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THE GEOMETRY OF SUM-PRESERVING PERMUTATIONS

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Geometric characterizations of the semigroup of permutations which preserve convergence of series are presented.

1. Introduction. A well-known result of Riemann asserts that the sum of a conditionally convergent series may be changed to any value by suitably permuting the terms of the series. We introduce geometric tools to prove, among other results, that the set S of permutations which do not change the value of any convergent series is exactly the set of permutations which fix a type of asymptotic density of subsets of the natural numbers (Theorem 1.6).

Several authors ([Le], [A], see Schafer's survey article [Sch]) have given characterizations of the set S of permutations. The ideas of Levi [Le] are combinatoric in nature. Those of Agnew, on the other hand, come from the theory of summability of series; see e.g. Chapter III of [H], especially Theorems 1–3 of Schur and Toeplitz. In fact, consideration of Theorem 1 leads to certain geometric notions (Definitions 1.2 and 2.1) with which we express our characterizations of S.

1.1. NOTATION. If (a_i) is a sequence of real numbers, $\sum a_i$ denotes the limit of partial sums $\lim_{n\to\infty}\sum_{k=1}^{n} a_i$. Let $\mathscr{C} = \{(a_i): \sum a_i \in \mathbf{R}\}$ be the set of convergent series.

Let P denote the group of permutations of the natural numbers $N = \{1, 2, \dots\}$. If $\sigma \in P$ and (a_i) is a sequence, $\sigma(a_i)$ is the sequence whose ith term is $a_{\sigma^{-1}(i)}$. Let

$$S^* = \{\sigma \in P : (a_i) \in \mathscr{C} \text{ implies } \alpha(a_i) \in \mathscr{C}\},$$

$$S = \left\{\sigma \in S^* : (a_i) \in \mathscr{C} \text{ implies } \sum(a_i) = \sum \sigma(a_i)\right\}.$$

It is proven in [Sch] that $S = S^*$; we see this as Corollary 3.5. Note, however, that S and S^* are clearly semigroups.

Last, $\#X$ denotes the cardinality of a set X, and if $n, m \in \mathbf{N}$, $n \le m$, then $I(n, m) = \{n, n+1, \dots, m\}$.

We now give definitions necessary for the statement of our main results.

1.2. DEFINITION. Let $N = \{n_1 < \cdots < n_s\}$ and $M = \{m_1 < \cdots < m_s\}$, be subsets of the natural numbers. We say that M and N are *collated* if either $m_1 < n_1 < m_2 < n_2 < \cdots < m_s < n_s$ or else $n_1 < m_1 < n_2 < m_2 < \cdots n_s < m_s$. We say that M and N are *separated* if either $m_s < n_1$ or $n_s < m_1$. From now on, saying that M and N are separated or collated will imply that $\#M = \#N$.

1.3. DEFINITION. If $\sigma \in P$, *σ satisfies condition* A if there exists a natural number H so that if N and M are collated, and σN, σM separated, then $\#N = \#M < H$.

1.4. DEFINITION. Let $K = \{K_1 < K_2 < \cdots\}$ be an infinite subset of the natural numbers, and let $B \subseteq K$. Let $\alpha \in [0, 1]$. Then *the asymptotic density of B in K is α* and we write $D_K(B) = \alpha$ if, given $\varepsilon > 0$ there exists $N \in \mathbf{N}$ such that if $m - n > N$,

$$\left|\frac{\#(B \cap \{K_i\}_{i=n+1}^{m})}{m-n} - \alpha\right| < \varepsilon.$$

Note that $D_K(B)$ may not exist. If $\sigma \in P$, so that for all pairs $B \subseteq K$ so that $D_K(B)$ exists we have $D_{\sigma K}(\sigma B) = D_K(B)$, we say that σ preserves asymptotic density.

