

HARMONIC GAUSS MAPS

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A construction is given whereby a Riemannian manifold induces a Riemannian metric on the total space of a large class of fibre bundles over it. Using this metric on the appropriate bundles, necessary and sufficient conditions are given for the Gauss map and the spherical Gauss map to be harmonic. A weak maximum principle is applied to the Gauss map of an isometric immersion into Euclidean space in order to prove a sufficient condition for when such an immersion with parallel mean curvature vector must be minimal.

1. Introduction. For a Riemannian manifold M^m and an isometric immersion $f: M \rightarrow \mathbf{R}^n$, Ruh-Vilms [13] proved that the Gauss map of f is harmonic if and only if f has parallel mean curvature vector. Here the Gauss map assigns to a point $p \in M$ the m -dimensional subspace of \mathbf{R}^n obtained from the parallel translation of f_*T_pM to the origin. It thus takes values in the Grassmannian $G_m(n)$, endowed with an $O(n)$ -invariant Riemannian metric.

In this paper we generalize the Ruh-Vilms theorem to isometric immersions $f: M \rightarrow N$, where N^n is a Riemannian manifold. There are two natural ways in which a Gauss map can be defined. The first, which we call simply the Gauss map, $\gamma_f: M \rightarrow G_m(TN)$, sends a point $p \in M$ to the tangent m -plane f_*T_pM in the Grassmann bundle of tangent m -planes of N . The second, which we call the spherical Gauss map, $\nu_f: TM_1^\perp \rightarrow TN_1$, maps a unit normal vector of M to itself as a unit tangent vector of N .

The notion of harmonicity of these Gauss maps requires some Riemannian metric on the, generally non-trivial, fibre bundles $G_m(TN)$, TM_1^\perp , and TN_1 . In §2 we present a natural construction of a Riemannian metric on the total space of a large class of fibre bundles over a Riemannian manifold. As an immediate application of this construction we analyse the geometry of this metric on the tangent bundle, where the Sasaki metric is obtained. To illustrate our formalism we give a global version of Raychauduri's equation on the tangent bundle level.

Using the metrics constructed by this method, we are then able to prove a generalized Ruh-Vilms theorem for the Gauss map γ_f in

Theorem 3.2 of §3; and for the spherical Gauss map ν_f in Theorem 4.1 of §4. Our results in §3 extend earlier results obtained by M. Obata in [11], for the case when N has constant curvature, and by C. M. Wood in [15], where he introduces the notion of vertically harmonic section. In §4 we also extend some of our previous work and take the opportunity to correct an error in [12].

In §5 we apply L. Karp's weak maximum principle in [8] to the geometry of the Gauss map. In Theorem 5.1 we give an estimate on the size of the image of a harmonic map in terms of its tension field, while in Theorem 5.2 we extend a result of Y. L. Xin in [16] which gives a sufficient condition for an isometric immersion $f: M \rightarrow \mathbf{R}^n$ with parallel mean curvature to be minimal.

Part of the technique used in §4 to analyse the spherical Gauss map has been used by A. Sanini in [14] in a different context.

2. A metric construction. Let N be a Riemannian manifold of dimension n . We describe here a natural construction of a Riemannian metric on the total space of fibre bundles $E \rightarrow N$ associated to a large class of principal K -bundles over N .

Let $\pi: P \rightarrow N$ be a principal K -bundle, where K is a closed subgroup of a Lie group G such that the dimension of G/K is n . Let $\tilde{\omega}$ be a Cartan connection on P (cf. Kobayashi [9]). We assume that the Lie algebra \mathfrak{g} of G decomposes as

$$\mathfrak{g} = \mathfrak{k} + V$$

where \mathfrak{k} is the Lie algebra of K and V is an $\text{Ad}(K)$ -invariant complementary subspace which possesses an $\text{Ad}(K)$ -invariant inner product $\langle \cdot, \cdot \rangle$. Then the \mathfrak{g} -valued 1-form $\tilde{\omega}$ on P decomposes as

$$\tilde{\omega} = \omega + \theta,$$

where ω is \mathfrak{k} -valued and θ is V -valued. If $\{E_A\}_1^n$ is an orthonormal basis of V , we can write

$$\theta = \sum \theta^A E_A.$$

Let F be a Riemannian manifold on which K acts as isometries. Each element $A \in \mathfrak{k}$ induces a vector field on F whose value at $x \in F$ we denote

$$Ax = \left. \frac{d}{dt} \right|_0 (\exp tA)x.$$

Then for any $x \in F$, ωx is a $T_x F$ -valued 1-form on P .

Let $\tau: P \times_K F \rightarrow N$ denote the fibre bundle associated to P with standard fibre F (see Kobayashi-Nomizu [10]). For brevity we write $FN = P \times_k F$, and we let

$$\begin{aligned} \sigma: P \times F &\rightarrow FN \\ (e, x) &\rightarrow ex \end{aligned}$$

denote the projection. Thus ex is the equivalence class of (e, x) under the action of K on $P \times F$ given by $a(e, x) = (ea^{-1}, ax)$, for all $a \in K$.

If $\{\varphi^b\}_1^s$ is a local orthonormal coframe field defined on an open subset $U \subset F$, then for any $x \in U$, $\varphi^b \circ (\omega x)$ is a 1-form on P and depends smoothly on x . We obtain a symmetric bilinear form h on $P \times U$ by

$$h = \langle \theta, \theta \rangle + \sum (\varphi^b + \varphi^b \circ (\omega x))^2.$$

As h does not depend on the choice of orthonormal coframe $\{\varphi^b\}$, it is well-defined on all of $P \times F$.

PROPOSITION 1. *There exists a unique Riemannian metric ds_{FN}^2 on FN such that $\sigma^* ds_{FN}^2 = h$.*

Proof. The assertion of the proposition is equivalent to the following easily verified properties of h .

- (i) h is invariant under the action of K on $P \times F$;
- (ii) h is horizontal, meaning that $h(u, v) = 0$ whenever one of the vectors u or v is tangent to a fibre of σ ;
- (iii) $h(v, v) = 0$ if and only if v is tangent to a fibre of σ . □

REMARKS. (1) This construction of ds_{FN}^2 depends on the choice of K -invariant inner product on V . The first term of h can be multiplied by any positive constant t , but that is the same as replacing $\langle \cdot, \cdot \rangle$ by $t\langle \cdot, \cdot \rangle$.

(2) If K possesses a bi-invariant Riemannian metric, then $h + |\omega|^2$ is a K -invariant Riemannian metric on $P \times F$ with respect to which σ is a Riemannian submersion with totally geodesic fibres.

At present we are interested in two special cases:

1. F is a vector space with an inner product and K acts by a linear representation into the orthogonal group of this space.
2. F is a homogeneous space K/K_0 .

Case 1. Let $F = \mathbf{R}^s$ with the standard inner product, and let $\rho: K \rightarrow O(s)$ be a representation. Then

$$\rho_* \circ \omega = (\omega_b^a), \quad 1 \leq a, b \leq s$$

is an $o(s)$ -valued matrix of 1-forms on P .

