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# RANGE TRANSFORMATIONS ON A BANACH FUNCTION ALGEBRA. II

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*Dedicated to Professor Junzo Wada on his 60th birthday*

**In this paper, localization for ultraseparability is introduced and a local version of Bernard's lemma is proven. By using these results it is shown that a function in $\mathrm{Op}(I_D, \mathrm{Re}\,A)$ is harmonic near the origin for a uniformly closed subalgebra $A$ of $C_0(Y)$ and an ideal $I$ of $A$ unless the uniform closure $\mathrm{cl}\,I$ of $I$ is self-adjoint; in particular, it is shown that $\mathrm{cl}\,I$ is self-adjoint if $\mathrm{Re}\,I \cdot \mathrm{Re}\,I \subset \mathrm{Re}\,A$, which is not true when $I$ is merely a closed subalgebra of $A$.**

**1. Introduction.** Let $Y$ be a locally compact Hausdorff space, and $C_0(Y)$ (resp. $C_{0,R}(Y)$) be the Banach algebra of all complex (resp. real) valued continuous functions on $Y$ which vanish at infinity. If $Y$ is compact, we write $C(Y)$ and $C_R(Y)$ instead of $C_0(Y)$ and $C_{0,R}(Y)$ respectively. Thus $C(Y)$ (resp. $C_R(Y)$) is the algebra of all complex (resp. real) valued continuous functions on $Y$ if $Y$ is compact. For a function $f$ in $C_0(Y)$, $\|f\|_\infty$ denotes the supremum norm on $Y$. We say that $A$ is a Banach algebra (resp. space) included in $C_0(Y)$ with the norm $\|\cdot\|_A$ if $A$ is a complex subalgebra (resp. space) of $C_0(Y)$ which is a complex Banach algebra (resp. space) with respect to the norm $\|\cdot\|_A$ (resp. such that $\|f\|_\infty \le \|f\|_A$ holds for every $f$ in $A$). It is well known that the inequality $\|f\|_\infty \le \|f\|_A$ holds for every $f$ in a Banach algebra $A$ included in $C_0(Y)$ with the norm $\|\cdot\|_A$. Thus we may suppose that a Banach algebra included in $C_0(Y)$ is a Banach space included in $C_0(Y)$. We say that $E$ is a real Banach space included in $C_{0,R}(Y)$ with the norm $\|\cdot\|_E$ if $E$ is a real subspace of $C_{0,R}(Y)$ which is a real Banach space with respect to the norm $\|\cdot\|_E$ such that $\|u\|_\infty \le \|u\|_E$ holds for every $u$ in $E$. A (resp. real) Banach space or algebra included in $C_0(Y)$ (resp. $C_{0,R}(Y)$) is said to be trivial if it coincides with $C_0(Y)$ (resp. $C_{0,R}(Y)$).

If $A$ is a Banach space included in $C_0(Y)$ with the norm $\|\cdot\|_A$ for a locally compact Hausdorff space $Y$, $\mathrm{Re}\,A = \{u \in C_{0,R}(Y) \colon \exists v \in C_{0,R}(Y)$ such that $u + iv \in A\}$ is a real Banach space with respect to the quotient norm $\|\cdot\|_{\mathrm{Re}\,A}$ defined by

$$\|u\|_{\mathrm{Re}\,A} = \inf\{\|f\|_A : f \in A,\ \mathrm{Re}\,f = u\}$$

for $u$ in $\mathrm{Re}\,A$. Since the inequality

$$\|u\|_\infty \le \|u\|_{\mathrm{Re}\,A}$$

holds for every $u$ in $\mathrm{Re}\,A$ by the definition of $\|u\|_{\mathrm{Re}\,A}$, $\mathrm{Re}\,A$ is a real Banach space included in $C_{0,R}(Y)$ with the norm $\|\cdot\|_{\mathrm{Re}\,A}$. Let $B$ be a (resp. real) Banach space included in $C_0(Y)$ (resp. $C_{0,R}(Y)$) with the norm $\|\cdot\|_B$ for a locally compact Hausdorff space $Y$ and $K$ be a compact subset of $Y$. We denote

$$\{f \in C(K) \text{ (resp. } C_R(K) : \exists F \in B,\ F|K = f\}$$

by $B|K$, where $F|K$ is the restriction of the function $F$ to $K$. $B|K$ is a (resp. real) Banach space included in $C(K)$ (resp. $C_R(K)$) with the quotient norm $\|\cdot\|_{B|K}$ defined by

$$\|f\|_{B|K} = \inf\{\|F\|_B : F \in B,\ F|K = f\}$$

for $f$ in $B|K$; in particular, $B|K$ is a Banach algebra included in $C(K)$ if $B$ is a Banach algebra included in $C_0(Y)$. For a point $x$ in $Y$, $B_x = \{f \in B : f(x) = 0\}$ is a (resp. real) Banach space included in $C_0(Y)$ (resp. $C_{0,R}(Y)$) with the norm $\|\cdot\|_B$; in particular, $B_x$ is a Banach algebra included in $C_0(Y)$ if $B$ is a Banach algebra included in $C_0(Y)$.

$A$ is said to be a Banach function algebra on $X$ if $X$ is a compact Hausdorff space and $A$ is a Banach algebra included in $C(X)$ which separates the points of $X$ and contains constant functions on $X$. A function algebra on $X$ is a Banach function algebra on $X$ with the supremum norm as the Banach algebra norm.

For any subsets $S$ and $T$ of $C_0(Y)$ and for a point $x$ in $Y$ and for a compact subset $K$ of a locally compact Hausdorff space $Y$, we use the following notations and a terminology in this paper.

$$S|K = \{f \in C(K) : \exists F \in S \text{ such that } F|K = f\},$$
$$S_x = \{f \in S : f(x) = 0\},$$

where $F|K$ denotes the restriction of the function $F$ to $K$.

$$\mathrm{Re}\,S = \{u \in C_{0,R}(Y) : \exists v \in C_{0,R}(Y) \text{ such that } u + iv \in S\},$$

where $i = \sqrt{-1}$.

$$\mathrm{cl}\,S = \text{ the uniform closure of } S \text{ in } C_0(Y),$$
$$\overline{S} = \{\overline{f} : f \in S\},$$

where $^-$ denotes the complex conjugation.

$$S \cdot T = \{fg : f \in S, g \in T\},$$
$$S + T = \{f + g : f \in S, g \in T\},$$
$$\operatorname{Ker} S = \{y \in Y : f(y) = 0\}.$$

We say that $S$ separates the points near $x$ if there is a compact neighborhood $U$ of $x$ in $Y$ such that $S$ separates the points in $U$.

It is a natural question to ask when a (resp. real) Banach space included in $C_0(Y)$ (resp. $C_{0,R}(Y)$) coincides with $C_0(Y)$ (resp. $C_{0,R}(Y)$). The Stone-Weierstrass theorem is classical: A self-adjoint function algebra on $X$ coincides with $C(X)$. Hoffmann-Wermer-Bernard's theorem on the uniformly closed real part of a Banach function algebra [**2**, **8**] is well known: If $A$ is a Banach function algebra on $X$ and $\operatorname{Re} A$ is uniformly closed, then $A = C(X)$. I. Glicksberg [**4**] generalized their theorem in the case of a function algebra on a metrizable $X$. J. Wada [**14**] removed the metrizability on $X$. S. Saeki [**10**] extended the results of J. Wada in the case of a Banach algebra included in $C_0(Y)$ with certain conditions (cf. [**13**]). One of Saeki's theorems in [**10**] is as follows: Let $A$ be a Banach algebra included in $C_0(Y)$, and $I$ be a closed subalgebra of $A$ such that $I \cdot A_R \subset I$, where $A_R = A \cap C_{0,R}(Y)$. If $\operatorname{cl}(\operatorname{Re} I) \subset \operatorname{Re} A$, then we have that $\operatorname{cl} I$ is closed under complex conjugation. If in addition, $A \cap \overline{A}$ is closed in $A$, then $I$ is uniformly closed.

