PACIFIC JOURNAL OF MATHEMATICS
Vol. 135, No. 2, 1988

A LOCALIZED ERDÖS-WINTNER THEOREM

P.D.T.A. ELLIOTT

In this paper I show that a form of the well-known Erdös-Wintner theorem for additive arithmetic functions holds, even if the information is only given on widely separated intervals.

For $y \geq x \geq 2$ let

$$\nu_{x,y}(n; f(n) \leq z) \tag{1}$$

denote the frequency amongst the integers n in the interval $(x-y, x]$, of those for which the real additive function $f(n)$ does not exceed z.

THEOREM. *Let $c > 1$. Let N_j be an increasing sequence of positive integers for which $N_{j+1} \leq N_j^c$. Let M_j be a further sequence of integers, $M_j \leq N_j$, $\log M_j / \log N_j \to 1$, as $j \to \infty$.*

In order that the frequencies

$$\nu_{N_j,M_j}(n; f(n) \leq z) \tag{2}$$

converge weakly, as $j \to \infty$, it is necessary and sufficient that the three series

$$\sum_{|f(p)|>1} \frac{1}{p}, \quad \sum_{|f(p)|\leq 1} \frac{f(p)}{p}, \quad \sum_{|f(p)|\leq 1} \frac{f(p)^2}{p} \tag{3}$$

converge.

When $N_j = j$, $M_j = j$ this is the well-known theorem of Erdös, Erdös and Wintner [5]. For $N_j = j$ and any M_j which satisfies $M_j/N_j \to 0$, together with the above condition $\log M_j \sim \log N_j$, it was proved by Hildebrand [7].

The present argument differs from theirs in many respects.

2. Preliminary results. It is convenient to introduce the Lévy-distance $\rho(F, G)$ between distributions $F(z)$ and $G(z)$ on the line, defined as the greatest lower bound of those real h for which

$$F(z-h) - h \leq G(z) \leq F(z+h) + h$$

for all z. Convergence in the topology which this induces on the space of distribution functions, is equivalent to the usual weak-convergence of measures.

For primes $p \leq x$ let Y_p be independent random variables distributed according to

$$Y_p = \begin{cases} f(p^\alpha) \text{ with probability } \dfrac{1}{p^\alpha}\left(1 - \dfrac{1}{p}\right), & 0 \leq \alpha < \gamma_p, \\ f(p^{\gamma_p}) \text{ with probability } \dfrac{1}{p^{\gamma_p}} & \end{cases}$$

where $\gamma_p = [\log x / \log p]$.

Let

$$G_x(z) = P\left(\sum_{p \leq x} Y_p \leq z\right),$$

and let $F_x(z)$ denote the frequency distribution function (1).

LEMMA 1. *There is a positive absolute constant c so that*

$$\rho(F_x, G_x) \leq c\left(\sum_{\substack{y^\varepsilon < q \leq y \\ |f(q)| > u}} \frac{1}{q} + \frac{u}{\varepsilon} + \exp\left(-\frac{1}{80\varepsilon}\log\frac{1}{\varepsilon}\right) + \frac{1}{\log y} + \frac{\log\frac{x}{y}}{\log x}\right)$$

holds uniformly for all $u > 0$, $x \geq y \geq x^{2/3} \geq 3$, $x^\varepsilon \geq (\log x)^3$, $0 < \varepsilon \leq 1$, *and* $f(q)$, *where* q *denotes a prime-power.*

Proof. Inequalities of this type are obtained in Elliott [**1**] Chapter 12, [**2**] Lemma 6. In the main they depend upon the application of a finite probability model constructed with the aid of Selberg's sieve method. The necessary background results can be found in Elliott [**1**], Chapter 3.

For an arithmetic function g, $M(g, x)$ will denote

$$\sum_{n \leq x} g(n).$$

For real α, g_α will denote the modified arithmetic function $n \mapsto g(n)n^{i\alpha}$.

