d(v_1, v_2)$ is defined as the length of the geodesic $[v_1, v_2]$. Let $\mathrm{Aut}(X)$ be the locally compact group of all isometries of X and, for $x \in V$, let K_x be the stability subgroup of x; K_x is a compact open subgroup of $\mathrm{Aut}(X)$. Let Ω be the boundary of the tree, that is, the set of equivalence classes of sequences of distinct vertices $\{s_n\}$, $n = 0, 1, 2, \ldots$, such that $[s_i, s_{i+1}]$ is an edge for every $i = 0, 1, 2, \ldots, n, \ldots$ (two such sequences are said to be equivalent if they have infinitely many common vertices). Ω is a compact metric space; every class of Ω is called an end of the tree. If $x_0 \in V$ and $\omega_0 \in \Omega$, there exists a unique geodesic $[x_0, \omega_0]$ from x_0 to ω_0, that is, a unique sequence $\{s_n\}$ of distinct vertices $\{s_0, s_1, \ldots, s_n, \ldots\}$ in the class ω_0 such that $s_0 = x_0$. Hence Ω can also be regarded, as the set of infinite sequences starting from any fixed vertex $x_0 \in V$.

In the same way, for $\omega_1, \omega_2 \in \Omega$ with $\omega_1 \neq \omega_2$, let $[\omega_1, \omega_2]$ be the unique geodesic joining ω_1 to ω_2; $[\omega_1, \omega_2]$ is a line, that is, a sequence $\{s_n\}$, $n = 0, \mp 1, \mp 2, \mp 3, \ldots$, of distinct vertices such that $[s_i, s_{i+1}]$ is an edge for every i. Conversely, every line is associated with a pair of ends of X. The reader is referred to [2, 3] for more details.

For $g \in \mathrm{Aut}(X)$, J. Tits has proved in [6] that one and only one of the following holds:

(1) There exists a vertex $v \in V$ such that $g(v) = v$ (in this case g is called a rotation).

(2) There exists an edge $[x, y]$ such that $g(x) = y$ and $g(y) = x$ (g is called an inversion).

(3) There exist a line $\{s_n\}$ and an integer $j \neq 0$ such that $g(s_n) = s_{n+j}$ for every n (g is called a step $|j|$ translation on the line $C = \{s_n\}$. Moreover $|j| = \min\{d(v, g(v)): v \in V\}$ and $C = \{v \in V: d(v, g(v)) = |j|\}$). Furthermore, J. Tits proved in [7] that a solvable fixed point free group of isometries of a tree leaves invariant an end or a pair of ends of X. But the stability subgroup of an end or a pair of ends, in general, is not solvable. In this paper we consider the larger class of amenable groups and we prove that, if X is a locally finite tree and G is a closed noncompact subgroup of $\mathrm{Aut}(X)$, then G is amenable iff G leaves invariant an end or a pair of ends of X. In particular, we deduce that a group G which acts transitively on the vertices of a homogeneous tree of order $r > 2$ (that is, a tree where every vertex belongs to r edges) is amenable iff G fixes one end. In this case we give another characterization of amenability, and we observe that G acts transitively on an open subset of the boundary Ω.

We recall that a locally compact group G satisfies the "Kunze-Stein property" if $L^p(G)^*L^2(G) \subset L^2(G)$ for every $1 < p < 2$ (a group of this type will be called a "Kunze-Stein group"). A closed subgroup of $\mathrm{Aut}(X)$ (X is a homogeneous or semihomogeneous tree) which acts transitively on Ω is a Kunze-Stein group [5]. In §3 below we show that if G acts transitively on the vertices and on an open subset of Ω then G is either amenable or a Kunze-Stein group. If no orbit of G on Ω is open, then G is not amenable and we conjecture that G is not a Kunze-Stein group. This amounts to conjecture that, for groups acting transitively on the vertices, the Kunze-Stein property is equivalent to the fact that the group acts transitively on the tree boundary Ω.

Finally, we prove this conjecture in the special case of a homogeneous tree of order three.

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2. Amenable groups acting on a tree. We consider the following four properties:

(P_1) G fixes one vertex.