Wermer's theorem about the ring condition on the real part of a function algebra [**15**] is also well known: If the real part of a function algebra is a ring, then the algebra is the trivial one. The theorem is generalized in the setting of range transformations [**7**]. Suppose that $S$ and $T$ are sets of complex or real valued functions on a set $Z$ and $D$ is a subset of the complex plane. We denote

$$\operatorname{Op}(S_D, T) = \{h : h \text{ is a complex valued function on } D \text{ such that } h \circ f \in T \text{ whenever } f \in S \text{ has range in } D\}.$$

The central problem on range transformations is to determine the class $\operatorname{Op}(S_D, T)$ (cf. [**1**]). The Stone-Weierstrass theorem asserts that if $\operatorname{Op}(A_C, A)$ for a function algebra $A$ on $X$ and for the complex plane $C$ contains the function $z \mapsto \overline{z}$, then $A = C(X)$. A theorem of de Leeuw-Katznelson [**9**], which is one of the generalizations of the Stone-Weierstrass theorem, states that a continuous nonanalytic function is not contained in $\operatorname{Op}(A_D, A)$ for a non-trivial function algebra $A$ on $X$ and a plane domain $D$. W. Spraglin [**12**] removed the continuity assumption for functions in $\mathrm{Op}(A_D, A)$ by showing that every function in $\mathrm{Op}(A_D, A)$ is continuous if $X$ is infinite. Wermer's theorem is generalized as follows [5, 11]: $\mathrm{Op}((\mathrm{Re}\,A)_I, \mathrm{Re}\,A)$ consists of only affine functions on an interval $I$ for a non-trivial function algebra $A$. Either of these theorems are generalized as the following.

THEOREM [7; *Corollary* 1.1]. *Let $A$ be a non-trivial function algebra and $D$ be a plane domain. Then every function in* $\mathrm{Op}(A_D, \mathrm{Re}\,A)$ *is harmonic.*

For certain non-trivial function algebras $A$ and $B$, $\mathrm{Op}(A_D, \mathrm{Re}\,B)$ contains non-harmonic functions (cf. [7]). In this paper we show that a result analogous to the above theorem holds when $B$ is uniformly closed and $A$ is an ideal of $B$. Our main result is the following.

THEOREM 2. *Let $A$ be a uniformly closed subalgebra of $C_0(Y)$ for a locally compact Hausdorff space $Y$ and $I$ be an ideal of $A$. Let $D$ be a plane domain containing the origin. Suppose that* $\mathrm{Op}(I_D, \mathrm{Re}\,A)$ *contains a function which is not harmonic on any neighborhood of the origin. Then, for every compact subset $K$ of $Y - \mathrm{Ker}\,I$, $I|K$ is uniformly closed and self-adjoint (i.e., closed under complex conjugation) and* $\mathrm{cl}\,I$ *is self-adjoint.*

As a corollary of Theorem 2 we prove a result analogous to a theorem of Saeki: Let $A$ be a uniformly closed subalgebra of $C_0(Y)$ and $I$ be an ideal of $A$. If $\mathrm{Re}\,I \cdot \mathrm{Re}\,I \subset \mathrm{Re}\,A$, then $\mathrm{cl}\,I$ is self-adjoint.

The concept of ultraseparation was introduced by A. Bernard and it was used to provide, for example, a solution of a problem on range transformations (cf. [2]). The so-called Bernard's lemma is the essential tool there. In the next section we introduce localization of ultraseparability and prove a "local" Bernard's lemma, which is used to prove Theorem 2 in the last section.

**2. Local property of functions in a Banach space included in $C_0(Y)$ or $C_{0,R}(Y)$.** Let $E$ be a (resp. real) Banach space included in $C_0(Y)$ (resp. $C_{0,R}(Y)$) with the norm $N_E(\cdot)$, where $Y$ is a locally compact Hausdorff space. Let $\Lambda$ be a discrete topological space. We denote the space of all bounded (with respect to the norm $N_E(\cdot)$) $E$-valued functions on $\Lambda$ by $\tilde{E}^\Lambda$. Then we see that $\tilde{E}^\Lambda$ is a Banach space with the norm

$$(N_E)^{\sim\Lambda}(\tilde{f}) = \tilde{N}_E^\Lambda(\tilde{f}) = \sup\{N_E(\tilde{f}(\alpha)) \colon \alpha \in \Lambda\}$$

for $\tilde{f}$ in $\tilde{E}^{\Lambda}$. If $E$ is a Banach algebra, then $\tilde{E}^{\Lambda}$ is also a Banach algebra. Let $K$ be a compact subset of $Y$. Then $(E|K)^{\sim\Lambda} = \tilde{E}^{\Lambda}|\tilde{K}^{\Lambda}$ and $(N_{E|K})^{\sim\Lambda}(\cdot) = (\tilde{N}_E^{\Lambda})_{|\tilde{K}^{\Lambda}}(\cdot)$, where $N_{E|K}(\cdot)$ is the quotient norm with respect to $N_E(\cdot)$ and $K$ and $(\tilde{N}_E^{\Lambda})_{|\tilde{K}^{\Lambda}}(\cdot)$ is the quotient norm with respect to $\tilde{N}_E^{\Lambda}(\cdot)$ and $\tilde{K}^{\Lambda}$. On the other hand we may suppose that every $E$-valued function $\tilde{f}$ in $\tilde{E}^{\Lambda}$ is a complex (resp. real) valued function on $Y \times \Lambda$ by defining

$$\tilde{f}(x, \lambda) = (\tilde{f}(\lambda))(x)$$

for $(x, \lambda)$ in $Y \times \Lambda$. Since every function $f$ in $E$ satisfies the inequality $\|f\|_\infty \leq N_E(f)$ we may suppose that every $E$-valued function $\tilde{f}$ in $\tilde{E}^{\Lambda}$ is a complex (resp. real) valued bounded function with respect to the supremum norm on $Y \times \Lambda$. So we may suppose that

$$\tilde{E}^{\Lambda} \subset C(\tilde{Y}^{\Lambda}),$$

where we denote by $\tilde{Y}^{\Lambda}$ the Stone-Čech compactification of the direct product $Y \times \Lambda$ of $Y$ and $\Lambda$. Let $x$ be a point in $Y$. We denote

$$F_x^{\Lambda} = \bigcap [G \times \Lambda],$$

where $G$ varies over all the compact neighborhoods of $x$ and $[\cdot]$ denotes the closure in $\tilde{Y}^{\Lambda}$. We denote

$$Q^{\Lambda}(E_x) = \{p \in F_x^{\Lambda} : \tilde{f}(p) = 0 \text{ for } \forall \tilde{f} \in \tilde{E}_x^{\Lambda}\}.$$

Let $(x, \lambda)$ be a point in $\{x\} \times \Lambda$ and $\tilde{f}$ be a function in $\tilde{E}_x^{\Lambda}$. Then we have $\tilde{f}(\lambda) \in E_x$ for every $\lambda \in \Lambda$ and so $(\tilde{f}(\lambda))(x) = 0$. By the definition of $Q^{\Lambda}(E_x)$ we see that

$$\{x\} \times \Lambda \subset Q^{\Lambda}(E_x) \subset F_x^{\Lambda}$$

so

$$[\{x\} \times \Lambda] \subset Q^{\Lambda}(E_x) \subset F_x^{\Lambda}$$

since $Q^{\Lambda}(E_x)$ is closed in $\tilde{Y}^{\Lambda}$. For a function $f$ in $E$ we denote by $\langle f \rangle$ the function on $\Lambda$ with the constant value $f$.

We assume from Lemma 1 through Lemma 5 that $E$ is a (resp. real) Banach space included in $C_0(Y)$ (resp. $C_{0,R}(Y)$) for a locally compact Hausdorff space $Y$ and that $\Lambda$ is a discrete topological space.