If (x^a) denotes the standard coordinates on \mathbf{R}^s , then $\{dx^a\}$ is an orthonormal coframe on \mathbf{R}^s and h is the sum of the squares of

$$\theta^A, \quad dx^a + x^b \omega_b^a,$$

where $1 \leq A \leq n$ and $1 \leq a, b \leq s$. The structure equations for ds_{FN}^2 can be obtained by differentiating these forms on $P \times F$. In fact, if $e: U \subset N \rightarrow P$ is any section of $\pi: P \rightarrow N$, let $u: \tau^{-1}U \subset FN \rightarrow P \times F$ be the section of $\sigma: P \times F \rightarrow FN$ defined by

$$u(\bar{e}x) = (e(p), e(p)^{-1}\bar{e}x),$$

where $p = \tau(\bar{e}x)$ and $y = e(p)^{-1}\bar{e}x \in F$ is defined by $e(p)y = \bar{e}x$. Then $ds_{FN}^2 = u^*h$ and

$$(2.1) \quad u^*\theta^A, \quad u^*\left(dx^a + \sum x^b \omega_b^a\right)$$

is a local orthonormal coframe for ds_{FN}^2 .

Case 2. The assumption now is that K acts transitively on F . Fix a point $x_0 \in F$ as origin, and let K_0 denote the isotropy subgroup of K at x_0 . If

$$\begin{aligned} i: P &\rightarrow P \times F \\ e &\rightarrow (e, x_0) \end{aligned}$$

denotes the injection, then the composition

$$\begin{aligned} \sigma \circ i: P &\rightarrow FN \\ e &\rightarrow ex_0 \end{aligned}$$

is surjective. Indeed, it is a principal K_0 -bundle.

As the metric on F is K -invariant, there is a decomposition

$$\mathfrak{k} = \mathfrak{k}_0 + W$$

where \mathfrak{k}_0 is the Lie algebra of K_0 and W is an $\text{Ad}(K_0)$ -invariant complementary subspace possessing an $\text{Ad}(K_0)$ -invariant inner product $\langle \cdot, \cdot \rangle$ which defines the Riemannian metric on F . The \mathfrak{k} -valued 1-form ω then decomposes into

$$\omega = \omega_0 + \omega_1,$$

where ω_0 is \mathfrak{k}_0 -valued, and ω_1 is W -valued. One easily verifies that

$$i^*h = \langle \theta, \theta \rangle + \langle \omega_1, \omega_1 \rangle.$$

If $\{F_a\}$ is an orthonormal basis of W so that

$$\omega_1 = \omega^a F_a,$$

then i^*h is the sum of the squares of $\{\theta^A, \omega^a\}$. As in the previous case, the structure equations for ds_{FN}^2 can be obtained by differentiating these forms in P and using the structure equations of $\tilde{\omega}$. In fact, if $u: U \subset FN \rightarrow P$ is any local section of $\sigma \circ i: P \rightarrow FN$, then $ds_{FN}^2 = u^*i^*h$, from which it follows that

$$u^*\theta^A, \quad u^*\omega^a$$

is a local orthonormal coframe field for ds_{FN}^2 in U .

To see the effectiveness of the above construction, consider the well known case [17] of the tangent bundle $\pi: TN \rightarrow N$. According to Case 1, where now $s = n$, an orthonormal coframe for ds_{TN}^2 is given by (2.1), where the functions x^A , and the entire setting, are described in more detail in §4. In this case ds_{TN}^2 coincides with the Sasaki metric of the tangent bundle and basic results of its geometry can be easily deduced in the above formalism. For instance, a symplectic Hermitian structure on TN is given by defining, as a local basis for the type (1,0) forms, the forms σ^A given by

$$\sigma^A = \eta^A + i\eta^{n+A},$$

where η^A and η^{n+A} are defined in (4.3) below.

The Levi-Civita connection forms relative to the orthonormal coframe $\{\eta^A, \eta^{n+A}\}$ are immediately determined (see (4.5)), and using them and the structure equations on N we obtain

$$(2.2) \quad d\left(\frac{i}{2}\sigma^A \wedge \bar{\sigma}^A\right) = 0.$$

$$(2.3) \quad d\sigma^A = -\omega_B^A \wedge \sigma^B + ix^B \Omega_B^A,$$

where $\Omega_B^A = \frac{1}{2}R_{BCD}^A \theta^C \wedge \theta^D$ are the (pull-backs of the) curvature forms of N . Thus, from (2.3), the Hermitian metric defined by the unitary coframe σ^A is symplectic, while from (2.3) the complex structure is integrable if and only if N is flat. Local exactness of the Kaehler form given by (2.2) is in fact immediate once we remark that the Kaehler form itself is the negative of the differential of

$$\mu = x^A \eta^A,$$

which uniquely determines the Levi-Civita connection of N . Actually, μ is the Pfaffian form dual to the geodesic spray F in the sense of [1],

$$(2.4) \quad F = x^A F_A,$$

for $\{F_A, F_{n+A}\}$ the frame field dual to $\{\eta^A, \eta^{n+A}\}$. The relevance of F is that the geodesics in N are exactly the projection of integral curves of F in TN .

Consider on TN the quadratic form

$$(2.5) \quad G = \eta^A \otimes \eta^{n+A}.$$

A simple check shows that G is globally defined. Using equations (4.5) and (2.4), we compute the Lie derivative

$$(2.6) \quad L_F G = \eta^{n+A} \otimes \eta^{n+A} + x^C x^D R_{CDB}^A \eta^A \otimes \eta^B.$$

Let now $\zeta = \zeta^A e_A$ be any (local) vector field on N , where $\{e_A\}$ is the frame field dual to $\{\theta^A\}$. Define the quadratic form A^ζ by setting

$$(2.7) \quad A^\zeta = \zeta_{AB} \theta^A \otimes \theta^B,$$

where ζ_{AB} are the coefficients of the covariant differential of ζ . If we consider ζ as a (local) section of the tangent bundle, then from (4.3), (2.5) and (2.7) we have

$$(2.8) \quad \zeta^* G = A^\zeta.$$

Computing the Lie derivative of A^ζ with respect to ζ we obtain

$$(2.9) \quad L_\zeta A^\zeta = \{\zeta_C \zeta_{ABC} + \zeta_{CB} \zeta_{CA} + \zeta_{AC} \zeta_{CB}\} \theta^A \otimes \theta^B,$$

where ζ_{ABC} are the coefficients of the covariant differential of A^ζ . Suppose now that ζ is a geodesic vector field (i.e., integral curves of ζ are geodesics), so that $\zeta^* L_F G = L_\zeta \zeta^* G$, and hence from (2.8)

$$\zeta^* L_F G = L_\zeta A^\zeta.$$

We are now able to interpret at the level of the manifold N the meaning of the trace of the pull back of equation (2.6). Indeed, from (2.6), (2.9) and (4.3) for the geodesic vector field ζ we obtain

$$\zeta^C \zeta^B R_{CBA}^A = \zeta^C \zeta_{AAC} + \zeta_{AC} \zeta_{CA}.$$

Thus,

$$(2.10) \quad \zeta(D) = -\text{Ric}(\zeta, \zeta) - \frac{D^2}{n} - \text{tr } V^2 - \text{tr } S^2,$$

where $\zeta(D) = \zeta^C \zeta_{AAC}$, $D = \zeta_{AA}$, $S = \{\frac{1}{2}(\zeta_{AB} + \zeta_{BA}) - \delta_{AB}D/n\}\theta^B \otimes e_A$, and $V = \frac{1}{2}(\zeta_{AB} - \zeta_{BA})\theta^B \otimes e_A$ that is, D, S, V are respectively the divergence, the shear, and the vorticity of the vector field ζ . Equation (2.10) can be interpreted as a Riemannian version of Raychauduri's equation considered in Lorentzian geometry in the study of singularities of geodesics [2] and, in a sense, equation (2.6) globalizes the local information contained in (2.10). For a different treatment we refer to [3].