(P_2) G leaves invariant an edge (that is $g([x, y]) = [x, y]$).

(P_3) G fixes one end of X.
(P_4) G leaves invariant a pair of ends of X.
The main result of this section is the following theorem.

THEOREM 1. *Let X be a locally finite tree and G a closed subgroup of* $\mathrm{Aut}(X)$. *Then G is amenable if and only if G has property (P_i) for some i $(i = 1, 2, 3, 4)$.*

Proof. Let G be a closed amenable subgroup of $\mathrm{Aut}(X)$. Let us prove that there exists i $(1 \leq i \leq 4)$ such that G has property (P_i). Observe that if ϕ and ψ are translations on the lines C_1 and C_2, respectively, and $C_1 \cap C_2 = \emptyset$ then the subgroup $\langle\phi, \psi\rangle$ generated by ϕ and ψ is a discrete subgroup of $\mathrm{Aut}(X)$ isomorphic to the free group with two generators. This is a consequence of the following claim: $w(C_1) \cap C_1 = \emptyset$ for every reduced word w in ϕ and ψ such that $w \notin \langle\phi\rangle$, the subgroup generated by ϕ. Indeed this implies that, for $x \in C_1$, $w(x) \neq x$ for every reduced word in ϕ and ψ. A fortiori, $w \neq 1$ and $\langle\phi, \psi\rangle \cap K_x = \{1\}$; this means that $\langle\phi, \psi\rangle$ is a discrete subgroup of $\mathrm{Aut}(X)$ isomorphic to the free group with two generators. We prove now the claim. Let γ_1, γ_2 be two lines such that $\gamma_1 \cap \gamma_2 = \emptyset$; we define $[\gamma_1, \gamma_2] = [x_0, y_0]$ where $x_0 \in \gamma_1$, $y_0 \in \gamma_2$ and $d(x_0, y_0) = d(\gamma_1, \gamma_2) = \min\{d(x, y): x \in \gamma_1,\ y \in \gamma_2\}$.

Let γ be a line such that $\gamma \cap C_1 = \gamma \cap C_2 = \emptyset$; it is easy to see that $[\gamma, C_1] \subset [\gamma, C_2]$ implies that $\psi^m(\gamma) \cap C_1 = \psi^m(\gamma) \cap C_2 = \emptyset$ and $[\psi^m(\gamma), C_2] \subset [\psi^m(\gamma), C_1]$ for every $m \neq 0$. Similarly, if $[\gamma, C_2] \subset [\gamma, C_1]$, then $\phi^m(\gamma) \cap C_1 = \phi^m(\gamma) \cap C_2 = \emptyset$ and $[\phi^m(\gamma), C_1] \subset [\phi^m(\gamma), C_2]$.

Since, for every $m \neq 0$, $\phi^m(C_1) = C_1$, $\psi^m(C_1) \cap C_1 = \psi^m(C_1) \cap C_2 = \emptyset$ and $[\psi^m(C_1), C_2] \subset [\psi^m(C_1), C_1]$ it follows that $w(C_1) \cap C_1 = w(C_1) \cap C_2 = \emptyset$ for $w = \phi^{j_1}\psi^{i_1}\phi^{j_2}\psi^{i_2} \cdots \phi^{j_n}\psi^{i_n}\phi^{j}$ with $j_2, \ldots, j_n$, i_1, $i_2, \ldots, i_n$ nonzero integers, $n > 0$. This proves the claim. Hence, if ϕ, $\psi \in G$ then $C_1 \cap C_2 \neq \emptyset$. We can suppose that there exists a translation $\phi \in G$ on C, otherwise G satisfies (P_1) or (P_2) or (P_3) **[6**, Prop. 3.4**]**. For every $g \in G$, $g\phi g^{-1}$ is a translation on the line $g(C)$, and thus by the argument above $C \cap g(C) \neq \emptyset$.