LEMMA 1. *Let $a$ and $b$ be different points in $Y$. Then $F_a^{\Lambda} \cap F_b^{\Lambda} = \emptyset$.*

*Proof*. Since $Y$ is a locally compact Hausdorff space we can choose disjoint compact neighborhoods $G_a$ and $G_b$ for $a$ and $b$ respectively.

By the definition of $F_a^\Lambda$ and $F_b^\Lambda$ we have

$$F_a^\Lambda \cap F_b^\Lambda \subset [G_a \times \Lambda] \cap [G_b \times \Lambda]$$

while $[G_a \times \Lambda] \cap [G_b \times \Lambda] = \emptyset$ since $G_a \cap G_b = \emptyset$. Thus we have $F_a^\Lambda \cap F_b^\Lambda = \emptyset$.

LEMMA 2. *Let $K$ be a compact subset of $Y$. Then*

$$\bigcup_{y \in \operatorname{Int} K} F_y^\Lambda \subset [K \times \Lambda] \subset \bigcup_{y \in K} F_y^\Lambda,$$

*where* $\operatorname{Int} K$ *is the interior of* $K$.

*Proof*. Let $y$ be a point in $\operatorname{Int} K$. By the definition of $F_y^\Lambda$ we see that

$$F_y^\Lambda \subset [K \times \Lambda],$$

so we have

$$\bigcup_{y \in \operatorname{Int} K} F_y^\Lambda \subset [K \times \Lambda].$$

Let $p$ be a point in $[K \times \Lambda]$. The functional

$$f \mapsto \langle f \rangle (p)$$

on $C(K)$ is linear and multiplicative, so there is a unique $t(p)$ in $K$ such that

$$\langle f \rangle (p) = f(t(p))$$

for all $f$ in $C(K)$. We will show that $p \in F_{t(p)}^\Lambda$. Suppose not. By the definition of $F_{t(p)}^\Lambda$ there is a compact neighborhood $G$ of $t(p)$ in $Y$ such that

$$p \notin [G \times \Lambda].$$

Since $\tilde{Y}^\Lambda = [G \times \Lambda] \cup [G^c \times \Lambda]$, where $G^c$ is the complement of $G$ in $Y$, we see that

$$p \in [G^c \times \Lambda].$$

By Urysohn's lemma there is a function $g$ in $C_0(Y)$ such that

$$g(t(p)) = 1 \quad \text{and} \quad g(y) = 0$$

for every $y$ in $G^c$. Since $p$ is in $[G^c \times \Lambda]$ we have

$$\langle g \rangle (p) = 0.$$

On the other hand

$$\langle g\rangle(p) = g(t(p)) = 1,$$

which is a contradiction. Thus we conclude that $p \in F^{\Lambda}_{t(p)}$. It follows that

$$[K \times \Lambda] \subset \bigcup_{y\in K} F^{\Lambda}_{y}.$$

LEMMA 3. *$\bigcup_{y\in Y} F^{\Lambda}_{y} \subset \tilde{Y}^{\Lambda}$ where the union is disjoint. In particular, if $Y$ is compact, then*

$$\bigcup_{y\in Y} F^{\Lambda}_{y} = \tilde{Y}^{\Lambda}.$$

*Proof*. The first assertion is trivial by the definition of $F^{\Lambda}_{y}$ and Lemma 1. If $Y$ is compact, then by Lemma 2 we see

$$\bigcup_{y\in Y} F^{\Lambda}_{y} = \tilde{Y}^{\Lambda}$$

since $Y = \operatorname{Int} Y$.

LEMMA 4. *Let $a$ be a point in $Y$ and $G$ be a compact neighborhood of $a$ in $Y$. Then*

$$F^{\Lambda}_{a} \subset \{p \in [G \times \Lambda] \colon \langle f\rangle(p) = f(a) \text{ for } \forall f \in E\}.$$

*In particular, if $E$ separates the points near $a$, that is, there is a compact neighborhood $U$ of $a$ such that $E$ separates the points in $U$, then we see that*

$$F^{\Lambda}_{a} = \{p \in [U \times \Lambda] \colon \langle f\rangle(p) = f(a) \text{ for } \forall f \in E\}.$$

*Proof*. Let $p$ be a point in $F^{\Lambda}_{a}$. Then $p \in [G\times\Lambda]$ since $F^{\Lambda}_{a} \subset [G\times\Lambda]$. Suppose that there is a function $f_0$ in $E_a$ such that

$$\langle f_0\rangle(p) \neq f_0(a).$$

Then

$$G' = \{y \in G \colon |f_0(y) - f_0(a)| \leq \delta/2\},$$

where $\delta = |\langle f_0\rangle(p) - f_0(a)|$, is a compact neighborhood of $a$ and

$$p \notin [G' \times \Lambda].$$

Thus we have $p \notin F_a^\Lambda$ since $F_a^\Lambda \subset [G' \times \Lambda]$, which is a contradiction. We conclude that

$$F_a^\Lambda \subset \{p \in [G \times \Lambda]\colon \langle f\rangle(p) = f(a) \text{ for } \forall f \in E\}.$$

Suppose that $E$ separates the points in $U$. Let $p$ be a point in $[U \times \Lambda]$ such that $\langle f\rangle(p) = f(a)$ for every $f$ in $E$. By Lemma 2 there is $y \in U$ such that $p \in F_y^\Lambda$. By the above argument we see that

$$\langle f\rangle(p) = f(y)$$

for every $f$ in $E$. Since

$$\langle f\rangle(p) = f(a)$$

for every $f$ in $E$ and we see that $a = y$ since $E$ separates the points in $U$, so we conclude that $p \in F_a^\Lambda$.

LEMMA 5. *Let $a$ be a point in $Y$ and $G$ be a compact neighborhood of $a$ in $Y$. Then*

$$Q^\Lambda(E_a) \subset \{p \in [G \times \Lambda]\colon \tilde{f}(p) = 0 \textit{ for } \forall \tilde{f} \in (E_a)^{\sim\Lambda}\}.$$

*In particular, if $E_a$ separates the points near $a$, that is, there is a compact neighborhood $U$ of $a$ such that $E_a$ separates the points in $U$, then*

$$Q^\Lambda(E_a) = \{p \in [U \times \Lambda]\colon \tilde{f}(p) = 0 \textit{ for } \forall \tilde{f} \in (E_a)^{\sim\Lambda}\}.$$

*Proof*. The first assertion is trivial by the definition of $Q^\Lambda(E_x)$. Suppose that $E_a$ separates the points in $U$. Since $\langle f\rangle$ is in $(E_a)^{\sim\Lambda}$ for every $f$ in $E_a$ we have

$$\begin{aligned}&\{p \in [U \times \Lambda]\colon \tilde{f}(p) = 0 \text{ for } \forall \tilde{f} \in (E_a)^{\sim\Lambda}\}\\ &\quad \subset \{p \in [U \times \Lambda]\colon \langle f\rangle(p) = 0 \text{ for } \forall f \in E_a\}.\end{aligned}$$

$E_a$ is a (resp. real) Banach space included in $C_0(Y)$ (resp. $C_{0,R}(Y)$) with the restriction of the norm $E$ to $E_a$. We see by Lemma 4 that

$$\{p \in [U \times \Lambda]\colon \langle f\rangle(p) = 0 \text{ for } \forall f \in E_a\} = F_a^\Lambda$$

since $f(a) = 0$ for every $f$ in $E_a$. So we conclude that

$$\{p \in [U \times \Lambda]\colon \tilde{f}(p) = 0 \text{ for } \forall \tilde{f} \in (E_a)^{\sim\Lambda}\} \subset F_a^\Lambda.$$

We see that

$$Q^\Lambda(E_a) = \{p \in [U \times \Lambda]\colon \tilde{f}(p) = 0 \text{ for } \forall \tilde{f} \in (E_a)^{\sim\Lambda}\}.$$

When $\Lambda = N$, the space of all positive integers we write $\tilde{E}, \tilde{N}_E(\cdot)$, $Q(E_x)$, $\tilde{Y}$ and $F_x$ instead of $\tilde{E}^N$, $\tilde{N}_E^N(\cdot)$, $Q^N(E_x)$, $\tilde{Y}^N$ and $F_x^N$ respectively (cf. [7]).