LEMMA 2. *Let g be a complex-valued multiplicative function,* $|g(n)| \leq 1$ *for positive n; and* $x \geq y \geq 3$. *Then*

$$M(g, x) - M(g, x - y) = \frac{M(g_\alpha, x)}{x}\int_{x-y}^{x} t^{-i\alpha}\,dt + O(yR(x, y))$$

where α is any real number, $|\alpha| \le x$, for which

$$|M(g_\alpha, x)| = \max_{|\beta|\le x} |M(g_\beta, x)|$$

and

$$R(x, y) = \left(\log \frac{\log 2x}{\log 2x/y}\right)^{-1/4}$$

Proof. This is Theorem 4 of Hildebrand [7].

LEMMA 3. *In the notation of Lemma 2, define the Dirichlet series*

$$G(s) = \sum_{n=1}^{\infty} \frac{g(n)}{n^s}.$$

Then

$$M(g, x) \ll x \left(T^{-1} + \frac{1}{\log x} \max_{|\tau|\le T} \left|G\left(1 + \frac{1}{\log x} + i\tau\right)\right|\right)^{1/5}$$

uniformly in all multiplicative functions g with $|g(n)| \le 1$, and in x, $T \ge 2$.

Proof. This result is due essentially to Halász [6], a detailed proof may be found in Elliott [1], Lemma (6.10).

LEMMA 4. *If*

$$\operatorname{Re} \sum_{p\le x} p^{-1}(1 - p^{i\lambda}) \ll 1$$

for some real λ, $|\lambda| \le x$, then $\lambda \ll (\log x)^{-1}$.

Proof. If $\delta = 1 + 1/\log x$, then the hypothesis of this lemma asserts that the Riemann-function $\zeta(s)$ satisfies

$$\log \left| \frac{\zeta(\delta)}{\zeta(\delta + i\lambda)} \right| \ll 1$$

uniformly in $x \ge 3$. The conclusion now follows from application of the bounds

$$\zeta(\sigma + it) = \begin{cases} \dfrac{1}{\sigma + it - 1} + O(1) & \text{if } \sigma > 1, |t| \le 2, \\ O((\log|t|)^{2/3}) & \text{if } \sigma > 1, |t| > 2, \end{cases}$$

the proofs of which may be found in Ellison and Mendès-France [4].

LEMMA 5. *Let the bounded function u, defined on the interval* $[-1, 1]$, *satisfy*

$$|u(t_1 + t_2) - u(t_1) - u(t_2)| \le K$$

whenever t_1, t_2 *and* $t_1 + t_2$ *belong to the interval. Then*

$$|u(t) - u(1)t| \le 3K.$$

Proof. This is established in Ruzsa [**9**]. It extends an earlier result of Hyers [**8**].

LEMMA 6. *Suppose that for a sequence of real numbers* α_n *the limit (as* $n \to \infty$*) of* $\exp(it\alpha_n)$ *exists uniformly on some open interval of real* t*-values including* $t = 0$. *Then* $\lim \alpha_n$ *exists (finitely).*

Proof. (Cf. Elliott and Ryavec [**3**].) Since $(e^{it\alpha_n})^2 = \exp(i2t\alpha_n)$, we see that the hypothesis holds on every bounded set of t-values. Here $\exp(it\alpha_n)$ is the characteristic function of the improper distribution function $H_n(z)$ which has a jump at the point α_n. It follows from a standard theorem in the theory of probability that the $H_n(z)$ converge weakly to a distribution function $J(z)$, say.

It is now not difficult to deduce that the α_n are bounded uniformly for all n, that $J(z)$ is itself improper, with a jump at β, say; and that $\alpha_n \to \beta$ as $n \to \infty$.

LEMMA 7. *Let* $P_j(x)$ *be polynomials in* x *with complex coefficients, and* d_j *distinct real numbers,* $j = 1, \dots, k$. *If*

$$\theta(t) = \sum_{j=1}^{k} P_j(t)e^{id_jt} = 0$$

on a proper interval of real t*-values, then the polynomials are identically zero.*

Proof. Without loss of generality $0 = d_1 > d_2 > \cdots > d_k$. As a function of the *complex-variable* t, $\theta(t)$ is everywhere analytic. After the hypothesis, analytic continuation shows that $\theta(t)$ is identically zero. We set $t = -iy$ for real y, and consider

$$\lim_{y \to \infty} y^{-m}\theta(-iy)$$

where m is the degree of P_1.