1.5. THEOREM. *If $\sigma \in P$, then $\sigma \in S$ if and only if σ satisfies condition* A.

1.6. THEOREM. *If $\sigma \in P$, then $\sigma \in S$ if and only if σ preserves asymptotic density.*

It seems difficult (see Example 1.8 and [St]) to arrive at combinatorial descriptions of the group G of elements of S whose inverses also are in S. However, Theorem 1.6 has the following corollary:

1.7. COROLLARY. *$\sigma \in G$ if and only if, given $B \subseteq K$, $D_K(B)$ is defined if and only if $D_{\sigma K}(\sigma B)$ is, and $D_K(B) = D_{\sigma K}(\sigma B)$.*

1.8. EXAMPLE. Let $\sigma_n : I(1, 2n) \to I(1, 2n)$ be defined by

$$\sigma_n(2i + 1) = i + 1, \qquad \sigma_n(2i) = n + i.$$

Then σ_n separates the (collated) odd and even numbers in $I(1,2n)$. Further, using the graph of σ (see 2.1) one sees that σ_n^{-1} never separates collated sets whose cardinality exceeds 2.

Let $\sigma \in P$ be defined as σ_1 on $I(1,2)$, σ_2 on $I(3,6)$, σ_3 on $I(7,12)$ on so on. By Theorem 1.5, $\sigma \notin S$, but $\sigma^{-1} \in S$. (Note also that $D_{\mathbf{N}}(2\mathbf{N}) = \frac{1}{2}$, but that $D_{\mathbf{N}}(\sigma 2\mathbf{N})$ is not defined.) Further, let τ be defined as σ_1 on $I(1,2)$, σ_2^{-1} on $I(3,6)$, σ_3 on $I(7,12)$, σ_4^{-1} on $I(13,20)$ and so on. Then neither τ nor τ^{-1} is in S.

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1.10. ORGANIZATION. In §2 we introduce the graph of a permutation and establish some results. In §3 the idea of total family helps complete the proof of 1.5. In §4 we discuss asymptotic density, and its relation with equidistribution.

2. The graph of a permutation.

2.1. DEFINITION. Let $\sigma \in P$. Consider, in the plane $\mathbf{R} \times \mathbf{R}$ the polygonal path obtained by connecting the points $(i, \sigma(i))$ and $(i+1, \sigma(i+1))$ for all i. This path is the graph of a function called $g(\sigma)$. The path is called the *graph* of σ.

2.2. PROPOSITION. *If $\sigma \in P$, the following are equivalent:*

(i) *there exists $H \in \mathbf{N}$ so that $\#(g(\sigma)^{-1}(r)) < H$ for all $r \in [1, \infty)$.*

(ii) *σ satisfies condition* A.

(iii) *There exists $H' \in \mathbf{N}$ so that, given $n, m \in \mathbf{N}$, if $\sigma^{-1}I(n,m)$ is written as the union $\bigcup_{j=1}^{s} I(n_j, m_j)$ with $m_j < n_{j+1} - 1$, then $s \leq H'$ (R. P. Agnew, see* [Sch]*).*

To prove 2.2 we use the following lemma, whose proof we leave for the reader.

2.3. LEMMA. *If σ does not satisfy condition* A *then there is an increasing sequence $K_1 < K_2 < \cdots$ of natural numbers, and sequences $\{A_i\}$, $\{B_i\}$ of subsets of* $\mathbf{N}$ *so that*

(i) A_i, B_i, $\sigma(A_i)$, $\sigma(B_i) \subseteq I(K_i, K_{i+1})$,

(ii) *A_i and B_i are collated, and σA_i and σB_i are separated, with $\sigma a < \sigma b$ for $a \in A_i$, $b \in B_i$,*

(iii) *$\#A_i < \#A_{i+1}$ for all i.*

Proof of 2.2. (i)⇒(ii) Let $\sigma \in P$ so that σ does not satisfy condition A; there exist $\{A_i\}$, $\{B_i\}$, $\{K_i\}$ as in Lemma 2.3. For each i, let $M_i = \min\{\sigma(b) : b \in B_i\}$ and $m_i = \max\{\sigma(a) : a \in A_i\}$, and set $r_i = (m_i + M_i)/2$. Since $\sigma(a) < r_i < \sigma(b)$ for $a \in A_i$, $b \in B_i$ we have that $g(\sigma)^{-1}(r_i) \geq \#A_i$. By 2.3 (iii), 2.2 (i) does not hold.