DEFINITION 1. Let $E$ be a (resp. real) Banach space included in $C_0(Y)$ (resp. $C_{0,R}(Y)$). We say that $E$ is ultraseparating if $\tilde{E}$ separates the points of $\tilde{Y}$. We say that $E$ is ultraseparating near a point $x$ in $Y$ if there is a compact neighborhood $K$ of $x$ such that $E|K$ is ultraseparating with respect to the quotient norm, that is, $(E|K)^{\sim}$ of $E|K$ with the quotient norm separates the points of $\tilde{K}$.

It is easy to see that if $E$ is ultraseparating on $Y$, then $Y$ is compact and $E$ separates the points of $Y$ and $E \neq E_y$ for every point $y$ in $Y$.

LEMMA 6. *Let $E$ be a (resp. real) Banach space included in $C(X)$ (resp. $C_R(X)$) for a compact Hausdorff space $X$. Then the following are equivalent.*

(1) *$E$ is ultraseparating.*

(2) *$E$ separates the points in $X$ and $E$ is ultraseparating near $x$ for every $x$ in $X$.*

(3) *$E$ separates the points in $X$ and $\tilde{E}$ separates the points in $F_x$ for every $x$ in $X$.*

*Proof*. (1) → (2) and (2) → (3) are trivial. So we show (3) → (1). Suppose that (3) is satisfied. By Lemma 3 we have $\tilde{X} = \bigcup_{x \in X} F_x$, where the union is disjoint. Let $p$ and $q$ be different points in $\tilde{X}$. We consider two cases. If there is $x \in X$ such that $p$ and $q$ are points in $F_x$, then $\tilde{E}$ separates $p$ and $q$ by (3). If $p \in F_x$ and $q \in F_y$ for different points $x$ and $y$ in $X$, then there is a function $f$ in $E$ such that $f(x) \neq f(y)$ since we suppose that (3) is satisfied. It follows that

$$\langle f \rangle(p) \neq \langle f \rangle(q)$$

since $\langle f \rangle(p) = f(x)$ and $\langle f \rangle(q) = f(y)$. In any case we see that $\tilde{E}$ separates $p$ and $q$.

PROPOSITION 1. *Let $E$ be a real Banach space included in $C_{0,R}(Y)$ for a locally compact Hausdorff space $Y$. Let $x$ be a point in $Y$. If $E$*

*is ultraseparating near* $x$, *then the following condition is satisfied:*

(*) *There is a compact neighborhood* $G$ *of* $x$ *which satisfies the condition that there are a natural number* $m$ *and a* $\delta > 0$ *such that if* $Y_1$ *and* $Y_2$ *are disjoint compact subsets of* $G$, *then we can choose* $f_1, f_2, \dots, f_m$ *and* $g_1, g_2, \dots, g_m$ *in the unit ball of* $E$ *satisfying*

$$\sum_{i=1}^{m}(|f_i| - |g_i|) > \delta \quad \text{on } Y_1,$$

$$\sum_{i=1}^{m}(|f_i| - |g_i|) < -\delta \quad \text{on } Y_2.$$

*If* $(*)$ *is satisfied, then* $E|G$ *is ultraseparating.*

LEMMA 7. *Let* $E$ *be a real Banach space included in* $C_R(X)$ *for a compact Hausdorff space* $X$ *such that* $E$ *separates the points of* $X$ *and* $E$ *contains constant functions. Then the space of all linear combinations of* $|f|$ *for* $f$ *in the unit ball of* $E$ *is uniformly dense in* $C_R(X)$.

*Proof*. Let $\delta > 0$ and $\sigma_\delta$ be a $C^\infty$-smoothing operator supported in $(-\delta, \delta)$, that is, $\sigma_\delta$ is a nonnegative real valued function of class $C^\infty$ on the real line supported in $(-\delta, \delta)$ with integral 1. Put

$$h_\delta(x) = \int_{-\delta}^{\delta} |x - t|\sigma_\delta(t)\,dt.$$

Then $h_\delta$ is a function of class $C^\infty$. For every positive $\varepsilon$ and for every positive integer $m$ there exist a $\delta > 0$, a $C^\infty$-smoothing operator $\sigma_\delta$ and a real number $t$ with $|t| < \varepsilon$ such that

$$(d^m/dx^m)h_\delta(t) \neq 0$$

since $|\cdot|$ is not a polynomial near the origin. We denote the uniform closure of the space of all linear combinations of the absolute value of functions in the unit ball of $E$ by $V$. Let $g_1, g_2, \dots, g_n$ be functions in the unit ball of $E$. Then

$$h_\delta(g_1 s_1 + g_2 s_2 + \cdots + g_n s_n + t) \in V$$

for real numbers $s_1, s_2, \dots, s_n$, $t$ with sufficiently small absolute values, provided $\delta < 1$. Thus we see that

$$\{h_\delta(g_1 s_1 + g_2 s_2 + \cdots + g_n s_n + t) - h_\delta(g_2 s_2 + g_3 s_3 + \cdots + g_n s_n + t)\}/s_1$$

is in $V$. In particular, fixing $s_2, s_3, \dots, s_n$ and letting $s_1 \to 0$ we have

$$g_1(d/dx)h_\delta(g_2 s_2 + \cdots + g_n s_n + t) \in V.$$

Continuing in this manner,

$$g_1 g_2 \cdots g_n (d^n/dx^n) h_\delta(t) \in V$$

and since we may suppose that $(d^n/dx^n)h_\delta(t) \neq 0$ we have

$$g_1 g_2 \cdots g_n \in V.$$

It follows by the Stone-Weierstrass theorem that

$$V = C_R(X).$$

*Proof of Proposition* 1. Suppose that the condition $(*)$ is satisfied. We show that $E|G$ with the quotient norm is ultraseparating on $G$. Let $a$ and $b$ be different points of $\tilde{G}$ and $U_a$ and $U_b$ be disjoint compact neighborhoods of $a$ and $b$ respectively. Let

$$U_a^k = U_a \cap (G \times \{k\})$$

and

$$U_b^k = U_b \cap (G \times \{k\}).$$

Then we see that $U_a^k \cap U_b^k = \varnothing$ and $a \in [\bigcup_{k=1}^\infty U_a^k]$, $b \in [\bigcup_{k=1}^\infty U_b^k]$. Let $t$ be the map

$$t \colon \tilde{G} \to G$$

which satisfies

$$\langle f \rangle (p) = f(t(p))$$

for every $f$ in $C(G)$ and for every $p$ in $\tilde{G}$. Since $t(U_a^k)$ and $t(U_b^k)$ are disjoint compact subsets of $G$, by the condition $(*)$ and by the definition of the quotient space we can choose $f_{1,k}, f_{2,k}, \dots, f_{m,k}$ and $g_{1,k}, g_{2,k}, \dots, g_{m,k}$ in the unit ball of $E|G$ for every positive integer $k$ satisfying

$$\sum_{i=1}^m (|f_{i,k}| - |g_{i,k}|) > \delta/2 \quad \text{on } t(U_a^k),$$

$$\sum_{i=1}^m (|f_{i,k}| - |g_{i,k}|) < -\delta/2 \quad \text{on } t(U_b^k).$$

It follows that

$$\sum_{i=1}^m (|\langle f_{i,n} \rangle (a)| - |\langle g_{i,n} \rangle (a)|) \geq \delta/2$$

and

$$\sum_{i=1}^{m}(|\langle f_{i,n}\rangle(b)| - |\langle g_{i,n}\rangle(b)|) \leq -\delta/2,$$

where $\langle f_{i,n}\rangle$ and $\langle g_{i,n}\rangle$ are functions in $\tilde{E}$ such that $\langle f_{i,n}\rangle(y,k) = f_{i,k}(y)$ and $\langle g_{i,n}\rangle(y,k) = g_{i,k}(y)$ for every $(y,k)$ in $G\times N$ respectively. Thus we see that at least one of $\langle f_{1,n}\rangle, \langle f_{2,n}\rangle, \dots, \langle f_{m,n}\rangle$ and $\langle g_{1,n}\rangle, \langle g_{2,n}\rangle, \dots, \langle g_{m,n}\rangle$ separates $a$ and $b$. We conclude that $E|G$ is ultraseparating.