3. The Gauss map. Let N^n be a Riemannian manifold. Denote its $O(n)$ -bundle of orthonormal frames by

$$O(N) \rightarrow N$$

on which live the canonical form and Levi-Civita connection, respectively,

$$\theta = (\theta^A), \quad \omega = (\omega_B^A), \quad \omega_B^A = -\omega_A^B.$$

Throughout this section we use the index conventions

$$1 \leq i, j, k \leq m; \quad m + 1 \leq \alpha, \beta, \gamma \leq n; \quad 1 \leq A, B, C, D \leq n;$$

where m is a fixed integer, $1 \leq m \leq n$. The structure equations are

$$d\theta^A = -\omega_B^A \wedge \theta^B, \quad d\omega_B^A = -\omega_C^A \wedge \omega_B^C + \Omega_B^A,$$

and the curvature forms Ω_B^A are given by

$$\Omega_B^A = \frac{1}{2}R_{BCD}^A \theta^C \wedge \theta^D,$$

where the functions R_{BCD}^A satisfy the usual symmetry relations of the Riemann curvature tensor. N has constant sectional curvature c if and only if

$$R_{BCD}^A = c(\delta_C^A \delta_{BD} - \delta_D^A \delta_{BC}).$$

Let

$$\pi: G_m(TN) \rightarrow N$$

denote the Grassmann bundle over N of m -dimensional tangent subspaces to N . It is a fibre bundle over N associated to $O(N)$ with standard fibre the Grassmann manifold

$$G_m(n) = O(n)/O(m) \times O(n - m)$$

on which $O(n)$ acts on the left by multiplication.

Let $\varepsilon_1, \dots, \varepsilon_n$ denote the standard basis of \mathbf{R}^n , and for the origin of $G_m(n)$ we choose the subspace of \mathbf{R}^n spanned by $\varepsilon_1, \dots, \varepsilon_m$, which we denote

$$o = [\varepsilon_1, \dots, \varepsilon_m].$$

Using the construction of Case (ii) in §2, we now have

$$K = O(n), \quad K_0 = O(m) \times O(n - m),$$

and the decomposition of ω is $\omega = \omega_0 + \omega_1$ where

$$\omega_0 = \begin{bmatrix} \omega_j^i & 0 \\ 0 & \omega_\beta^\alpha \end{bmatrix}, \quad \omega_1 = \begin{bmatrix} 0 & \omega_\beta^i \\ \omega_j^\alpha & 0 \end{bmatrix}.$$

The $O(n)$ -invariant metric on $G_m(n)$ is unique up to constant positive factor. If we let

$$\mu: O(N) \rightarrow G_m(TN)$$

be defined by $\mu(p; e_1, \dots, e_n) = (p; [e_1, \dots, e_m])$, then the metric ds_{FN}^2 on $G_m(TN)$, which we denote ds_t^2 , is characterized by

$$(3.1) \quad \mu^* ds_t^2 = \sum (\theta^A)^2 + t^2 \sum (\omega_i^\alpha)^2,$$

where t is any positive constant.

If $U \subset G_m(TN)$ is an open subset containing o and

$$(3.2) \quad u: U \rightarrow O(N)$$

is any local section, then

$$(3.3) \quad \{\varphi^A = u^* \theta^A, \varphi^{\alpha i} = tu^* \omega_i^\alpha\}$$

is an orthonormal coframe from ds_t^2 on U . From the structure equations of $O(N)$ we find that the pull-back by u^* of the forms

$$(3.4) \quad \begin{aligned} \varphi_B^A &= \omega_B^A + \frac{t^2}{2} R_{iBA}^\alpha \omega_i^\alpha, \\ \varphi_B^{\alpha i} &= -\frac{t}{2} R_{iAB}^\alpha \theta^A = -\varphi_{\alpha i}^B, \\ \varphi_{\beta j}^{\alpha i} &= \delta_{\alpha\beta} \omega_j^i + \delta_{ij} \omega_\beta^\alpha \end{aligned}$$

gives the Levi-Civita connection forms of ds_t^2 with respect to this orthonormal coframe field.

Consider now an isometric immersion

$$f: M \rightarrow N$$

of an m -dimensional Riemannian manifold M , ds^2 . If $e: 0 \subset M \rightarrow O(N)$ is a local Darboux frame field along f , then $e^* \theta^\alpha = 0$ and

$$(3.5) \quad e^* \omega_i^\alpha = h_{ij}^\alpha e^* \theta^j$$

where $h_{ij}^\alpha = h_{ji}^\alpha$ are the components of the second fundamental tensor II of f . Its mean curvature is $H = H^\alpha e_\alpha$ where

$$H^\alpha = h_{ii}^\alpha / m,$$

and we set

$$\mathbb{I}_H = \langle \mathbb{I}, H \rangle = \frac{1}{m} \sum h_{jk}^\alpha h_{ii}^\alpha e^*(\theta^j \theta^k).$$

The Gauss map of f is defined to be

$$\begin{aligned} \gamma_f: M &\rightarrow G_m(TN) \\ p &\rightarrow f_*T_pM. \end{aligned}$$

Observe that $\pi \circ \gamma_f = f$ and that $\mu \circ e = \gamma_f$ for any Darboux frame field e along f . These compositions are illustrated by the following commuting diagram

$$\begin{array}{ccc} & & O(N) \\ & e \nearrow & \downarrow \mu \\ M & \xrightarrow{\gamma_f} & G_m(TN) \\ & f \searrow & \downarrow \pi \\ & & N \end{array}$$

Using (3.1) and a local Darboux frame e along f , we find

$$\gamma_f^* ds_i^2 = \sum (\theta^i)^2 + t^2 \sum (e^* \omega_i^\alpha)^2,$$

which, when combined with (3.5) and the Gauss equations gives

$$(3.6) \quad \gamma_f^* ds_i^2 = ds^2 + t^2(m\mathbb{I}_H - \text{Ric}_M + \text{Ric}(f)),$$

where Ric_M is the Ricci tensor of ds^2 and we have defined a symmetric (0,2) tensor on M , given with respect to any Darboux frame field e , by

$$\text{Ric}(f) = e^*(R_{jik}^i \theta^j \theta^k).$$

From (3.6) one immediately deduces the following result of C. M. Wood [15].

THEOREM 3.1. *Any three of the following properties imply the fourth:*

- (1) γ_f is conformal;
- (2) f is Einsteinian (i.e., $\text{Ric}(f)$ is a scalar multiple of ds^2);
- (3) f is pseudo-umbilical (i.e., \mathbb{I}_H is a multiple of ds^2);
- (4) M , ds^2 is Einstein (i.e., Ric_M is a multiple of ds^2).

REMARK. If N has constant sectional curvature c , then $\text{Ric}(f) = c(m - 1)ds^2$, i.e., any isometric immersion into N is Einsteinian.