The terms $P_j(-iy)\exp(d_jy)$ with $j \ge 2$ converge exponentially to zero, whilst $y^{-m}P_1(-iy)$ approaches $(-i)^m$ times the coefficient of x^m

in P_1. Since the value of this limit is zero, $P_1(x)$ is identically zero. An argument by induction completes the proof of the lemma.

3. Proof of the theorem: (3) *implies* (2). Define independent random variables Z_p by

$$Z_p = \begin{cases} Y_p & \text{if } Y = f(p), \\ 0 & \text{otherwise.} \end{cases}$$

The convergence of the three series at (3) is precisely Kolmogorov's condition that the series $Z_2 + Z_3 + \cdots$ be almost surely convergent. Moreover,

$$\sum_p P(Z_p \neq Y_p) \leq \sum_p \sum_{m=2}^{\infty} \frac{1}{p^m} < \infty,$$

so that by the Borel-Cantelli lemma, $Y_2 + Y_3 + \cdots$ is also almost surely convergent. This is equivalent to the weak convergence of the distribution functions $G_x(z)$ appearing in Lemma 1. The relevant background results from the theory of probability may be found in Elliott **[1]**, Lemma (1.18).

We apply Lemma 1 with $x = N_j$, $y = M_j$. Since the series $\sum p^{-1}$ taken over those primes p for which $|f(p)| > u$ converges for each positive u,

$$\limsup_{j\to\infty} \rho(F_{N_j}, G_{N_j}) \leq c\left(\frac{u}{\varepsilon} + \exp\left(-\frac{1}{80\varepsilon}\log\frac{1}{\varepsilon}\right)\right)$$

for all $u > 0$, $0 < \varepsilon < 1$. Letting $u \to 0+$, $\varepsilon \to 0+$ we obtain the weak convergence of the frequencies (2).

In this direction no restriction upon the rate of growth of the N_j need be assumed.

4. Proof of the theorem: (2) *implies* (3). The characteristic function of a typical frequency (2) is given by

$$\phi_j(t) = M_j^{-1} \sum_{N_j - M_j < n \leq N_j} g(n),$$

where $g(n) = \exp(itf(n))$ is a multiplicative function, and t is real. If the frequencies (2) converge weakly to a distribution function with characteristic function $\phi(t)$, then by a standard result in the theory of probability, $\phi_j(t) \to \phi(t)$ as $j \to \infty$, uniformly on any bounded interval of t-values.

If we temporarily use x, y to denote N_j, M_j respectively, then it follows from Lemma 2 that

$$\phi(t) = x^{-1}M(g_\alpha, x)y^{-1}\int_{x-y}^{x} v^{-i\alpha}\,dv + o(1), \quad x \to \infty, \tag{4}$$

for some real α, $|\alpha| \le x$. Since $\phi(t)$ is continuous in t, and $\phi(0) = 1$, there is a proper interval $|t| \le \tau$, on which $|\phi(t)| \ge 1/2$. On this same interval $|M(g_\alpha, x)| \ge x/4$ for all sufficiently large x $(= N_j)$. The parameter α may depend upon both t and x.

Applying Lemma 3 with $T = \log x$ gives

$$M(g_\alpha, x) \ll x \exp\left(-\frac{1}{5} \operatorname{Re} \sum_{p \le x} \frac{1 - g(p)p^{i\psi}}{p}\right) + x(\log x)^{-1/5}$$

for some real ψ, $|\psi(x) - \alpha| \le \log x$. Thus $|\psi(x)| \le x + \log x$. In view of the lower bound for $|M(g_\alpha, x)|$

$$\operatorname{Re} \sum_{p \le x} \frac{1 - g(p)p^{i\psi}}{p} \ll 1.$$

We first show that $\psi = \psi(t)$ is essentially linear in t.