To prove the reverse implication we suppose that $E|G'$ is ultraseparating for a compact neighborhood $G'$ of $x$. We consider two cases:

(1) $E|G'$ contains constant functions.

(2) $E|G'$ does not contain non-zero constant functions.

First we treat the case (1). Suppose that $(*)$ is not satisfied with $G = G'$. Then for every positive integer $n$ there are disjoint compact subsets $Y_{1,n}$ and $Y_{2,n}$ of $G'$ such that

$$\sum_{i=1}^{n}(|f_i| - |g_i|) > 1/n \quad \text{on } Y_{1,n}$$

or

$$\sum_{i=1}^{n}(|f_i| - |g_i|) < -1/n \quad \text{on } Y_{2,n}$$

are not satisfied for every $f_1, f_2, \dots, f_n$ and $g_1, g_2, \dots, g_n$ in the unit ball of $E|G'$. Put

$$\tilde{Y}_1 = \left[\bigcup_{n=1}^{\infty}(Y_{1,n}\times\{n\})\right]$$

and

$$\tilde{Y}_2 = \left[\bigcup_{n=1}^{\infty}(Y_{2,n}\times\{n\})\right].$$

Since $\tilde{Y}_1$ and $\tilde{Y}_2$ are disjoint compact subsets of $\tilde{G}'$ there are $\tilde{f}$ in the unit ball of $C(\tilde{G}')$ such that

$$\tilde{f}(\tilde{y}) = 1 \quad \text{for every } \tilde{y} \text{ in } \tilde{Y}_1,$$
$$\tilde{f}(\tilde{y}) = -1 \quad \text{for every } \tilde{y} \text{ in } \tilde{Y}_2.$$

By Lemma 7 there are a finite number of functions $\tilde{f}_1, \tilde{f}_2, \dots, \tilde{f}_\nu$ and $\tilde{g}_1, \tilde{g}_2, \dots, \tilde{g}_\nu$ in $(E|G')^{\sim}$ with the norm less than $1/2$ respectively which satisfy

$$\left|\sum_{i=1}^{\nu}(|\tilde{f}_i| - |\tilde{g}_i|) - \tilde{f}\right| < 1/3.$$

Thus we see that

$$\sum_{i=1}^{\nu}(|\tilde{f}_i| - |\tilde{g}_i|) > 2/3 \quad \text{on } \tilde{Y}_1$$

and

$$\sum_{i=1}^{\nu}(|\tilde{f}_i| - |\tilde{g}_i|) < -2/3 \quad \text{on } \tilde{Y}_2.$$

By the definition of the norm of $(E|G')^{\sim}$ there are functions $f_{1,n}, f_{2,n}, \ldots, f_{\nu,n}$ and $g_{1,n}, g_{2,n}, \ldots, g_{\nu,n}$ in the unit ball of $E$ such that

$$\tilde{f}_i(n) = f_{i,n}|G',$$
$$\tilde{g}_i(n) = g_{i,n}|G'$$

for every positive integer $n$ and $i = 1, 2, \ldots, \nu$. It follows that

$$\sum_{i=1}^{\nu}(|f_{i,n}| - |g_{i,n}|) > 2/3 \quad \text{on } Y_{1,n}$$

and

$$\sum_{i=1}^{\nu}(|f_{i,n}| - |g_{i,n}|) < -2/3 \quad \text{on } Y_{2,n},$$

which is a contradiction to the definition of $Y_{1,n}$ and $Y_{2,n}$ for large $n$. Thus we have that $(*)$ is satisfied with $G = G'$.

Next we consider the case (2). Let $E' = E|G' + C$, where $C$ is the space of all the real valued constant functions on $G'$. We identify a real number $c$ and the function on $G'$ with constant value $c$. Then $B$ is a real Banach space included in $C_R(G')$ with the norm defined by

$$\|f + c\|_{E'} = \|f\|_{E|G'} + |c|,$$

where $\|f\|_{E|G'}$ is the quotient norm for $f$ in $E|G'$ and $|c|$ is absolute value of a real number $c$. By (1) we see the following:

There are a natural number $m$ and a $\delta' > 0$ such that if $Y_1'$ and $Y_2'$ are disjoint compact subsets of $G'$, then we can choose $f_1' + c_1$, $f_2' + c_2, \ldots, f_m' + c_m$ and $g_1' + d_1$, $g_2' + d_2, \ldots, g_m' + d_m$ in the unit ball of $E'$ satisfying

$$\sum_{i=1}^{m}(|f_i' + c_i| - |g_i' + d_i|) > \delta' \quad \text{on } Y_1',$$
$$\sum_{i=1}^{m}(|f_i' + c_i| - |g_i' + d_i|) < -\delta' \quad \text{on } Y_2'.$$

There is a function $u$ in $E|G'$ such that $u(x) = 1$ since $E|G'$ is ultraseparating. Put $M = \|u\|_{E|G'}$. Take the compact neighborhood

$$G = \{y \in G' : |1 - u(y)| \leq \delta'/4m\}$$

of $x$. Then we see the following:

If $Y_1$ and $Y_2$ are disjoint compact subsets of $G$, there are functions $(f_i' + c_i u)/(M+1)$ and $(g_i' + d_i u)/(M+1)$ in the unit ball of $E'$ and that

$$\sum_{i=1}^{m} \{|(f_i' + c_i u)/2(M+1)| - |(g_i' + d_i u)/2(M+1)|\} > \delta'/4(M+1)$$

on $Y_1$ and

$$\sum_{i=1}^{m} \{|(f_i' + c_i u)/2(M+1)| - |(g_i' + d_i u)/2(M+1)|\} < -\delta'/4(M+1)$$

on $Y_2$. By the definition of the quotient norm of $E|G$ there are functions $f_1, f_2, \ldots, f_m$ and $g_1, g_2, \ldots, g_m$ in the unit ball of $E$ which satisfy

$$f_i|G = (f_i' + c_i u)/2(M+1),$$
$$g_i|G = (g_i' + d_i u)/2(M+1)$$

for $i = 1, 2, \ldots, m$. Put $\delta = \delta'/4(M+1)$. The condition (*) holds on $G$ with $m$ and $\delta$.

COROLLARY 1. *Let $E$ be a (resp. real) Banach space included in $C_0(Y)$ (resp. $C_{0,R}(Y)$). Let $K$ be a compact subset of $Y$. Then the following are equivalent.*

(1) *$E|K$ is ultraseparating.*

(2) *$(E|K)^{\sim\Lambda}$ is ultraseparating for a discrete topological space $\Lambda$.*

(3) *$(E|K)^{\sim\Lambda}$ separates the points of $\tilde{K}^{\Lambda}$ for a discrete topological space $\Lambda$ whose cardinality is infinite.*

(4) *$((E|K)^{\sim\Lambda})^{\sim\Lambda'}$ separates the points of $(\tilde{K}^{\Lambda})^{\sim\Lambda'}$ for discrete topological spaces $\Lambda$ and $\Lambda'$, where at least one of the cardinalities of $\Lambda$ and $\Lambda'$ is infinite.*

*Proof*. Suppose that $E$ is a Banach space included in $C_0(Y)$. By the definition of the quotient norm of $\mathrm{Re}\,E$ we see that $(\mathrm{Re}\,E|K)^{\sim\Lambda} = \mathrm{Re}((E|K)^{\sim\Lambda})$. Thus (1), (2), (3) and (4) are equivalent to the following respectively.