But if $C \cap g(C)$ is finite for a $g \in G$, then there exists j such that $\phi^j(C \cap g(C)) \cap (C \cap g(C)) = \emptyset$, and thus, since $g(C) \cap \{\phi^j(g(C))\} \neq \emptyset$ would force existence of a circuit joining $C \cap g(C)$ and $\phi^j(g(C))$, $g(C) \cap \{\phi^j(g(C))\} = \emptyset$ which is a contradiction. Thus $C \cap g(C)$ is infinite for every $g \in G$. We denote by ω_1, ω_2 the two ends of

C. Then $g\omega_1$ and $g\omega_2$ are the two ends of $g(C)$. If $C = g(C)$, then $g\omega_1 = \omega_1$ and $g\omega_2 = \omega_2$ or $g\omega_1 = \omega_2$ and $g\omega_2 = \omega_1$. If $C \neq g(C)$ then the set: $A_g = \{\omega_1, \omega_2\} \cap \{g\omega_1, g\omega_2\}$ is a singleton because $C \cap g(C)$ is infinite. We prove now that $A_g = A_h$ for every g and h in G such that $g(C) \neq C \neq h(C)$. To prove this, we observe that if $g(C) \neq C$ and $A_g = \{\omega_2\}$ then $g\omega_2 = \omega_2$. Indeed let $g\omega_2 \neq \omega_2$ and $A_g = \{\omega_2\}$, then $g\omega_1 = \omega_2$ and $g\omega_2 \neq \omega_1$; therefore $gg\omega_2 = \omega_1$ (otherwise $gg\omega_2 \neq \omega_2$ and $A_{gg} = \varnothing$ which is impossible). This means that g is a cyclic permutation of the set $\{\omega_1, \omega_2, g\omega_2\}$. We consider the element $\phi g \in G$; it follows that $\phi g(C) \neq C$ and $\phi g\omega_1 = \phi\omega_2 = \omega_2$, $\phi g\omega_2 \neq \omega_2$. The reasoning above implies that ϕg is a cyclic permutation of the set $\{\omega_1, \omega_2, \phi g\omega_2\}$. In particular, $\phi g(\phi g\omega_2) = \omega_1$, that is, $\phi g\omega_2 = g^{-1}\phi^{-1}\omega_1 = g\omega_2$. This is a contradiction because ϕ is a translation on C and $\phi\omega = \omega$ iff $\omega = \omega_1$ or $\omega = \omega_2$. This proves that $g\omega_2 = \omega_2$.

The proof of the fact that $A_g = \{\omega_1\}$ implies that $g\omega_1 = \omega_1$ is similar. We suppose now that there exist g and h in G such that $A_g = \{\omega_1\}$ and $A_h = \{\omega_2\}$, then $gh\omega_2 \neq \omega_2$ and $gh\omega_1 \neq \omega_1$ which implies that $gh(C) = C$ and so $gh\omega_2 = \omega_1$ and $g\omega_2 = \omega_1$, contrary to the assumption that $g\omega_1 = \omega_1$. Thus $A_g = A_h$ for every g and h such that $g(C) \neq C \neq h(C)$, say $A_g = \{\omega_2\}$. Therefore for any given g in G, we have the following two mutually exclusive possibilities: (1) $g\omega_2 = \omega_2$; (2) $g\omega_1 = \omega_2$ and $g\omega_2 = \omega_1$. Hence G has property (P_3) or (P_4) because it is easy to see that if there exists an isometry of type (2) then $g\omega_2 = \omega_2$ implies $g\omega_1 = \omega_1$. Conversely, we now prove that every group which has property (P_i), $i = 1, 2, 3, 4$, is amenable. If G is the stability subgroup of a vertex or of an edge then G is compact because X is locally finite. Now, let G be the stability subgroup of an end $\omega_0 \in \Omega$. Let $\{s_0, s_1, \ldots, s_n, \ldots\}$ be the geodesic from s_0 to ω_0 and B the group of all isometries b such that $b(s_n) = s_n$ for n sufficiently large. Then $B \subset G$ and $B = \bigcup_{n=0}^{\infty} B_n$ where $B_n = G \cap K_{s_n}$; since B_n is compact open in G the group B is a closed amenable subgroup of $\mathrm{Aut}(X)$, open in G. B is the subset of rotations of G: indeed, if $g \in G$ fixes a vertex v, it also fixes the geodesic $[v, \omega_0]$, hence $g \in B$. Moreover, by definition, G contains no inversions. Therefore, if $G \neq B$ there exists a step j translation ϕ on $C = \{s_n\}$ and we can choose ϕ in such a way that j is smallest possible. In particular if G contains translations of step j' then j' is a multiple of j. Hence $G = \langle\phi\rangle B$ where $\langle\phi\rangle$ is the group generated by ϕ. Because $B \cap \langle\phi\rangle = \{1\}$ and B is a normal amenable subgroup of G, it follows that G is amenable. Finally, let $\omega_1, \omega_2 \in \Omega$ with $\omega_1 \neq \omega_2$ and denote by $C = \{s_n\}$, $n \in \mathbf{Z}$, the geodesic joining ω_1 to ω_2, and by G the group which leaves invariant the set $\{\omega_1, \omega_2\}$. Let $K_0 = \{h \in G: h(s_n) = s_n \text{ for every } n\}$; by [6, Proposition 3.4] we can suppose that there exists a step j translation ϕ on C and j is smallest possible. Then $K_0\langle\phi\rangle$ is an amenable closed subgroup of G (K_0 is compact normal in $K_0\langle\phi\rangle$). It is easy to see that $K_0\langle\phi\rangle$ is the set of isometries of G such that $g\omega_1 = \omega_1$ and $g\omega_2 = \omega_2$. Hence, if $G \neq K_0\langle\phi\rangle$ then $K_0\langle\phi\rangle$ is a closed normal amenable subgroup of index 2 in G. Then G is amenable and the theorem is proved.