We define a 1-form on M with values in the normal bundle TM^\perp by

$$\text{Ric}^\perp(f) = R_{kik}^\alpha e_\alpha \otimes e^* \theta^i.$$

We define a section T of $TM^\perp \otimes TM^{\perp*} \otimes f^{-1}TN$ by

$$T = T_\beta^{\alpha A} e_\alpha \otimes \theta^\beta \otimes e_A$$

where

$$(3.7) \quad T_\beta^{\alpha A} = h_{ij}^\alpha R_{\beta j A}^i.$$

Contracting T we obtain a vector field along f ,

$$\text{Tr } T = T_\alpha^{\alpha A} e_A.$$

REMARK. If N has constant sectional curvature c , then

$$(3.8) \quad \begin{aligned} \text{Ric}^\perp(f) &= 0, \\ T &= mcH \otimes \theta^\alpha \otimes e_\alpha, \\ \text{Tr } T &= mcH. \end{aligned}$$

The tensor fields $\text{Ric}^\perp(f)$ and T give the obstruction to a generalization of the Ruh-Vilms Theorem.

THEOREM 3.2. *Let $f: M \rightarrow N$ be an isometric immersion with Gauss map $\gamma_f: M \rightarrow G_m(TN)$. Then γ_f is harmonic if and only if*

$$t^2 \text{Tr } T = mH \quad \text{and} \quad \text{Ric}^\perp(f) = m\nabla H.$$

Proof. Let u be a local section (3.2). Then $e = u \circ \gamma_f$ is a local Darboux frame field along f . If $\{E_A, E_{\alpha i}\}$ denotes the local frame field dual to the orthonormal coframe field (3.3), then

$$(3.9) \quad d\gamma_f = (e^* \theta^A) E_A + t(e^* \omega_i^\alpha) E_{\alpha i},$$

and the tension field of γ_f is

$$\tau(\gamma_f) = t^2 R_{jik}^\alpha h_{ji}^\alpha E_k + (mH^\beta + t^2 R_{ji\beta}^\alpha h_{ji}^\alpha) E_\beta + t h_{ijj}^\alpha E_{\alpha i},$$

where h_{ijk}^α are the components of the covariant derivative of II . Effectively, all these calculations can be made on $O(N)$ using (3.4).

From the Codazzi equations of f we have

$$(3.10) \quad h_{ijk}^\alpha = h_{ikj}^\alpha - R_{ijk}^\alpha.$$

Thus, letting $H_i^\alpha = h_{jji}^\alpha/m$ denote the components of the covariant derivative of H , we find that

$$(3.11) \quad \tau(\gamma_f) = t^2 R_{jiA}^\alpha h_{ji}^\alpha E_A + mH^\alpha E_\alpha + tmH_i^\alpha E_{\alpha i} - tR_{jij}^\alpha E_{\alpha i}.$$

We see then that for $\tau(\gamma_f)$, the:

- coefficient of E_A is the coefficient of e_A in $-t^2 \text{Tr } T$;
- coefficient of E_α is the coefficient of e_α in mH ;
- coefficient of $E_{\alpha i}$ is the coefficient of $e_\alpha \otimes e^* \theta^i$ in $t(m\nabla H - \text{Ric}^\perp(f))$.

The theorem now follows immediately. □

COROLLARY. *Suppose that N is a space of constant sectional curvature c . If $1 - t^2c \neq 0$, then γ_f is harmonic if and only if f is minimal. If $c > 0$ and if $t = 1/\sqrt{c}$, then γ_f is harmonic if and only if f has parallel mean curvature vector.*

Proof. In this case the tension field of γ_f is

$$\tau(\gamma_f) = (1 - t^2c)mH^\alpha E_\alpha + tmH_i^\alpha E_{\alpha i},$$

from which the corollary follows immediately. □

REMARKS. (1) If N is Euclidean space, then $G_m(TN) = N \times G_m(n)$, and ds_t^2 is the product metric (g_N, t^2g_G) , where g_G is a fixed $O(n)$ -invariant metric on $G_m(n)$. Then $\gamma_f = (f, \gamma)$, where $\gamma: M \rightarrow G_m(n)$ is the Gauss map of Ruh-Vilms. As in this case $\tau(\gamma_f) = (\tau(f), \tau(\gamma))$, our theorem generalizes that of Ruh and Vilms. The factor $\tau(f)$ in $\tau(\gamma_f)$ explains the apparent discrepancy between our result and that of Ruh-Vilms. From (3.11) it follows that in this case $\tau(\gamma) = 0$ if and only if f has parallel mean curvature vector.

(2) In C. M. Wood's [15] concept of vertical variation of γ_f , we have for the vertical component of $d\gamma_f$ the second term of (3.9), and for the vertical tension, $\tau^V(\gamma_f)$, the last two terms of (3.11).

(3) M. Obata's [11] Gauss map, which is defined for the case when N has constant sectional curvature $c \neq 0$, is our Gauss map γ_f composed with a natural projection $O(N) \rightarrow G_{m+1}(n+1)$ which exists in the constant curvature case.

4. The spherical Gauss map. To the notation and index conventions of §3 we add the conventions $1 \leq \sigma, \tau \leq 2n$. Let $\text{Dar}(f) \rightarrow M$ denote the $O(m) \times O(q)$ -bundle of Darboux frames along f , where $q = n - m$. Taking the standard left action of $O(m) \times O(q)$ on $\mathbf{R}^q = \{0\} \times \mathbf{R}^q \xrightarrow{i_q} \mathbf{R}^m \times \mathbf{R}^q$, we obtain the normal bundle of f as an associated vector bundle

$$\text{Dar}(f) \times \mathbf{R}^q \xrightarrow{\sigma_1} TM^\perp = \text{Dar}(f) \times_{O(q)} \mathbf{R}^q \xrightarrow{\pi_1} M.$$

The unit normal bundle of f is the hypersurface

$$TM_1^\perp = \{(p, v) \in TM^\perp : |v| = 1\} = \text{Dar}(f) \times_{O(q)} S^{q-1}$$

of TM^\perp .

For the tangent bundle TN of N we define projections

$$O(N) \times \mathbf{R}^n \xrightarrow{\sigma_3} TN = O(N) \times_{O(n)} \mathbf{R}^n \xrightarrow{\pi_2} N,$$

and the unit tangent bundle of N is the hypersurface in TN

$$TN_1 = \{(p, v) \in TN : |v| = 1\} = O(N) \times_{O(n)} S^{n-1}.$$

The spherical Gauss map of f is

$$\nu_f: TM_1^\perp \rightarrow TN_1$$

defined by $\nu_f(p, v) = (f(p), v)$. It is the restriction to $TM_1^\perp \subset TM^\perp$ of the normal map

$$\nu: TM^\perp \rightarrow TN$$

defined by $\nu(p, v) = (f(p), v)$. It is easier to work with ν and then restrict the results to TM_1^\perp , rather than work directly with ν_f .