(1)$'$ $\mathrm{Re}\,E|K$ is ultraseparating.

(2)′ $(\mathrm{Re}\, E|K)^{\sim\Lambda}$ is ultraseparating for a discrete topological space $\Lambda$.

(3)′ $(\mathrm{Re}\, E|K)^{\sim\Lambda}$ separates the points of $\tilde{K}^{\Lambda}$ for a discrete topological space $\Lambda$ with infinite cardinality.

(4)′ $((\mathrm{Re}\, E|K)^{\sim\Lambda})^{\sim\Lambda'}$ separates the points of $(\tilde{K}^{\Lambda})^{\sim\Lambda'}$ for discrete topological spaces $\Lambda$ and $\Lambda'$, where at least one of the cardinalities of $\Lambda$ and $\Lambda'$ is infinite.

So without loss of generality we may consider only the case that $E$ is a real Banach space included in $C_{0,R}(Y)$. By Lemma 6 (1) is equivalent to the condition that $E|K$ separates the points of $K$ and $E$ is ultraseparating near $x$ for every $x$ in $K$ with the relative topology induced by $Y$. Thus by Proposition 1 (1) is equivalent to the condition that $E|K$ separates the points of $K$ and $(*)$ of Proposition 1 is satisfied for every $x$ in $K$. In the same way as in the proof of Proposition 1 we see that (2), (3) and (4) are equivalent to the above condition respectively.

Now we show a local version of Bernard's lemma.

THEOREM 1. *Suppose that $E$ is a (resp. real) Banach space included in $C_0(Y)$ (resp. $C_{0,R}(Y)$) for a locally compact Hausdorff space $Y$. Let $x$ be a point in $Y$. Suppose that $\Lambda$ is a discrete topological space with cardinality not less than that of an open base for $x$. Then the following hold.*

(1) *$\tilde{E}^{\Lambda}$ separates the different points in $F_x^{\Lambda}$ if and only if $E$ is ultraseparating near $x$.*

(2) *$\tilde{E}^{\Lambda}|F_x^{\Lambda}$ is uniformly dense in $C(F_x^{\Lambda})$ (resp. $C_R(F_x^{\Lambda})$) if and only if there is an interpolating compact neighborhood $G$ of $x$ for $E$; i.e., $E|G = C(G)$ (resp. $C_R(G)$).*

*Proof*. First we prove (1). Since a Banach space $A$ included in $C_0(Y)$ is ultraseparating near a point $x$ in $Y$ if and only if $\mathrm{Re}\, A$ with the quotient norm is ultraseparating near $x$, so without loss of generality we may assume that $E$ is a real Banach space included in $C_{0,R}(Y)$. Suppose that $E$ is ultraseparating near $x$. By Proposition 1 we see that there is a compact neighborhood $G$ of $x$ which satisfies the condition that there are a positive integer $n$ and a positive real number $\delta$ such that for every pair of disjoint compact sets $G_1$ and $G_2$ of $G$, there are functions $f_1, f_2, \ldots, f_n$ and $g_1, g_2, \ldots, g_n$ in the unit ball of $E$ such that

$$\sum_{i=1}^{n}(|f_i| - |g_i|) > \delta \quad \text{on } G_1,$$

$$\sum_{i=1}^{n}(|f_i| - |g_i|) < -\delta \quad \text{on } G_2.$$

Let $p$ and $q$ be different points in $F_x^\Lambda$. Let $U_p$ and $U_q$ be disjoint compact neighborhoods in $\tilde{G}^\Lambda$ of $p$ and $q$ respectively. So we have that $t(U_p^\alpha)$ and $t(U_q^\alpha)$ are disjoint compact sets in $G$ for every $\alpha$ in $\Lambda$, where $U_p^\alpha = U_p \cap (G \times \{\alpha\})$ and $U_q^\alpha = U_q \cap (G \times \{\alpha\})$ and $t$ is the map from $[G \times \Lambda]$ onto $G$ satisfying

$$\langle f \rangle(a) = f(t(a))$$

for every $f$ in $C(G)$ and $a$ in $[G \times \Lambda]$. There are functions $f_{1,\alpha}, f_{2,\alpha}, \ldots, f_{n,\alpha}$ and $g_{1,\alpha}, g_{2,\alpha}, \ldots, g_{n,\alpha}$ in the unit ball of $E$ with

$$\sum_{i=1}^{n}(|f_{i,\alpha}| - |g_{i,\alpha}|) > \delta \quad \text{on } t(U_p^\alpha),$$

$$\sum_{i=1}^{n}(|f_{i,\alpha}| - |g_{i,\alpha}|) < -\delta \quad \text{on } t(U_q^\alpha)$$

for every $\alpha$ in $\Lambda$. Let $\tilde{f}_i$ and $\tilde{g}_i$ be $E$-valued functions in $\tilde{E}^\Lambda$ such that $\tilde{f}_i(\alpha) = f_{i,\alpha}$ and $\tilde{g}_i(\alpha) = g_{i,\alpha}$ for $i = 1, 2, \ldots, n$ and for every $\alpha$ in $\Lambda$. Since we may suppose that every $E$-valued function in $\tilde{E}^\Lambda$ is a function in $C(\tilde{Y}^\Lambda)$ by defining

$$\tilde{f}(x, \alpha) = (\tilde{f}(\alpha))(x)$$

for every $(x, \alpha)$ in $Y \times \Lambda$ and since $p$ is a point in $[\bigcup_\alpha U_p^\alpha]$ and $q$ is a point in $[\bigcup_\alpha U_q^\alpha]$ we have that $\tilde{f}_j(p) \neq \tilde{f}_j(q)$ or $\tilde{g}_j(p) \neq \tilde{g}_j(q)$ for some $1 \leq j \leq n$.

On the other hand, suppose that $\tilde{E}^\Lambda$ separates the points of $F_x^\Lambda$ so there is a $g$ in $E$ such that $g(x) = 1$ since $\tilde{E}^\Lambda$ separates the points in $\{x\} \times \Lambda$. Since $\langle g \rangle = 1$ on $F_x^\Lambda$ we see by Lemma 7 that the linear combinations of absolute value of functions in $\tilde{E}^\Lambda | F_x^\Lambda$, is uniformly dense in $C_R(F_x^\Lambda)$. Let $\{G_\alpha\}$ be a family of compact neighborhoods of $x$ such that $\{\operatorname{Int} G_\alpha\}$, the family of all the interiors of $G_\alpha$, is an open base for $x$ with the cardinality not greater than that of $\Lambda$. Without loss of generality we may assume that the two cardinalities coincide. We shall show that there are a compact neighborhood $G$ of $x$ and a positive integer $n_0$ with the following property: For every pair of disjoint compact subsets $Y_1$ and $Y_2$ of $G$, there are functions $f_1, f_2, \dots, f_{n_0}$ and $g_1, g_2, \dots, g_{n_0}$ in the unit ball of $E$ such that

$$\sum_{i=1}^{n_0}(|f_i| - |g_i|) > 1/2 \quad \text{on } Y_1,$$
$$\sum_{i=1}^{n_0}(|f_i| - |g_i|) < -1/2 \quad \text{on } Y_2.$$

Suppose not. For every compact neighborhood $G_\alpha$ in $\{G_\alpha\}$ and positive integer $n$, there are disjoint compact subsets $Y_1^{\alpha,n}$ and $Y_2^{\alpha,n}$ of $G_\alpha$ such that for every $f_1, f_2, \dots, f_n$ and $g_1, g_2, \dots, g_n$ in the unit ball of $E$ we have

$$\sum_{i=1}^{n}(|f_i| - |g_i|)(y_1) \le 1/2 \quad \text{for } \forall y_1 \in Y_1^{\alpha,n}$$

or

$$\sum_{i=1}^{n}(|f_i| - |g_i|)(y_2) \ge -1/2 \quad \text{for } \forall y_2 \in Y_2^{\alpha,n}.$$