(ii)⇒(i) Let $\sigma \in P$, and consider the function $g(\sigma)$. Local maxima and minima of $g(\sigma)$ occur only at natural numbers; further, a value $n \in \mathbf{N}$ can occur at most once as a local maximum or minimum of $g(\sigma)$. Thus, given $r \in [1, \infty]$, we can perturb r slightly, to r', so that r' is not a local maximum or minimum of $g(\sigma)$, and further $|\#g(\sigma)^{-1}(r) - \#g(\sigma)^{-1}(r')| \leq 1$.

Let $g(\sigma)^{-1}(r') = \{x_i : i = 1, \ldots, n\}$ with $x_i < x_{i+1}$. For each i, there exist $a, b \in \mathbf{N}$, $a < x_i < b$, so that $\sigma(a) < r' < \sigma(b)$; let a_i (resp. b_i) be the sup (resp. inf) of all such a (resp. b). Then the sets $A = \{a_{2i+1}\}$, $B = \{b_{2i+1}\}$ are disjoint, collated and σA and σB are separated, and $\#A = \#B > \frac{1}{2}(\#g(\sigma)^{-1}(r')) - 1$. Thus if the $\#g(\sigma)^{-1}(r)$ are unbounded, σ does not satisfy condition A.

(i)⇒(iii) Let $\sigma \in P$, with H as in 2.2 (i), and $n, m \in \mathbf{N}$. Suppose that $\sigma^{-1}I(n, m) = \bigcup_{i=1}^{s} I(n_i, m_i)$ with $m_i < n_{i+1} - 1$. By continuity, $g(\sigma)^{-1}[n, m] = \bigcup_{j=1}^{k}[x_i, y_i]$ (where $[a, a]$ denotes $\{a\}$ if necessary), and $k < s$. But

$$K < \#\{x_i, y_i, i = 1, \ldots, K\} = \#(g(\sigma)^{-1}(n) \cup g(\sigma)^{-1}(m)) < 2H,$$

so $S < 2H$.

(iii)⇒(ii) Let $\sigma \in P$, and suppose that A and B are collated, and σA and σB separated, with $\sigma(a) < \sigma(b)$ for $a \in A$, $b \in B$. Let $n = \sup\{\sigma(a) : a \in A\}$. Then $\sigma^{-1}I(1, m) = \bigcup_{j=1}^{s} I(n_j, m_j)$ with $s > \#A$. Hence if condition A is not satisfied, neither is 2.2 (iii).

This completes the proof of 2.2. We now prove part of Theorem 1.5. Actually, this is the same proof after the first sentence, as given in [**Sch**]; we include it because it is short, and for completeness.

2.4. Proposition. *If $\sigma \in P$ and σ satisfies condition* A, *then $\sigma \in S$.*

Proof. We show that if σ satisfies 2.2 (iii), then $\sigma \in S$. So, let $H' \in \mathbf{N}$ so that if $n, m \in \mathbf{N}$ and $\sigma^{-1}I(m, n) = \bigcup_{j=1}^{s} I(m_j, n_j)$ with

$m_j < n_{j+1} - 1$, then $s < H'$. Let $\varepsilon > 0$; we show that there exists an $N \in \mathbf{N}$ so that if $n \geq N$,

$$\left|\sum_{i=1}^{n} a_i - \sum_{i=1}^{n} a_{\sigma^{-1}(i)}\right| < \varepsilon.$$

Since $\sum a_i \in R$, there is some M so that if $n, m \geq M$, $\left|\sum_{i=m}^{n} a_i\right| < \varepsilon/2H'$. Let $N = \max\{\sigma(1), \sigma(2), \ldots, \sigma(M), M\}$. Note that if $n > N$, we can write $\sigma^{-1}I(1,n)$ as the disjoint union $\sigma^{-1}I(1,n) = I(1,m_1) \cup \bigcup_{i=2}^{k} I(m_i, n_i)$ where $M \leq m_1 \leq n$. Further, $K < H'$.