Let $e: U \subset N \rightarrow O(N)$ be a local orthonormal frame field for which $e \circ f: \tilde{U} \rightarrow \text{Dar}(f)$ is a local Darboux frame field along f on some neighborhood $\tilde{U} \subset f^{-1}U$. We define local sections

$$u_1: \pi_1^{-1}\tilde{U} \rightarrow \text{Dar}(f) \times \mathbf{R}^q \quad \text{and} \quad u_2: \pi_2^{-1}U \rightarrow O(N) \times \mathbf{R}^n$$

of σ_1 and σ_2 , respectively, by

$$\begin{aligned} u_1(p, v) &= (e(f(p)), e(f(p))^{-1}v), \\ u_2(p, v) &= (e(p), e(p)^{-1}v), \end{aligned}$$

where we interpret a basis e of an r -dimensional vector space V as an isomorphism $e: \mathbf{R}^r \rightarrow V$. It is easily checked that

$$(\hat{f} \times i_q) \circ u_1 = u_2 \circ \nu,$$

where $\hat{f} \times i_q: \text{Dar}(f) \times \mathbf{R}^q \rightarrow O(N) \times \mathbf{R}^n$ is the inclusion map. The following diagram commutes.

$$(4.1) \quad \begin{array}{ccc} \text{Dar}(f) \times \mathbf{R}^q & \xrightarrow{\hat{f} \times i_q} & O(N) \times \mathbf{R}^n \\ u_1 \uparrow \downarrow \sigma_1 & & \sigma_2 \downarrow \uparrow u_2 \\ TM^\perp & \xrightarrow{\nu} & TN \\ \pi_1 \downarrow & & \downarrow \pi_2 \\ M & \xrightarrow{f} & N \end{array}$$

From (3.5) we have on $\text{Dar}(f)$

$$\hat{f}^*\theta^\alpha = 0, \quad \hat{f}^*\omega_i^\alpha = h_{ij}^\alpha \hat{f}^*\theta^j,$$

where here $h_{ij}^\alpha = h_{ji}^\alpha$ are functions on $\text{Dar}(f)$. In what follows we will not write \hat{f}^* as the context will indicate when θ^A and ω_B^A have been restricted to $\text{Dar}(f)$.

From the construction of Case 1 in §2, we obtain a Riemannian metric $ds_{TM^\perp}^2$ on TM^\perp characterized by

$$\sigma_1^* ds_{TM^\perp}^2 = \sum (\varphi^A)^2,$$

where

$$(4.2) \quad \varphi^i = \theta^i, \quad \varphi^\alpha = dx^\alpha + x^\beta \omega_\beta^\alpha.$$

Similarly the metric ds_{TN}^2 on TN is characterized by

$$\sigma_2^* ds_{TN}^2 = \sum (\eta^A)^2 + \sum (\eta^{n+A})^2,$$

where

$$(4.3) \quad \eta^A = \theta^A, \quad \eta^{n+A} = dx^A + x^B \omega_B^A.$$

Thus

$$\{u_1^* \varphi^A\} \quad \text{and} \quad \{u_2^* \eta^\sigma\}$$

are local orthonormal coframes for $ds_{TM^\perp}^2$ and ds_{TN}^2 , respectively.

On $\text{Dar}(f) \times \mathbf{R}^q$ the following 1-forms φ_B^A satisfy $\varphi_B^A = -\varphi_A^B$ and $d\varphi^A = -\varphi_B^A \wedge \varphi^B$:

$$(4.4) \quad \begin{aligned} \varphi_j^i &= \omega_j^i - \frac{1}{2} x^\beta \perp R_{\beta ij}^\alpha \varphi^\alpha, \\ \varphi_i^\alpha &= \frac{1}{2} x^\beta \perp R_{\beta ij}^\alpha \varphi^j = -\varphi_\alpha^i, \\ \varphi_\beta^\alpha &= \omega_\beta^\alpha, \end{aligned}$$

where

$$d\omega_\beta^\alpha + \omega_\gamma^\alpha \wedge \omega_\beta^\gamma = \frac{1}{2} \perp R_{\beta ij}^\alpha \varphi^i \wedge \varphi^j$$

is the curvature of the normal bundle TM^\perp . It follows that $u_1^* \varphi_B^A$ are the Levi-Civita connection forms of $ds_{TM^\perp}^2$ with respect to the coframe (4.2).

In the same way $u_2^* \eta_\tau^\sigma$, are the Levi-Civita connection forms of ds_{TN}^2 with respect to the coframe (4.3), where

$$(4.5) \quad \begin{aligned} \eta_B^A &= \omega_B^A - \frac{1}{2} x^C R_{CAB}^D (dx^D + x^E \omega_E^D), \\ \eta_B^{n+A} &= \frac{1}{2} x^C R_{CBD}^A \theta^D = -\eta_{n+A}^B, \\ \eta_{n+B}^{n+A} &= \omega_B^A. \end{aligned}$$

If we apply $(\hat{f} \times i_q)^*$ to η^σ and η_τ^σ we obtain on $\text{Dar}(f) \times \mathbf{R}^q$

$$(4.6) \quad \begin{aligned} \eta^i &= \varphi^i, \quad \eta^\alpha = 0, \\ \eta^{n+i} &= x^\beta \omega_\beta^i = -x^\beta h_{ij}^\beta \varphi^j, \\ \eta^{n+\alpha} &= dx^\alpha + x^\beta \omega_\beta^\alpha = \varphi^\alpha. \end{aligned}$$

Thus, from the commutativity of (4.1) we obtain

$$\nu^* ds_{TN}^2 = ds_{TM^\perp}^2 + \sum (x^\beta \omega_\beta^i)^2,$$

and if we write

$$\nu^* u_2^* \eta^\sigma = B_A^\sigma u_1^* \varphi^A,$$

then

$$(4.7) \quad \begin{aligned} B_j^i &= \delta_j^i, \quad B_j^\alpha = 0, \quad B_j^{n+i} = -x^\beta h_{ij}^\beta, \quad B_i^{n+\alpha} = 0, \\ B_\alpha^i &= 0, \quad B_\beta^\alpha = 0, \quad B_\alpha^{n+i} = 0, \quad B_\beta^{n+\alpha} = \delta_\beta^\alpha. \end{aligned}$$

To compute the tension field of ν we need to compute $(\hat{f} \times i_q)^* \eta_\tau^\sigma$. These are (omitting $(\hat{f} \times i_q)^*$):

$$(4.8) \quad \begin{aligned} \eta_j^i &= \varphi_j^i + \frac{1}{2} x^\alpha (R_{\beta ij}^\alpha - {}^\perp R_{\beta ij}^\alpha) \varphi^\beta + \frac{1}{2} x^\alpha x^\beta h_{k1}^\beta R_{\alpha ij}^k \varphi^1, \\ \eta_j^\alpha &= \left(h_{jk}^\alpha + \frac{1}{2} x^\beta x^\gamma h_{ik}^\gamma R_{\beta \alpha j}^i \right) \varphi^k - \frac{1}{2} x^\beta R_{\beta \alpha j}^\gamma \varphi^\gamma, \\ \eta_\beta^\alpha &= \varphi_\beta^\alpha + \frac{1}{2} x^\gamma x^\delta h_{ij}^\delta R_{\gamma \alpha \beta}^i \varphi^j - \frac{1}{2} x^\gamma R_{\gamma \alpha \beta}^\delta \varphi^\delta, \\ \eta_B^{n+A} &= \frac{1}{2} x^\alpha R_{\alpha Bi}^A \varphi^i = -\eta_{n+A}^B, \\ \eta_{n+j}^{n+i} &= \varphi_j^i + \frac{1}{2} x^\beta {}^\perp R_{\beta ij}^\alpha \varphi^\alpha, \\ \eta_{n+j}^{n+\alpha} &= h_{jk}^\alpha \varphi^k = -\eta_{n+\alpha}^{n+j}, \\ \eta_{n+\beta}^{n+\alpha} &= \varphi_\beta^\alpha. \end{aligned}$$