Let $f_{\alpha,n}$ be a real valued continuous function on $Y$ with $\|f_{\alpha,n}\|_\infty \le 1$ and

$$f_{\alpha,n} = 1 \quad \text{on } Y_1^{\alpha,n},$$
$$f_{\alpha,n} = -1 \quad \text{on } Y_2^{\alpha,n}.$$

Let $\Phi$ be a homeomorphism from a discrete space $\Lambda$ onto a discrete space $\Lambda \times N$, where $N$ is the discrete space of all positive integers. Let $\tilde{f}$ be a $E$-valued function in $\tilde{E}^\Lambda$ such that

$$\tilde{f}(\gamma) = f_{\Phi(\gamma)}$$

for every $\gamma$ in $\Lambda$, so $\tilde{f}|F_x^\Lambda \in C(F_x^\Lambda)$. Thus by Lemma 7 there are a finite number of functions $\tilde{f}_1, \tilde{f}_2, \dots, \tilde{f}_m$ and $\tilde{g}_1, \tilde{g}_2, \dots, \tilde{g}_m$ in $\tilde{E}^\Lambda$ with $\tilde{N}_E^\Lambda(\tilde{f}_i) \le 1$ and $\tilde{N}_E^\Lambda(\tilde{g}_i) \le 1$ for $i = 1, 2, \dots, m$ such that

$$\left|\sum_{i=1}^{m}(|\tilde{f}_i| - |\tilde{g}_i|) - \tilde{f}\right| < 1/8$$

on $F_x^\Lambda$. Let $U$ be an open neighborhood of $F_x^\Lambda$ such that

$$\left|\sum_{i=1}^{m}(|\tilde{f}_i| - |\tilde{g}_i|) - \tilde{f}\right| < 1/4$$

on $U$. By the definition of $F_x^\Lambda$ there is a compact neighborhood $G_\beta$ in $\{G_\alpha\}$ such that

$$U \supset [G_\beta \times \Lambda].$$

Thus we see that

$$\left|\sum_{i=1}^{m}(|\tilde{f}_i(\gamma)| - |\tilde{g}_i(\gamma)|) - \tilde{f}(\gamma)\right| < 1/4$$

on $G_\beta$. We have that

$$\sum_{i=1}^{m}(|\tilde{f}_i(\Phi^{-1}(\beta, m))| - |\tilde{g}_i(\Phi^{-1}(\beta, m))|) > 3/4 \quad \text{on } Y^{\beta,m},$$

$$\sum_{i=1}^{m}(|\tilde{f}_i(\Phi^{-1}(\beta, m))| - |\tilde{g}_i(\Phi^{-1}(\beta, m))|) < -3/4 \quad \text{on } Y^{\beta,m},$$

which is a contradiction, proving (1).

To prove (2) we need the following. One can prove it by the standard argument on Banach spaces.

LEMMA 8. *Let $T_1$ and $T_2$ be Banach spaces with the norms $N_1(\cdot)$ and $N_2(\cdot)$ respectively. Let $\phi$ be a bounded linear transformation on $T_1$ into $T_2$. Suppose that there exist an $\varepsilon$ with $0 < \varepsilon < 1$ and a positive constant $M_0$ such that for every $u$ in the unit ball of $T_2$ there is $v$ in $T_1$ such that $N_1(v) \leq M_0$ and $N_2(u - \phi(v)) \leq \varepsilon$. Then $\phi$ is onto.*

*Proof of* (2) *in Theorem* 1. Clearly existence of an interpolating compact neighborhood of $x$ implies $\tilde{E}^\Lambda|F_x^\Lambda = C(F_x^\Lambda)$ (resp. $C_R(F_x^\Lambda)$), so we need only prove the reverse implication. Assume $\tilde{E}^\Lambda|F_x^\Lambda$ is uniformly dense in $C(F_x^\Lambda)$ (resp. $C_R(F_x^\Lambda)$). Without loss of generality we may suppose that $Y$ is compact. $\tilde{E}^\Lambda$ separates the points of $F_x^\Lambda$ since $\tilde{E}^\Lambda|F_x^\Lambda$ is uniformly dense in $C(F_x^\Lambda)$, so $E$ is ultraseparating near $x$ by (1). Thus without loss of generality we may suppose that $E$ separates the points of $Y$. Let $\{G_\alpha\}$ be a family of compact neighborhoods of $x$ such that $\{\operatorname{Int} G_\alpha\}$ is an open base for $x$. Without loss of generality we may assume that the cardinalities of $\{G_\alpha\}$ and $\Lambda$ coincide. First we show that there are a compact neighborhood $G_\beta$ in $\{G_\alpha\}$ and a natural number $n_1$ such that for every $f$ in the unit ball of $C(Y)$ (resp. $C_R(Y)$) there is an $h$ in $E$ with $N_E(h) \leq n_1$ and

$$\|f|G_\beta - h|G_\beta\|_\infty < 1/2.$$

Suppose that it is not true. Then for every compact neighborhood $G_\alpha$ in $\{G_\alpha\}$ and natural number $n$, there is an $f_{\alpha,n}$ in the unit ball of $C(Y)$ which satisfies the condition that $\|f_{\alpha,n}|G_\alpha - h|G_\alpha\|_\infty < 1/2$ for $h \in E$

implies $N_E(h) > n$. Let $\Phi$ be a homeomorphism from $\Lambda$ onto $\Lambda \times N$. Let $\tilde{f}$ be a $C(Y)$-valued function in $C(\tilde{Y}^\Lambda) = (C(Y))^{\sim\Lambda}$ such that

$$\tilde{f}(\gamma) = f_{\Phi(\gamma)}$$

for every $\gamma$ in $\Lambda$. Since $\tilde{E}^\Lambda | F_x^\Lambda$ is uniformly dense in $C(F_x^\Lambda)$, we see that

$$\|\tilde{f}|F_x^\Lambda - \tilde{g}|F_x^\Lambda\|_\infty < 1/4$$

for some $\tilde{g}$ in $\tilde{E}^\Lambda$. Thus we see that

$$U = \{\tilde{x} \in \tilde{Y}^\Lambda : |\tilde{f}(\tilde{x}) - \tilde{g}(\tilde{x})| < 1/3\}$$

is an open neighborhood of $F_x^\Lambda$. So there is a $G_\beta$ in $\{G_\alpha\}$ such that $U \supset [G_\beta \times \Lambda]$. Thus we have

$$\|\tilde{f}(\Phi^{-1}(\beta, n))|G_\beta - \tilde{g}(\Phi^{-1}(\beta, n))|G_\beta\|_\infty < 1/2,$$

so $N_E(\tilde{g}(\Phi^{-1}(\beta, n))) > n$, which is a contradiction since $\tilde{g} \in \tilde{E}^\Lambda$. Let $T$ be the linear transformation of $E|G_\beta$ into $C(G_\beta)$ (resp. $C_R(G_\beta)$) defined by

$$Tf = f$$

for $f$ in $E|G_\beta$. Then $T$ is bounded since the inequality

$$\|f\|_\infty \le \|f\|_{E|G_\beta}$$

holds for every $f$ in $E|G_\beta$. By the above argument the hypotheses of Lemma 8 hold with $\varepsilon = 1/2$ and $M_0 = n_1$. Thus we see that

$$E|G_\beta = C(G_\beta) \quad \text{(resp. } C_R(G_\beta)).$$

PROPOSITION 2. *Let $E$ be a (resp. real) Banach space included in $C_0(Y)$ (resp. $C_{0,R}(Y)$) for a locally compact Hausdorff space $Y$ and $x$ be a point in $Y$. Let $\Lambda$ be a discrete space. Suppose that $E$ is ultraseparating near $x$. Then we have that*

$$[\{x\} \times \Lambda] = Q^\Lambda(E_x).$$

*Proof*. Since $E$ is ultraseparating near $x$, $\tilde{E}^\Lambda$ separates the points of $\{x\} \times \Lambda$, so there is a $g$ in $E$ such that $g(x) = 1$. Suppose that $\tilde{f}$ is a $E$-valued function in $\tilde{E}^\Lambda$. We see that

$$\tilde{f} - \langle(\tilde{f}(\alpha))(x)g\rangle$$

is in $\tilde{E}_x^\Lambda$, where $\langle(\tilde{f}(\alpha))(x)g\rangle$ is an $E$-valued function such that $\langle(\tilde{f}(\alpha))(x)g\rangle(\gamma) = (\tilde{f}(\gamma))(x)g$ for every $\gamma$ in $\Lambda$. That does not prove Proposition 2 but the rest of the proof is the same as the proof of Lemma 4 in [7].