Therefore, if $n > N$ we have

$$\left|\sum_{i=1}^{n} a_i - \sum_{i=1}^{n} a_{\sigma^{-1}(i)}\right| = \left|\sum_{i=1}^{m_1} a_i - \sum_{i=m_1+1}^{n} a_i - \left(\sum_{i=1}^{m_1} a_i + \sum_{j=2}^{k} \sum_{i=m_j}^{n_j} a_i\right)\right|$$

$$\leq \left|\sum_{i=m_1+1}^{n} a_i\right| + \sum_{j=2}^{k} \left|\sum_{i=m_j}^{n_j} a_i\right| \leq \varepsilon/2H' + H'\varepsilon/2H' < \varepsilon. \qquad \square$$

3. Complete families. Recall that $\mathscr{C} = \{(a_i) : \sum a_i \in \mathbf{R}\}$.

3.1. Definition. If $B \subseteq \mathscr{C}$, B is a *complete family* if given $\sigma \in P$, $\sigma \notin S$ there exits $(b_i) \in B$ such that $\sum_{i=1}^{\infty} b_{\sigma^{-1}(i)}$ is undefined or unequal to $\sum b_i$.

3.2. Definition. Let (a_i) be a sequence, and $K = \{K_j : K_j \in \mathbf{N}, K_i < K_{i+1}\}$ an infinite subset of $\mathbf{N}$. By $K(a_i)$ we mean the sequence b_i, where $b_i = a_j$ if $i = K_j$, $b_i = 0$ if $i \notin K$. Let $E(a_i) = \{K(a_i) : K \subseteq \mathbf{N}$ infinite$\}$.

3.3. Proposition. *Let $(a_i) \in \mathscr{C}$ so that $(|a_i|) \notin \mathscr{C}$. If σ does not satisfy condition* A, *there exists $K \subset \mathbf{N}$ so that $\sigma(K(a_i)) \notin \mathscr{C}$.*

Proof. We can assume that $a_{2j} \geq 0$, $a_{2j+1} \leq 0$; if not, insert zeros in the sequence (a_i), obtaining (by "dilution", [H]) a $(b_i) = K(a_i)$ so that $b_{2j} \geq 0$, $b_{2j+1} \leq 0$. Let j_n, $n \geq 1$, be the smallest natural number so that $\sum_{i \leq j_n} a_{2i} > n$ and $\sum_{i \leq j_n} a_{2i+1} < -n$. Since $\lim_{n\to\infty} a_n = 0$, $\lim(j_n - j_{n-1}) = \infty$. By Lemma 2.3, there exist pairwise disjoint subsets $A_n, B_n \subseteq \mathbf{N}$ so that A_n, B_n are collated, σA_n, σB_n are separated, so that $\# A_n > j_n - j_{n-1}$, and such that if $a \in A_i$, $a' \in A_{i+1}$, $b \in B_i$, $b' \in B_{i+1}$ then $a < a'$, b' and $b < a'$, b'. Let $f_n : I(2_{j_{n-1}} + 1, 2_{j_n}) \to A_n \cup B_n$ be an order preserving one to one map taking even numbers to A_n and odd numbers to B_n. Let $K = \bigcup_{n=1}^{\infty} \text{Image}(f_n)$. It is not hard to see that the points of accumulation of the partial sums of $\sigma K(a_i)$ include 0 and 1. □

Proof of 1.5. This is immediate from 3.3 and 2.3, as are:

3.4. COROLLARY. *If* $(a_i) \in \mathscr{C}$, $(|a_i|) \notin \mathscr{C}$ *then* $E(a_i)$ *is a complete family.*

3.5. COROLLARY. $S = S^*$.

With respect to 3.4, the $E(a_i)$ are not *minimal* complete families. (For example, $E(a_i) - \{\mathbf{N}\}$ is complete.) However, we have the following:

3.6. PROPOSITION. *There are no countable complete families.*

Proof. Let, for $j > 1$, $j \in \mathbf{N}$, $(a_i^j) \in \mathscr{C}$. We will prove that there exists $\sigma \in P$, $\sigma \notin S$ so that for all j, $\sum a^j_{\sigma^{-1}(i)} = \sum a_i^j$.

First note that by Theorem 1.5, given any infinite subset $K \subseteq N$ there exists $\sigma \in P$, $\sigma \notin S$ so that $\sigma(j) = j$ if $j \notin K$. Thus, it suffices to find a sequence $K_1 < K_2 < \cdots$ such that for all j, the series $\sum a^j_{K_i}$ is absolutely convergent.