REMARKS. 1. The first part of Theorem 1 holds for a general tree; in other words a closed amenable subgroup of $\mathrm{Aut}(X)$, where X is a general tree, has property (P_i) for some $i = 1, \ldots, 4$. But the converse is not true for a general tree. Indeed, let X_0 be the tree with vertex set $V = \mathbf{N} \cup \{\infty\}$ and edge set $E = \{\{n, \infty\}: n \in \mathbf{N}\}$. X_0 is not a locally finite tree and $\mathrm{Aut}(X_0) = K_\infty \simeq S(\mathbf{N})$, the group of all permutations of $\mathbf{N}$, is not amenable.

2. We shall say that a vertex v is of homogeneity l if v belongs to exactly l edges. A semihomogeneous tree $X_{l,q}$ is a tree such that every vertex is of homogeneity l or q and two adjacent vertices are of homogeneity l and q, respectively. Let S_l [respectively S_q] be the subset of $X_{l,q}$ of vertices of homogeneity l [respectively q]. If $l = q = r$, then $X_{l,q}$ is a homogeneous tree of order r. Let X be a homogeneous tree and G be a closed subgroup of $\mathrm{Aut}(X)$ acting transitively on the vertices, or alternatively let $X = X_{l,q}$ and G be a closed subgroup of $\mathrm{Aut}(X_{l,q})$ acting transitively on S_l. It follows from Theorem 1 that G is amenable iff G fixes one end. In fact, if G has property (P_1), (P_2) or (P_4) then, for some vertex v, the orbit Gv is either finite or contained in a line.

3. We can deduce from the proof of Theorem 1 that a closed non-amenable subgroup of $\mathrm{Aut}(X)$ contains a discrete subgroup isomorphic to the free group with two generators. Indeed if G is not amenable, Proposition 3.4 of [6] implies that G contains a translation ϕ on C. The proof of Theorem 1 shows that if $g(C) \cap C \neq \emptyset$ for every g in G, then G is amenable. Therefore there exists $g \in G$ such that $g(C) \cap C = \emptyset$, and so the subgroup of G generated by ϕ and $g\phi g^{-1}$ is a discrete group isomorphic to the free group with two generators.

4. In [6] J. Tits proved that a solvable subgroup of $\mathrm{Aut}(X)$ has property (P_1), (P_3) or (P_4); but the vice-versa is not true. For example, the stabilizer of a vertex or an end, and the stabilizer of an edge or

a pair of ends of a homogeneous or semihomogeneous tree are not solvable. Since this group contains subgroups of type $K_C = \{g: g(x) = x \text{ for every } x \in C\}$ where $C = \{s_n: n \in \mathbf{Z}\}$ is a line, it is enough to prove that K_C is not solvable. Let E_m be the following set:

$$E_m = \{y \in X: d(y, s_0) = m \text{ and } [s_0, y] \cap C = \{s_0\}\}$$

for $m > 0$, and $K_m = \{g \in K_C: g(y) = y \text{ for every } y \in E_m\}$.