Let

$$S(f) = \sum_{p \le x} p^{-1} \left(\operatorname{Sin} \frac{f(p)}{2}\right)^2.$$

Then since $|\operatorname{Sin}(a + b)| \le |\operatorname{Sin} a| + |\operatorname{Sin} b|$,

$$S(f_1 + f_2) \le 2(S(f_1) + S(f_2)). \tag{5}$$

With $g(p) = \exp(itf(p))$,

$$\begin{aligned} \operatorname{Re}(1 - g(p)p^{i\psi}) &= \operatorname{Re}(1 - \exp(i(tf(p) + \psi(t) \log p))) \\ &= 2\left(\operatorname{Sin} \frac{1}{2}(tf(p) + \psi(t) \log p)\right)^2 \end{aligned}$$

so that

$$S(tf + \psi(t) \log) \ll 1$$

uniformly for $|t| \le \tau$.

In view of the inequality (5), whenever $|t_j| \le \tau$, $j = 1, 2$, $|t_1 + t_2| \le \tau$,

$$S((\psi(t_1 + t_2) - \psi(t_1) - \psi(t_2)) \log) \ll 1,$$

so that by Lemma 4

$$\psi(t_1 + t_2) - \psi(t_1) - \psi(t_2) \ll (\log x)^{-1}.$$

We can now apply Lemma 5, to deduce that

$$\psi(t) = t\psi(\tau)/\tau + O((\log x)^{-1}).$$

Then

$$\sum_{p\le x}\frac{1}{p}|p^{i\psi(t)} - p^{it\psi(\tau)/\tau}| \le |\psi(t) - t\psi(\tau)/\tau| \sum_{p\le x}\frac{\log p}{p} \ll 1$$

uniformly for $|t| \le \tau$. Thus

$$S(t(f - \omega(x)\log)) \ll 1 \tag{6}$$

holds, uniformly for $|t| \le \tau$, for some function $\omega(x)$ of x alone.

Up until this point the proof has followed Elliott [2]. The relative sizes of the N_j now comes into play.

For all sufficiently large integers j, the interval $(2^{c^j}, 2^{c^{j+1}}]$ contains at least one member, r_j say, of the sequence of N_i. Since $r_{j+2} \ge r_j^c$, by induction

$$\frac{\log r_m}{\log r_n} \ge (\sqrt{c})^{m-n-1} \tag{7}$$

for all $m \ge n \ge$ (some fixed) n_0.

From their definition $r_{m+1} \le r_m^{c^2}$. By an elementary estimate from number theory

$$\sum_{r_m < p \le r_{m+1}} \frac{1}{p} = \log\left(\frac{\log r_{m+1}}{\log r_m}\right) + O((\log r_m)^{-1}) \ll 1,$$

so that

$$\operatorname{Re}\sum_{p\le r_m}\frac{1}{p}(1 - g(p)p^{it\omega}) \ll 1$$

holds for both $\omega = \omega(r_m)$, and $\omega = \omega(r_{m+1})$. Another application of Lemma 4 yields

$$|\omega(r_{m+1}) - \omega(r_m)| \le \frac{D}{\log r_m}$$

for some D and all positive m.

Employing our lower bound (7), an argument by induction shows that

$$|\omega(r_m) - \omega(r_n)| \le \frac{2D}{\log r_n}\sum_{n<k\le m} c^{-(k-n-1)/2} \tag{8}$$

uniformly for $m \ge n \ge n_0$. In particular the $\omega(r_m)$ form a Cauchy sequence, and converge to a limit, A say. Letting $m \to \infty$ in (8) gives

$$\omega(r_n) - A \ll (\log r_n)^{-1}$$

for $n \ge n_0$.

Since every large enough N_j lies in an interval $(r_m, r_{m+1}]$,

$$\omega(N_j) - A \ll (\log N_j)^{-1}$$

for all j. In the way that we replaced $\psi(t)$ by $t\psi(\tau)/\tau$ we replace $\omega(N_j)$ by A, to obtain

$$S(t(f - A\log)) \ll 1$$

uniformly for $|t| \leq \tau$, for all sufficiently large (underlying) N_j.