Now let K_1 be the smallest natural number so that $|a^1_{K_1}| < \frac{1}{2}$. Assuming $K_1, \dots, K_{l-1}$ defined, let K_l be the smallest natural number, $K_l > K_{l-1}$ so that $|a^1_{K_l}|, |a^2_{K_l}|, \dots, |a^l_{K_l}| < 2^{-l}$. Then for all j, $\sum_{i=j}^{\infty} |a^j_{K_i}| < 2^{1-j}$. □

4. Density. In one direction, the proof of Theorem 1.6 is a straightforward application of Lemma 2.3.

4.1. PROPOSITION. *If* $\sigma \in P$, $\sigma \notin S$ *then there exists* $K \subseteq \mathbf{N}$, $A \subseteq K$ *so that* $D_K(A) = \frac{1}{2}$ *but* $D_{\sigma K}(\sigma A)$ *is not defined.*

Proof. Let A_n, B_n as in Lemma 2.3. Set $K = \bigcup_{n=1}^{\infty} (A_n \cup B_n)$, and $A = \bigcup A_n$. Then $D_K A = \frac{1}{2}$, but $D_{\sigma K}(\sigma A)$ is not defined. □

The following completes the proof of 1.6.

4.2. PROPOSITION. *If* $\sigma \in S$ *then if* $A \subseteq K \subseteq \mathbf{N}$ *and* $D_K(A) = \alpha \in [0,1]$ *then* $D_{\sigma K}(\sigma A) = \alpha$.

Proof. We first reduce to the case where $K = \mathbf{N}$, as follows. If $K \subseteq \mathbf{N}$ is any infinite subset, let $i_K \colon K \to \mathbf{N}$ be the unique order preserving bijection. Given $\sigma \in P$, define $\sigma_K \in P$ by $\sigma_K = i_{\sigma(K)} \sigma i_K^{-1}$. If σ satisfies condition A, so does σ_K; thus $\sigma \to \sigma_K$ is a function from S to S, a homomorphism of semigroups if $K = \sigma K$. Further, if $D_K(A) = \alpha$, $D_N(i_K A) = \alpha$ and $D_{\sigma K}(\sigma A) = \alpha$ if and only if $D_{\mathbf{N}}(\sigma_K(i_K A)) = \alpha$.

So assume that $K = \mathbf{N}$, and $A \subseteq \mathbf{N}$ with $D_{\mathbf{N}}(A) = \alpha$, and let $\sigma \in S$. We begin with a lemma: let H' be the bound guaranteed in Proposition 2.2 (iii). Let $m, n \in \mathbf{N}$ and write $\sigma^{-1} I(n,m) = \bigcup_{i=1}^k I(m_i, n_i)$ with $n_i < m_{i+1} - 1$. Let $N \in \mathbf{N}$, and let $L_N = \{i \colon n_i - m_i + 1 < N\}$. (Note that $\# I(m,n) = n - m + 1$.)

4.3. LEMMA. *Given* $\varepsilon > 0$ *there exists* $M \in \mathbf{N}$ *such that if* $n - m + 1 > M$,

$$\frac{1}{n-m+1} \sum_{i \in L_N} (n_i - m_i + 1) < \varepsilon.$$

Proof. Pick $M \geq H'N/\varepsilon$. Since $\# L_N \geq H'$,

$$\frac{1}{n-m+1} \sum_{i \in L_N} (n_i - m_i + 1) < \frac{1}{n-m+1} H'N < \varepsilon. \qquad \square$$

Now we proceed with the proof that $D_{\mathbf{N}}(\sigma A) = \alpha$. That is, let $\varepsilon > 0$. We show that there exists N so that if $n - m > N$ then

$$\left| \frac{\#(\sigma A \cap I(m,n))}{n-m+1} - \alpha \right| < \varepsilon.$$

Since $D_{\mathbf{N}}(A) = \alpha$, there exists an N' so that if $n - m > N'$ then

$$|\#(A \cap I(m,n))/(n-m+1) - \alpha| < \varepsilon/3H'. \tag{4.4}$$

By Lemma 4.3 there exists an M so that if $n - m > M$, then

$$\frac{1}{n-m+1} \sum_{i \in L_{N'}} (n_i - m_i + 1) < \varepsilon/3. \tag{4.5}$$