Using the formula for the covariant derivative of $d\nu$

$$(4.9) \quad \nabla B_A^\sigma = dB_A^\sigma - B_B^\sigma \varphi_A^B + B_A^\tau \eta_\tau^\sigma = B_{AB}^\sigma \varphi^B,$$

we obtain the functions B_{AB}^σ . If we let $\{F_\sigma\}$ denote the orthonormal frame field in TN dual to $\{\eta^\sigma\}$, then the tension field of ν is

$$\tau(\nu) = B_{AA}^\sigma F_\sigma.$$

Combining these calculations, together with the Codazzi equations (3.10), we finally obtain: at $x = x^\alpha e_\alpha \in TM_1^\perp$,

$$(4.10) \quad \begin{aligned} \tau(\nu) = & x^\alpha x^\beta h_{kj}^\beta R_{\alpha ij}^k F_i + (mH^\alpha + x^\beta x^\gamma h_{kj}^\beta R_{\gamma\alpha j}^k) F_\alpha \\ & - x^\alpha (mH_i^\alpha - R_{jij}^\alpha) F_{n+i} - x^\beta h_{kj}^\alpha h_{kj}^\beta F_{n+\alpha}. \end{aligned}$$

Consider now the hypersurfaces $i_1: TM_1^\perp \rightarrow TM^\perp$ and $i_2: TN_1 \rightarrow TN$. Let $g_1: TM^\perp \rightarrow \mathbf{R}$, $g_1(v) = |v|^2$, and let $\tilde{g}_1: \text{Dar}(f) \times \mathbf{R}^q \rightarrow \mathbf{R}$, $\tilde{g}_1(e, x) = |x|^2$. Then $g_1 \circ \sigma_1 = \tilde{g}_1$ and $TM_1^\perp = \{g_1 = 1\}$. Now $d\tilde{g}_1 = 2x^\alpha dx^\alpha = 2x^\alpha \varphi^\alpha$ and $\sigma_1 \circ u_1 = \text{identity}$ together imply that $dg_1 = u_1^* \sigma_1^* dg_1 = u_1^* d\tilde{g}_1 = 2x^\alpha u_1^* \varphi^\alpha$. If we define a unit vector field V in terms of the gradient of g_1 by

$$V = \frac{1}{2} \text{grad}(g_1) = (x^\alpha \circ u_1) E_\alpha,$$

where $\{E_A\}$ is the local orthonormal frame field dual to $\{u_1^* \varphi^A\}$ in TM^\perp , then $V \circ i_1$ is the unit normal vector of TM_1^\perp .

In the same way, if

$$W = (x^A \circ u_2) F_{n+A},$$

then $W \circ i_2$ is the unit normal vector of TN_1 .

In general, for a composition of maps between Riemannian manifolds,

$$(L, g) \xrightarrow{\psi} (M, h) \xrightarrow{\varphi} (N, k),$$

the tension satisfies (cf. [4])

$$\tau(\varphi \circ \psi) = d\varphi(\tau(\psi)) + \text{Tr}_g \nabla d\varphi(d\psi, d\psi).$$

Thus, as $\nu \circ i_1 = i_2 \circ \nu_f$, we have

$$\tau(\nu \circ i_1) = (n - 1)d\nu(\tilde{H}) + \tau(\nu) \circ i_1 - \nabla d\nu(V, V),$$

where \tilde{H} is the mean curvature vector of the hypersurface $TM_1^\perp \subset TM^\perp$, and thus is a multiple of $V \circ i_1$. From (4.6), one finds that $d\nu(V \circ i_1)$ is a multiple of $W \circ \nu$, and consequently, $d\nu(\tilde{H})$ is a multiple of $W \circ \nu$, which means that $d\nu(\tilde{H})$ is normal to TN_1 in TN .

In the same way,

$$\tau(i_2 \circ \nu_f) = i_{2*} \tau(\nu_f) + \text{Tr}_{ds^2_{TM_1^\perp}} \nabla di_2(d\nu_f, d\nu_f),$$

the first term of which is tangent to TN_1 and the second term of which is normal to TN_1 .

As $B_{\alpha\beta}^\sigma = 0$ for all α, β and σ , a direct calculation shows that $\nabla d\nu(V, V) = 0$. Combining these observations we obtain

$$\begin{aligned} i_{2*}\tau(\nu_f) &= \text{tangential component of } \tau(i_2 \circ \nu_f) \\ &= \text{tangential component of } \tau(\nu \circ i_1) \\ &= \text{tangential component of } \tau(\nu) \circ i_1 \\ &= \tau(\nu) \circ i_1 - \langle \tau(\nu) \circ i_1, W \circ i_2 \circ \nu_f \rangle W \circ i_2 \circ \nu_f. \end{aligned}$$

Hence, at $x \in TM_1^\perp$, (and leaving the i_{2*} tacit)

$$(4.11) \quad \tau(\nu_f) = \tau(\nu) + x^\alpha x^\beta h_{ij}^\alpha h_{ij}^\beta x^\gamma F_{n+\gamma},$$

where $\tau(\nu)$ is given in (4.10).

DEFINITION. For any $x \in TM^\perp$, a symmetric bilinear form \mathbb{I}_x is defined on T_pM by

$$\mathbb{I}_x = \langle x, \mathbb{I} \rangle = x^\alpha h_{ij}^\alpha \theta^i \theta^j,$$

where $p = \pi_1(x)$. Notice that if $x, y \in T_pM^\perp$, then $\mathbb{I}_{ax+by} = a\mathbb{I}_x + b\mathbb{I}_y$, for any $a, b \in \mathbf{R}$.

LEMMA. *The following are equivalent.*

$$(4.12a) \quad x^\alpha x^\beta h_{ij}^\alpha h_{ij}^\beta x^\gamma = x^\alpha h_{ij}^\alpha h_{ij}^\gamma$$

for every $x = x^\alpha e_\alpha \in TM_1^\perp$;

$$(4.12b) \quad \langle \mathbb{I}_x, \mathbb{I}_y \rangle = \lambda(p)^2 \langle x, y \rangle$$

for every $x, y \in T_pM_1^\perp$, for every $p \in M$, where $\lambda \geq 0$ is a function on M , $p = \pi_1(x)$, and $\langle \mathbb{I}_x, \mathbb{I}_y \rangle$ denotes the inner product on $S^2 T_p^*M$ given by

$$\langle \mathbb{I}_x, \mathbb{I}_y \rangle = a_{ij} b_{ij},$$

where $a_{ij} = x^\alpha h_{ij}^\alpha$ and $b_{ij} = y^\alpha h_{ij}^\alpha$.