Again we argue as in Elliott [2]. Let d denote $\pi/|\tau|$. The inequality $|\operatorname{Sin}\theta| \geq 2|\theta|/\pi$ holds for $|\theta| \leq \pi/2$. With $h(p) = f(p) - A\log p$, $x = N_j$, we deduce that

$$\frac{\tau^2}{\pi^2} \sum_{\substack{p\leq x \\ |h(p)|\leq d}} \frac{|h(p)|^2}{p} \leq S(\tau h) \ll 1.$$

Moreover,

$$\begin{aligned}\left(1 - \frac{1}{\pi}\right) \sum_{\substack{p\leq x \\ |h(p)|>d}} \frac{1}{p} &\leq \sum_{p\leq x} \frac{1}{p}\left(1 - \frac{\operatorname{Sin}\tau h(p)}{\tau h(p)}\right) \\ &= \frac{1}{2\tau}\int_{-\tau}^{\tau} S(th)\,dt \ll 1.\end{aligned}$$

Together these inequalities imply the convergence of the series

$$\sum_{|h(p)|>u} \frac{1}{p}, \quad \sum_{|h(p)|\leq u} \frac{h(p)^2}{p} \tag{9}$$

for each positive u. We shall use this to estimate $M(g_\alpha, x)$ for all large x, whether of the form N_j or not.

Let

$$\mu(x) = \sum_{\substack{p\leq x \\ |h(p)|\leq 1}} \frac{h(p)}{p}.$$

If $x^{1/2} \leq w \leq x$, $u > 0$,

$$\begin{aligned}|\mu(x) - \mu(w)| &\leq \sum_{\substack{w<p\leq x \\ |h(p)|>u}} \frac{1}{p} + u \sum_{\substack{w<p\leq x \\ |h(p)|\leq u}} \frac{1}{p} \\ &= o(1) + O\left(u\log\left(\frac{\log x}{\frac{1}{2}\log x}\right)\right)\end{aligned}$$

as $x \to \infty$. Since u may be chosen arbitrarily small, $\mu(x) - \mu(w) \to 0$ as $x \to \infty$, uniformly for $x^{1/2} \le w \le x$.

In the same way that the convergence of the three series (3) implies the weak convergence of the distribution functions $G_x(z)$, the convergence of the two series at (9) implies the weak convergence of

$$P\left(\sum_{p \le x} Z_p - \mu(x) \le z\right),$$

where the random variables Z_p are defined like the Y_p, but with $f(p^\alpha)$ everywhere replaced by $f(p^\alpha) - A\log p^\alpha$.

Another application of Lemma 1, this time with $y = x$, and to the function $f(n) - A\log n$, shows that

$$\nu_{x,x}(n; f(n) - A\log n - \mu(x) \le z) \Rightarrow H(z), \qquad x \to \infty,$$

for some distribution function $H(z)$. If $h(t)$ is the characteristic function of $H(z)$, we can express this last assertion in the form of the asymptotic estimate:

$$x^{-1}M(g_{-A}, x)e^{-it\mu(x)} \to h(t), \qquad x \to \infty,$$

uniformly on every bounded set of t-values.

An integration by parts shows that

$$M(g_\alpha, x) = x^{i(\alpha+At)}M(g_{-A}, x) - i(\alpha + At)\int_{1-}^{x} v^{i(\alpha+At)-1}M(g_{-A}, v)\,dv.$$

The integral term is small. In fact, from our hypothesis (4) (with $x = N_j$),

$$\operatorname{Re}\sum_{p \le x} p^{-1}(1 - g(p)p^{i\alpha}) \ll 1,$$

and we have shown that a similar relation holds with α replaced by $-At$. Arguing with the function S (as earlier), we see that $\alpha + At \ll (\log x)^{-1}$, $x = N_j$. Thus as $x\ (= N_j) \to \infty$,

$$M(g_\alpha, x) = xh(t)\exp(i(\alpha + At)\log x + it\mu(x)) + o(x).$$

Combining this result with that of (4),

$$e^{it(\mu(x)+A\log x)}\left(\frac{1-(1-y/x)^{1-i\alpha}}{(1-i\alpha)y/x}\right) \to \frac{\phi(t)}{h(t)}, \qquad x \to \infty, \tag{10}$$

uniformly on a proper interval $|t| \le t_0$. Here $x = N_j$, $y = M_j$.