Let $N = \max(N', M)$. Since $\sum_{i=1}^{k} n_i - m_i + 1 = n - m + 1$, we obtain the following estimation:

$$\begin{aligned}
\left|\frac{\#\sigma A \cap I(m,n))}{n-m+1} - \alpha\right| &= \left|\frac{1}{n-m+1}\sum_{i=1}^{k}\#(A \cap I(m_i, n_i)) - \alpha\right| \\
&\leq \left|\sum_{i \in L_{N'}} \frac{\#(A \cap I(m_i, n_i))}{n-m+1}\right| \\
&\quad + \left|\sum_{i \notin L_{N'}} \frac{\#(A \cap I(m_i, n_i))}{n-m+1} - \alpha \sum_{i=1}^{k} \frac{n_i - m_i + 1}{n-m+1}\right| \\
&\leq \left|\sum_{i \in L_{N'}} \frac{\#(A \cap I(m_i, n_i))}{n-m+1}\right| \\
&\quad + \sum_{i \notin L_{N'}} \left|\frac{\#(A \cap I(m_i, n_i))}{n_i - m_i + 1} - \alpha\right| \frac{(n_i - m_i + 1)}{n-m+1} \\
&\quad + \alpha \sum_{i \in L_{N'}} \frac{n_i + m_i + 1}{n-m+1} \\
&< \varepsilon/3 + H'\varepsilon/3H' + \varepsilon/3 = \varepsilon.
\end{aligned}$$

where (4.4) is used to estimate the first and third terms, and (4.5) for the estimation of the second term. □

For examples of sets of natural numbers with arbitrary asymptotic density, let us consider the circle, as $[0, 1]$ with 0 and 1 identified. If $a, b \in [0, 1]$, let $[a, b]$ be the interval from a to b in the circle. If $a < b$, the length $l[a, b] = b - a$, while if $b < a$, $l[a, b] = 1 - a + b$. Let $r \in \mathbf{R}$, and let $nr \in [0, 1]$ be the value of nr modulo 1. Let

$$J^r_{[a,b]} = \{n \in \mathbf{N}\colon nr \in [a, b]\}.$$

4.6. Proposition. *If r is irrational, $D_{\mathbf{N}}(J^r_{[a,b]}) = l[a, b]$.*

Proof.

$$\frac{\#\left(I(m,n) \cap J^r_{[a,b]}\right)}{n-m+1} = \frac{\#\{k; m < k < n, kr \in [a, b]\}}{n-m+1}.$$

The fact that, as $n - m$ tends to ∞ the latter expression approaches $l[a, b]$ is the Weyl Equidistribution Theorem when $m = 1$; the same proof works for general m (see e.g. **[DMc**, p. 54**]**).

4.7. DEFINITION. Let $\sigma \in P$, $a, b \in [0, 1]$, and $r \in \mathbf{R}$ irrational. Let

$$\mu^{\sigma,r}[a, b] = \lim_{n\to\infty} \frac{\#\{[a, b] \cap \{\sigma^{-1}(k)r\}_{k=1}^{n}\}}{n}$$

when the limit exists.

The theorem of Weyl asserts that if σ is the identity, $\mu^{\sigma,r}[a, b] = l[a, b]$.

4.8. THEOREM. *If* $\sigma \in S$, $\mu^{\sigma,r}[a, b] = l[a, b]$.

Proof. As in the proof of Proposition 4.6,

$$\begin{aligned}\mu^{\sigma,r}[a, b] &= \lim_{n\to\infty} \frac{\#\{[a, b] \cap \{\sigma^{-1}(k)r\}_{k=1}^{n}\}}{n} \\ &= \lim_{n\to\infty} \frac{\#\{I(1, n) \cap \sigma J_{[a,b]}^{r}\}}{n} \\ &= l[a, b] \quad \text{by Proposition 4.6.}\end{aligned}$$

One might well ask: What is the set of $\sigma \in P$ such that $\mu^{\sigma,r}[a, b] = l[a, b]$? For which σ does $\mu^{\sigma,r}$ extend to some measure; when is the limit in 4.7 well defined? And if so, when is the measure continuous, smooth, etc., with respect to the usual Lebesgue measure?

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