K_m is a closed normal subgroup of K_C and K_C/K_m is isomorphic to $S(E_m)$, the group of all permutations of E_m.

When m is sufficiently large, $|E_m| \geq 5$ and $S(E_m)$ is not solvable. This implies that K_C is not solvable.

Now we give another characterization of amenability for groups acting transitively on a homogeneous tree X. For $x \in V$ and for an integer $n \geq 0$ let $S_n^x = \{y \in V: d(x, y) = n\}$.

We define the following "local property (*)":

(*) For every $x \in V$ there exists $\tilde{x} \in V$ with $d(x, \tilde{x}) = 1$ such that:

(1) $G \cap K_x \subsetneq G \cap K_{\tilde{x}}$,

(2) $G \cap K_x$ acts transitively on $S_1^x \backslash \{\tilde{x}\}$.

THEOREM 2. *Let X be a homogeneous tree of order $r > 2$ and G a closed subgroup of* Aut(X) *which acts transitively on the vertices, then the following are equivalent.*

(a) *G is amenable.*

(b) *G fixes one end of X.*

(c) *G has property (*).*

Proof. As observed in Remark 2, (a)$\Longleftrightarrow$(b) follows from Theorem 1. If (*) holds then for every $x \in V$ there exists a unique $y = \tilde{x}$ such that $G \cap K_x \subsetneq G \cap K_{\tilde{x}}$; in particular $x \neq \tilde{\tilde{x}}$. This implies that there exists a unique G-invariant path $C_x = \{s_0, s_1, \ldots, s_n, \ldots\}$ starting at $s_0 = x$ such that $s_{i+1} = \tilde{s}_i$ for every i. To prove that G fixes one end of X it is enough to prove that $C_x \cap C_y$ is infinite for every $x, y \in V$.

If $x, y \in V$ and $d(x, y) = 1$ then $C_x \cap C_y$ is infinite because $\tilde{x} = y$ or $\tilde{y} = x$. In fact, let $W_x = S_1^x \backslash \{\tilde{x}\}$. Because $g(\tilde{x}) = [g(x)]^{\sim}$ for every $g \in G$ and $x \in V$, we have the following two mutually exclusive possibilities: (1) $\tilde{v} = x$ for every $v \in W_x$ and for every $x \in V$; (2) $\tilde{v} \neq x$ for every $v \in W_x$ and for every $x \in V$. But if $x \in V$ satisfies (2) then $\tilde{v}$ satisfies (1) because $\tilde{\tilde{v}} \neq v$ (by property (*)) and $v \in W_{\tilde{v}}$. This means that every vertex $x \in V$ satisfies (1). In particular $y = \tilde{x}$ or $x = \tilde{y}$. By induction on the distance $d(x, y)$ we have that $C_x \cap C_y$ is infinite for every $x, y \in V$, and (c)$\Rightarrow$(b). Conversely, we now show that (b)⇒(c). Let G be a closed amenable subgroup of $\mathrm{Aut}(X)$ which acts transitively on the vertices. Let $s_0 \in V$. Then G must fix some end $\omega = [\{s_0, s_1, s_2, \dots\}]$. We claim $(*)$ holds with $\tilde{s}_0 = s_1$. The fact that $G \cap K_{s_0} \subset G \cap K_{s_1}$ follows from the fact that G fixes ω. Suppose $u, v \in S_1^{s_0} \backslash \{s_1\}$. There exists $g \in G$ such that $g(u) = v$ because G acts transitively on the vertices. Since g fixes the end $\omega = [\{u, s_0, s_1, \dots\}] = [\{v, s_0, s_1, \dots\}]$, $g \in K_{s_0}$; proving (2) of $(*)$. The argument above implies that $G \cap K_{s_1}$ acts transitively on $S_1^{s_1} \backslash \{s_2\} \ni s_0$. If the tree is homogeneous of order > 2, then there exists an element $w \in S_1^{s_1} \backslash \{s_0, s_2\}$, this implies that there exists $k \in G \cap K_{s_1}$ such that $k(s_0) = w \neq s_0$. This means that $G \cap K_{s_0} \neq G \cap K_{s_1}$. Since G acts transitively on the vertices, property $(*)$ holds.