Suppose now that for a sequence of j-values, $M_j/N_j \to \rho$. Then for this sequence of values the coefficient of the exponential at (10) converges to

$$\rho^{-1}(1-(1-\rho)^{1+iAt}) \text{ if } \rho \neq 0; \quad 1+iAt \text{ if } \rho = 0.$$

This convergence is uniform on some bounded interval of t-values which includes $t = 0$. Here we have again applied the estimate $\alpha + At \ll (\log x)^{-1}$. It follows from this and an application of Lemma 6, that on this same sequence of j-values, $\beta(\rho) = \lim(\mu(x) + A\log x)$ exists. Moreover, for all sufficiently small t,

$$e^{it\beta(\rho)}\rho^{-1}(1-(1-\rho)^{1+iAt}) = \phi(t)h(t)^{-1}$$

if $\rho > 0$, with a similar (modified) relation if $\rho = 0$.

We next show that the value of $\beta(\rho)$ does not depend upon ρ.

Assume that for an interval of real t-values

$$(11) \qquad \rho_1^{-1}e^{it\beta_1}(1-(1-\rho_1)^{1+iAt}) = \rho_2^{-1}e^{it\beta_2}(1-(1-\rho_2)^{1+iAt}),$$

where each ρ_j is positive and < 1. Suppose that $\beta_1 \neq \beta_2$. Then $A \neq 0$, and the coefficient of $e^{it\beta_2}$ on the right-hand side is ρ_2^{-1}. It follows from Lemma 7 that

$$\beta_2 = \beta_1 + A\log(1-\rho_1), \qquad \beta_1 = \beta_2 + A\log(1-\rho_2),$$

which is impossible. A similar argument may be made when the restrictions upon the values of ρ_1, ρ_2 are removed.

We have now proved that

$$\lim_{j\to\infty}(\mu(N_j) - A\log N_j)$$

exists, the variable j running through all positive integers. By an elementary estimate

$$|\mu(N_j)| \leq \sum_{p \leq N_j} \frac{1}{p} \ll \log\log N_j,$$

so that $A\log N_j \ll \log\log N_j$ for all j, and $A = 0$. A look back at (11) shows that $A = 0$ removes the possibility of comparing the values of ρ_1 and ρ_2.

Thus the series

$$\sum_{|f(p)|>1} \frac{1}{p}, \qquad \sum_{|f(p)|\leq 1} \frac{f(p)^2}{p}$$

converge, and

$$\lim_{j\to\infty} \sum_{\substack{p\le N_j \\ |f(p)|\le 1}} \frac{f(p)}{p}$$

exists. Since every sufficiently large real w lies in an interval $(N_j, N_{j+1}]$, and (now with $A = 0$) $\mu(N_{j+1}) - \mu(w) \to 0$ as $j \to \infty$, uniformly for $N_j < w \le N_{j+1}$, the series

$$\sum_{|f(p)|\le 1} \frac{f(p)}{p}$$

also converges.

The proof of the theorem is complete.

REFERENCES

[1] P.D.T.A. Elliott, *Probabilistic Number Theory*, Grund. der math. Wiss., **239** (1979), **240** (1980), Springer-Verlag, New York, Heidelberg, Berlin.

[2] ———, *Localised value-distribution of additive arithmetic functions*, J. für die reine und ang. Math., **379** (1987), 64–75.

[3] P.D.T.A. Elliott and C. Ryavec, *The distribution of the values of additive arithmetical functions*, Acta Math., **126** (1971), 143–164.

[4] W. J. Ellison and M. Mendès-France, *Les Nombres Premiers*, Hermann, Paris, 1975.

[5] P. Erdös and A. Wintner, *Additive arithmetical functions and statistical independence*, Amer. J. Math., **61** (1939), 713–721.

[6] G. Halász, *On the distribution of additive arithmetic functions*, Acta Arith., **27** (1975), 143–152.

[7] A. Hildebrand, *Multiplicative functions in short intervals*, Canad. J. Math., **39** (1987), 646–672.

[8] D. H. Hyers, *On the stability of the linear functional equation*, Proc. Nat. Acad. Sci. U.S.A., **27** (1941), 222–224.

[9] I. Z. Ruzsa, *On the concentration of additive functions*, Acta Math. Acad. Sci. Hungar., **36** (1980), 215–232.

Received May 22, 1986 and in revised form March 15, 1988. Partially supported by N.S.F. contract DMS-8500949.

UNIVERSITY OF COLORADO
BOULDER, CO 80309