Proof. (4.12a) implies (4.12b): In the notation of (4.12b), (4.12a) says

$$(4.13) \quad \langle \mathbb{I}_x, \mathbb{I}_x \rangle x^\gamma = \langle \mathbb{I}_x, h^\gamma \rangle,$$

where $h^\gamma = (h_{ij}^\gamma)$. Multiplying both sides by y^γ and summing on γ gives (4.12b) in the case when $\langle x, y \rangle = 0$. As it suffices to verify (4.12b) for the cases $x = y$ and $x \perp y$, it remains to show that $\langle \mathbb{I}_x, \mathbb{I}_x \rangle = \lambda(p)^2$

for any $x \in T_p M_1^\perp$. For this it suffices to show that whenever $x, y \in T_p M_1^\perp$ and $\langle x, y \rangle = 0$, then

$$(4.14) \quad \langle \mathbb{I}_x, \mathbb{I}_x \rangle = \langle \mathbb{I}_y, \mathbb{I}_y \rangle.$$

Let $a, b \in \mathbf{R}$ be such that $a^2 + b^2 = 1$. Then

$$\tilde{x} = ax + by, \quad \tilde{y} = -bx + ay$$

are orthonormal in $T_p M^\perp$ and, as (4.12b) holds for x, y and for \tilde{x}, \tilde{y} , we have

$$0 = \langle \mathbb{I}_{\tilde{x}}, \mathbb{I}_{\tilde{y}} \rangle = -ab(\langle \mathbb{I}_x, \mathbb{I}_x \rangle - \langle \mathbb{I}_y, \mathbb{I}_y \rangle).$$

Hence (4.14) must hold.

(4.12b) implies (4.12a): by (4.12b), for any $x \in TM_1^\perp$, the left side of (4.13) becomes

$$\langle \mathbb{I}_x, \mathbb{I}_x \rangle x^\gamma = \lambda(p)^2 x^\gamma,$$

while the right side of (4.13) becomes

$$\langle \mathbb{I}_x, \mathbb{I}_{e_\gamma} \rangle = x^\alpha \langle \mathbb{I}_{e_\alpha}, \mathbb{I}_{e_\gamma} \rangle = x^\gamma \lambda(p)^2.$$

This completes the proof of the lemma. □

DEFINITION. An isometric immersion $f: M \rightarrow N$ for which (4.12a) or (4.12b) holds is said to have conformal second fundamental tensor.

Observe that any hypersurface has conformal second fundamental tensor.

Recall that T defined in (3.7) is a section of

$$\text{Hom}(TM^\perp, TM^\perp \otimes f^{-1}TN).$$

If $x = x^\alpha e_\alpha \in TM_1^\perp$, then $\langle Tx, x \rangle = x^\alpha x^\beta T_\beta^{\alpha A} e_A = \langle Tx, x \rangle_M + \langle Tx, x \rangle_\perp$ with respect to the orthogonal direct sum $f^{-1}TN = TM \oplus TM^\perp$.

THEOREM 4.1. *Let $f: M \rightarrow N$ be an isometric immersion. Then the spherical Gauss map ν_f is harmonic if and only if*

- (a) $\langle Tx, x \rangle_M = 0$ for every $x \in TM_1^\perp$, and
- (b) $mH = \langle Tx, x \rangle_\perp$ for every $x \in TM_1^\perp$, and
- (c) $\text{Ric}^\perp(f) = m\nabla H$, and
- (d) \mathbb{I} is conformal.

In particular, if f is minimal, then ν_f is harmonic if and only if $T = 0$, $\text{Ric}^\perp(f) = 0$, and \mathbb{I} is conformal.

Proof. At $x \in TM_1^\perp$, the expansion of $\tau(\nu_f)$ with respect to the orthonormal frame $\{F_k, F_\alpha, F_{n+i}, F_{n+\alpha}\}$ has as coefficients, respectively:

- of F_k the coefficient of e_k in $-\langle Tx, x \rangle_M$;
- of F_α the coefficient of e_α in $mH - \langle Tx, x \rangle_\perp$;
- of F_{n+i} the coefficient of $(e \circ f)^* \theta^i$ in $\langle x, \text{Ric}^\perp(f) - m\nabla H \rangle$; and
- of $F_{n+\alpha}$ the number $\langle \Pi_x, \Pi_x \rangle x^\alpha - \langle \Pi_x, h^\alpha \rangle$.

The proof of the theorem follows immediately. □

COROLLARY. *Let $f: M \rightarrow N$ be an isometric immersion, and suppose that N has constant sectional curvature c . If $c = 1$, then ν_f is harmonic if and only if f has parallel mean curvature vector and conformal second fundamental tensor. If $c \neq 1$, then ν_f is harmonic if and only if f is minimal and has conformal second fundamental tensor.*

Proof. As we remarked in §3, when N has constant curvature c , then $\text{Ric}^\perp(f) = 0$, and $T = mcH \otimes \theta^\alpha \otimes e_\alpha$. Thus, for any $x \in TM_1^\perp$, $\langle Tx, x \rangle = mcH$. Hence, for any $x \in TM_1^\perp$,

- (i) $\langle Tx, x \rangle_M = 0$,
- (ii) $mH - \langle Tx, x \rangle_\perp = m(1 - c)H$,
- (iii) $\text{Ric}^\perp(f) - m\nabla H = -m\nabla H$.

The corollary now follows easily from Theorem 4.1. □

REMARKS. 1. There is nothing special about the value of $c = 1$ (as long as it is positive). The metric on TM^\perp can be scaled as we did with ds_t^2 on $G_m(TN)$ in (3.1). Namely, in (4.2), replace φ^α by $\varphi^\alpha = t(dx^\alpha + x^\beta \omega_\beta^\alpha)$, for some constant $t > 0$, in order to obtain a metric $ds_{TM^\perp}^2(t)$. The Corollary will then hold when N has constant positive curvature $c > 0$, provided that we use the metric $ds_{TM^\perp}^2(t)$ with $t^2 = 1/c$.

2. We take the opportunity here to correct an error in [12]. The Theorem there should be changed to: Let $f: M^m \rightarrow \mathbf{R}^n$ be an isometric immersion. Then its spherical Gauss map $\nu: TM_1^\perp \rightarrow S^{n-1}$ is harmonic if and only if f has parallel mean curvature and conformal second fundamental tensor. The error comes from equation (21) of that paper where one term is missing. From the proof of the above Corollary for the case $c = 0$, and from the fact that $TR_1^n = \mathbf{R}^n \times S^{n-1}$

(Riemannian direct product), our Theorem 4.1 generalizes the theorem in [12].

5. A maximum principle applied to the tension field. We conclude with an application of Karp's weak maximum principle [8] to the geometry of the Gauss map. A complete Riemannian manifold M is said to have subquadratic exponential volume growth if there exists a point $x_0 \in M$ such that if $r(y) = \text{dist}(y, x_0)$, for $y \in M$, and if $B_r(x_0)$ is the geodesic ball of radius r centered at x_0 , then

$$\gamma_2 = \overline{\lim}_{r \rightarrow \infty} \frac{1}{r^2} \log(\text{vol } B_r(x_0)) < +\infty.$$

This property is independent of the point x_0 .

The following result is due to L. Karp [8]:

Let M be a complete Riemannian manifold with subquadratic exponential volume growth. If u is a real function on M such that $\sup_M u < +\infty$ then $\inf_M \Delta u \leq 0$, where Δ is the Laplace-Beltrami operator on M .