REMARK 5. If G and ϕ are as in the proof of Theorem 1 then $G = \langle\phi\rangle K_0 \langle\phi\rangle$. Since $G(s_0) = V$ and $\langle\phi\rangle(s_0) = \{s_n : n \in \mathbf{Z}\}$, it follows that, for every $v \in V$ with $d(s_0, v) = n$ and $v \neq s_{-n}$, there exists $k \in K_0$ such that $k(s_{-n}) = v$. In other words, G acts transitively on an open subset of Ω, in fact on $\Omega \backslash \{\omega_0\}$ where ω_0 is the fixed end of X.

3. Groups acting transitively on an open subset of Ω. In this section we prove the following result:

THEOREM 3. *Let X be a homogeneous tree and let G be a closed subgroup of* $\mathrm{Aut}(X)$ *which acts transitively on the vertices and on an open subset of Ω. Then either G fixes one end of X (i.e. G is amenable) or G acts transitively on Ω (i.e. G is a Kunze-Stein group).*

Proof. It is enough to prove that there exists $\omega_0 \in \Omega$ such that G acts transitively on $\Omega \backslash \{\omega_0\}$; then the theorem follows from Theorem 2 and [5].

It follows, by [6], that there exists a translation $\phi \in G$ on a line $C = \{s_n\}$. In this proof we realize Ω as the set of all infinite paths issued from s_0. Then the sets: $E(x) = \{\{t_n\} \in \Omega : t_j = x\}$ with $x \in V$ and $d(s_0, x) = j$ form a basis for the topology of Ω. Also, we observe that, for $\omega_0 \in \Omega$, the orbit $G\omega_0$ is open iff $(G \cap K_{s_0})\omega_0$ is open: this follows from Baire's theorem and the fact that $G/(G \cap K_{s_0})$ is countable.

In particular $G \cap K_{s_0}$ acts transitively on a set $E(x)$. Let $h \in G$ such that $h(s_0) = x$. Then $h\phi h^{-1}$ is a translation and, without loss of generality, we can suppose that ϕ is a translation on $C = \{s_n\}$ and $G \cap K_{s_0}$ acts transitively on $\complement E(s_{-1})$. Let $\omega_0 = \{s_0, s_{-1}, s_{-2}, \dots\}$.

Since G contains the translation ϕ, then G acts transitively on $\complement E(s_{-n})$ for every $n \geq 0$, hence on $\Omega \backslash \{\omega_0\} = \bigcup_{n \geq 0} \complement E(s_{-n})$.

REMARK 6. The same statement holds, with the same proof, for semihomogeneous tree $X_{l,q}$ and groups acting transitively on S_l.

Theorem 3 means that if there exists an open orbit of G on Ω then G is either amenable or a Kunze-Stein group. What happens if no orbit is open? As observed in Remark 5, G is not amenable.

We conjecture that G is not a Kunze-Stein group. However, we are able to prove this conjecture only in the special case of a homogeneous tree of order three.

Denote by Γ the simply transitive subgroup of Aut(X) isomorphic with $\mathbf{Z}_2 * \mathbf{Z}_2 * \cdots \mathbf{Z}_2$ r-times considered in [**1**, **4**].

LEMMA 1. *Let G act transitively on the vertices of a homogeneous tree. Then G contains a step one translation iff $G \neq \Gamma$.*

Proof. By [**1**, **4**] it is enough to prove that if $G \cap K_x \neq \{1\}$ then there exists a step one translation in G.

If $G \cap K_x \neq \{1\}$ then there exists $k \in G \cap K_x$ and $a \neq b$ with $a, b \in S_1^x$ such that $k(a) = b$; this follows from the fact that G acts transitively on the vertices. Choose g in G such that $g(x) = b$. Then g is either a step one translation or an inversion. If g is not a step one translation then $g(b) = x$; hence gk is a step one translation because $gk(a) = x$ and $gk(x) = b$.