We apply Karp's theorem as follows.

THEOREM 5.1. *Let M be a complete Riemannian manifold with subquadratic volume growth and let N be a complete Riemannian manifold with sectional curvatures bounded above by a constant K . Let $f: M \rightarrow N$ be a smooth map such that $f(M) \subset B_R(y_0)$, a geodesic ball of radius R , $0 \leq R \leq +\infty$, inside the cut locus of a point $y_0 \in N$. Let $\tau(f)$ denote the tension field of f and suppose that*

$$\tau_0 = \sup_M |\tau(f)| < +\infty.$$

Let $e(f)$ denote the energy density of f . Then:

(1) *If $K > 0$ and $R < \pi/2\sqrt{K}$, then*

$$R \geq \frac{1}{\sqrt{K}} \arctan(2\sqrt{K} \inf e(f)/\tau_0).$$

(2) *If $K = 0$, then $R \geq (2/\tau_0) \inf e(f)$.*

(3) *If $K < 0$, then*

$$R \geq \frac{1}{\sqrt{-K}} \text{argtanh}(2\sqrt{-K} \inf e(f)/\tau_0).$$

REMARKS. (i) In Cases (2) and (3), with N simply connected, the Cartan-Hadamard theorem implies that $B_R(y_0)$ is automatically inside the cut locus of $y_0 \in N$.

(ii) The theorem generalizes a result of Karp’s [8] proved under the assumption that f is an isometry and that $K \leq 0$. It also generalizes the main result proved in Jorge-Xavier [7], for, when f is an isometry, it follows from the assumptions of their theorem that M has subquadratic exponential volume growth.

Proof. (1) Let $\rho(x)$ be the distance function from $y_0 \in N$ restricted to the ball $B_R(y_0)$. Being inside the cut locus of y_0 , ρ is smooth. On $B_R(y_0)$ consider the function $\varphi = 1 - \cos(\sqrt{K}\rho)$, where we remark that $0 \leq \rho \leq R < \pi/2\sqrt{K}$. By applying the Hessian comparison theorem of Greene-Wu [6], we obtain

$$\text{Hess } \varphi \geq K \cos(\sqrt{K}\rho) ds_N^2,$$

where ds_N^2 is the metric on N . Furthermore, from the gradient we obtain

$$\nabla \varphi = \sqrt{K} \sin(\sqrt{K}\rho) \nabla \rho.$$

Recall that if $\{e_i\}$ is an orthonormal frame field in M , then

$$\Delta(\varphi \circ f) = \text{Hess } \varphi(df(e_i), df(e_i)) + ds_N^2(\tau(f), \nabla \varphi).$$

Therefore

$$\begin{aligned} \Delta(\varphi \circ f) &\geq K \cos(\sqrt{K}\rho) |\nabla f|^2 - |\tau(f)| \sqrt{K} \sin(\sqrt{K}\rho) \\ &\geq 2K \cos(\sqrt{K}R) e(f) - \tau_0 \sqrt{K} \sin(\sqrt{K}R) \\ &\geq 2K \cos(\sqrt{K}R) \inf_M e(f) - \tau_0 \sqrt{K} \sin(\sqrt{K}R). \end{aligned}$$

Now apply Karp’s theorem to deduce

$$0 \geq \inf \Delta(\varphi \circ f) \geq 2\sqrt{K} \cos(\sqrt{K}R) \inf_M e(f) - \tau_0 \sin(\sqrt{K}R),$$

that is,

$$R \geq \frac{1}{\sqrt{K}} \arctan \left(\frac{2\sqrt{K} \inf e(f)}{\tau_0} \right).$$

The proofs of (2) and (3) are similar. □

We apply Theorem 5.1 to the case when N is the Euclidean space \mathbf{R}^n . As the Grassmann bundle of \mathbf{R}^n with any of the metrics (3.1) is a Riemannian product $\mathbf{R}^n \times G_m(n)$, the Gauss map of f decomposes into $\gamma_f = (f, \tilde{\gamma}_f)$, where $\tilde{\gamma}_f: M \rightarrow G_m(n)$ is the Gauss map considered by Ruh-Vilms [13]. Thus $\tilde{\gamma}_f$ is harmonic if and only if the isometric immersion f has parallel mean curvature vector. Furthermore, equation (3.6) reduces to Obata’s equation [11]

$$(5.1) \quad \tilde{\gamma}_f^* d\Sigma^2 = -\text{Ric}_M + m\text{II}_H.$$

where $d\Sigma^2$ is the standard $O(n)$ -invariant metric on $G_m(n)$. Thus, we calculate the energy density

$$(5.2) \quad 2e(\tilde{\gamma}_f) = \text{tr } \tilde{\gamma}_f^* d\Sigma^2 = |\mathbf{II}|^2.$$

THEOREM 5.2. *Let M be a complete Riemannian manifold with subquadratic exponential volume growth, and let $f : M \rightarrow \mathbf{R}^n$ be an isometric immersion with parallel mean curvature vector. Let $\tilde{\gamma}_f : M \rightarrow G_m(n)$ be its Gauss map. If $\tilde{\gamma}_f(M) \subset B_R(y_0)$ for some $y_0 \in G_m(n)$, and if $R < \pi/2\sqrt{K}$, (where $K = 1$ if $n - m = 1$ and $K = 2$ otherwise), then f is minimal.*

The proof will be given at the end of this section.

REMARKS. (1) This theorem extends a result of Xin [16] which was proved under the additional assumption that $\text{Ric}_M \geq 0$.

(2) By the hypotheses of the theorem the geodesic ball $B_R(y_0)$ does not contain any of the cut points of y_0 . It is, however, not trivial to describe such geodesic balls in algebraic terms. The following result is due to Fisher-Colbrie [5].

LEMMA. *Let $\beta_R(y_0) = \{q \in G_m(n) : \langle q, y_0 \rangle \geq \cos^m(R/\sqrt{m})\}$, where $\langle \cdot, \cdot \rangle$ is the inner product on unit length decomposable m -vectors of \mathbf{R}^n . Then $\beta_R(y_0) \subset B_R(y_0)$.*

From this lemma and Theorem 5.2 we obtain:

COROLLARY. *Let M be a complete Riemannian manifold with subquadratic exponential volume growth, and let $f : M \rightarrow \mathbf{R}^n$ be an isometric immersion with parallel mean curvature. Suppose there exists a decomposable m -vector y_0 such that $\langle \tilde{\gamma}_f(p), y_0 \rangle \geq \cos^m(R/\sqrt{m})$, for $R < \pi/2\sqrt{K}$, where K is as in the theorem. Then f is minimal.*

Proof of Theorem 5.2. As $\tilde{\gamma}_f$ is harmonic, from Theorem 5.1 we deduce that, for all $\tau_0 > 0$,

$$\inf_M e(\tilde{\gamma}_f) \leq \frac{\tau_0}{2\sqrt{K}} \tan(\sqrt{K}R);$$

that is, $\inf_M e(\tilde{\gamma}_f) = 0$. By (5.2) this means that $\inf_M |\mathbf{II}|^2 = 0$. On the other hand, $|H|^2 \leq |\mathbf{II}|^2$, and therefore $\inf |H|^2 = 0$. But H is parallel, hence $|H|$ is constant, and we thus have H identically zero on M . \square

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