In particular this translation exists if G is not discrete or if G is amenable.

LEMMA 2. *Let G be a closed subgroup of* Aut(X), *let $\phi \in G$ be a translation on a line $C = \{s_n\}$ and ω an end of C. If the orbit $(G \cap K_{s_0})\omega$ is finite, then G is not a Kunze-Stein group.*

Proof. Let K' be the stabilizer of ω in $G \cap K_{s_0}$. Let H be the subgroup of G generated by K' and ϕ. By assumptions $(G \cap K_{s_0})/K'$ is finite, therefore K' is open in $G \cap K_{s_0}$, hence in G. This implies that H is open in G and closed in Aut(X). By Theorem 1, H is amenable.

Therefore H is an open amenable noncompact subgroup of G. The result now follows from the facts that amenable noncompact groups are not Kunze-Stein groups and that every open subgroup of a Kunze-Stein group is a Kunze-Stein group.

PROPOSITION. *If G is a closed Kunze-Stein group which acts transitively on the vertices of a homogeneous tree of order three, then G acts transitively on Ω.*

Proof. Let G be a closed Kunze-Stein subgroup of $\mathrm{Aut}(X)$ which acts transitively on the vertices of X. Then G contains a step one translation ϕ on a line $C = \{s_n\}$ and $(G \cap K_{s_0})\omega_0$ is infinite (here $\omega_0 = \{s_0, s_1, \dots\} \in \Omega$). This follows from Lemma 1 and Lemma 2.

As ω_0 is an accumulation point of the orbit $(G \cap K_{s_0})\omega_0$, then there exists a sequence $n_m \to +\infty$ and a sequence $k_{n_m} \in G \cap K_{s_0}$ such that $k_{n_m}(s_{n_m}) = s_{n_m}$ but $k_{n_m}(s_{n_m+1}) \neq s_{n_m+1}$. Now the rotations $\phi^i k_{n_m} \phi^{-i}$ in $G \cap K_{s_0}$, for $i = 0, 1, \dots, n_m$, yield a sequence $\tilde{k}_n \in G \cap K_{s_0}$ such that $\tilde{k}_n(s_n) = s_n$ but $\tilde{k}_n(s_{n+1}) \neq s_{n+1}$ for every $n \geq 0$. The same argument applies to every point $\omega' = k\omega_0$ of the orbit $(G \cap K_{s_0})\omega_0$ by replacing C with $k(C)$ and ϕ with the step one translation $k\phi k^{-1}$ on $k(C)$. Since $G \cap K_{s_0}$ is compact, the orbit $(G \cap K_{s_0})\omega_0$ is closed. If $r = 3$, the argument above shows that ω_0 is an interior point of this orbit. Thus the orbit is open and, by Theorem 3, G acts transitively on Ω.

REMARK 7. Finally, we provide an example of a nondiscrete subgroup of $\mathrm{Aut}(X)$ acting transitively on the vertices and such that no orbit on Ω is open. Let E_1 be a subset of edges of X such that there are no two adjacent edges in E_1 and every vertex belongs to exactly one edge of E_1. Let G_1 be the stability group of E_1; then it is easy to see that G_1 is a closed subgroup of $\mathrm{Aut}(X)$ which acts transitively on the vertices but G_1 is not discrete because $G_1 \cap K_x$ is infinite.

If $g \in G_1$ and $\omega \in \Omega$ is a path which contains a finite number of edges of E_1 then also $g\omega$ contains a finite number of edges of E_1, in particular no orbit of G on Ω is open.

Observe, also that G_1 is not amenable, hence it does not satisfy the local property $(*)$ of Theorem 2. On the other hand, it satisfies the following property $(**)$:

$(**)$ For every $x \in V$ there exists $\tilde{x} \in V$ with $d(x, \tilde{x}) = 1$ and such that:

(1) $G \cap K_x = G \cap K_{\tilde{x}}$,

(2) $G \cap K_x$ acts transitively on $S_1^x \backslash \{\tilde{x}\}$.

In this case, if $y = \tilde{x}$ then $\tilde{y} = x$.

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