**3. Results of range transformations.** In this section we prove the main results.

THEOREM 2. *Let $A$ be a uniformly closed subalgebra of $C_0(Y)$ for a locally compact Hausdorff space $Y$ and $I$ be an ideal of $A$. Let $D$ be a plane domain containing the origin. Suppose that* $\mathrm{Op}(I_D, \mathrm{Re}\, A)$ *contains a function which is not harmonic on any neighborhood of the origin. Then $I|K$ is uniformly closed and self-adjoint for every compact subset $K$ of $Y - \mathrm{Ker}\, I$ and $\mathrm{cl}\, I$ is self-adjoint.*

*Proof*. Let $h$ be a function in $\mathrm{Op}(I_D, \mathrm{Re}\, A)$ which is not harmonic on any neighborhood of the origin. If $Y$ is not compact, then $\overline{Y}$ denotes the one point compactification of $Y$ and $\infty$ denotes the point in $\overline{Y} - Y$. If $Y$ is compact, then we add $\infty$ as an isolated point and $\overline{Y}$ denotes $Y \cup \{\infty\}$. We may suppose that $A$ is a closed subalgebra of $C(\overline{Y})$ such that $f(\infty) = 0$ for every $f$ in $A$. Let $\overline{Y}_1$ be the quotient space obtained by identifying the points in $\overline{Y}$ which cannot be separated by $A$. Let $\overline{Y}_0$ be the quotient space obtained by identifying the points in $\overline{Y}$ which cannot be separated by $I$. Let $p$ be the point in $\overline{Y}_0$ which corresponds to the equivalence class in $\overline{Y}$ containing $\infty$. We may suppose that $\overline{Y}_0$ is the quotient space obtained by identifying points in $\overline{Y}_1$ which cannot be separated by $I$ and that $p$ corresponds to $\mathrm{Ker}\, I$. We may also suppose that each point in $\overline{Y}_0 - \{p\}$ corresponds to a point in $\overline{Y}_1 - \mathrm{Ker}\, I$, that is, we may suppose that $\overline{Y}_0 - \{p\} = \overline{Y}_1 - \mathrm{Ker}\, I$. Let $I' = \mathrm{cl}\, I + C$ be the sum of the uniform closure of $I$ and the space of constant functions $C$. Then $I'$ is a function algebra on $\overline{Y}_0$. Let $\mathrm{Ch}(I')$ be the Choquet boundary for $I'$. We consider two cases: (1) There is no accumulation point of $\mathrm{Ch}(I')$ or $p$ is the only accumulation point of $\mathrm{Ch}(I')$ in $\overline{Y}_0$. (2) There is an accumulation point of $\mathrm{Ch}(I')$ which is not $p$.

*Case* (1). Let $S$ denote the closure of $\mathrm{Ch}(I')$ in $\overline{Y}_0$, that is, $S$ denotes the Shilov boundary for $I'$. By the condition every point in $S - \{p\}$ is isolated. Thus $I'|S = C(S)$, so we have $I' = C(\overline{Y}_0)$ since $S$ is the Shilov boundary for $I'$. It follows that $\mathrm{cl}\, I$ is self-adjoint. Since $A$ is uniformly closed and $I$ is an ideal of $A$ we have

$$I \cdot C(\overline{Y}_0) \subset I.$$

Thus we conclude that $I|K$ is uniformly closed and self-adjoint for every compact subset $K$ of $Y - \mathrm{Ker}\, I$.

*Case* (2). First we show that $h$ is continuous near the origin. Suppose that $h$ is not continuous on any neighborhood of the origin. There exists a positive number $\delta$ such that $\{z\colon |z| < \delta\} \subset D$. Take $q$ to be an accumulation point of $\mathrm{Ch}(I')$ other than $p$. There is a function $k$ in $I$ such that

$$k(q) = d > 0, \qquad \|k\|_\infty \leq 1$$

for a positive number $d$ since $p \neq q$. There is a point of discontinuity $z_0$ of $h$ with $|z_0| < \delta/2$ such that there is a function $g$ in $I$ such that

$$g(q) = z_0, \qquad \|g\|_\infty < \delta/2$$

since $p \neq q$ and we suppose that $h$ is not continuous near the origin. Since $q$ is an accumulation point of $\mathrm{Ch}(I')$ we can choose a sequence $\{y_n\}$ of $\mathrm{Ch}(I')$ which satisfies the condition that $p$ is not contained in the closure of $\{y_n\}$ and

$$|g(y_n) - g(q)| < 1/n$$

and

$$|k(y_n) - k(q)| < 1/n$$

for every positive integer $n$ and the $y_n$ have disjoint neighborhoods $V_n$ for every positive integer $n$. Let $q_0$ be an accumulation point of $\{y_n\}$. So we have $q_0 \notin \{y_n\}$ since $V_n \cap V_k = \varnothing$ if $n \neq k$. Then we have $g(q) = g(q_0)$ and $k(q) = k(q_0)$ so

$$|g(y_n) - g(q_0)| < 1/n, \qquad |k(y_n) - k(q_0)| < 1/n$$

for every positive integer $n$. Now we need Lemma 9.

LEMMA 9. *There are a positive number $M$ and a subsequence $\{y_{m(n)}\}$ of $\{y_n\}$ such that for every convergent sequence $\{\alpha_n\}$ of complex numbers with limit* 0 *there is a function $f$ in* $\mathrm{cl}\, I$ *such that*

$$f(y_{m(n)}) = \alpha_n, \qquad \|f\|_\infty \leq M \cdot \sup_n |\alpha_n|.$$

*Proof*. Since $\{y_n\}$ is a sequence in $\mathrm{Ch}(I')$ there is a function $f_n$ in $I$ for every positive integer $n$ with the property

$$f_n(y_n) = 1, \quad f_n(q_0) = 0, \quad \|f_n\|_\infty \leq 2$$
$$|f_n(y)| < 1/2^{n+1} \quad \text{for } \forall y \in V_n^c$$

since each $y_n$ is a point in the Choquet boundary, where $V_n^c$ is the complement of $V_n$ in $\overline{Y}_0$. Let $\{g_n\}$ be the countable set of all polynomials of $\{f_n\}$ with rational coefficients and vanishing constant term. For integers $m$ and $j$ put

$$K_{m,j} = \{x \in \overline{Y}_0 \colon |g_j(q_0) - g_j(x)| < 1/m\},$$
$$K_m = \bigcap_{j=1}^{m} K_{m,j}.$$

Choose a subsequence $\{y_{m(n)}\}$ of $\{y_n\}$ such that

$$y_{m(k)} \in K_k \cap \{y_n\}$$

for every positive integer $k$. Let $q_0'$ be an accumulation point of $\{y_{m(n)}\}$. Let $I_1$ be the uniform closure of $\{g_n\}$. Then $I_1$ is a closed subalgebra of $\mathrm{cl}\, I$ and

$$\lim_{n\to\infty} \mathfrak{g}(y_{m(n)}) = 0 = \mathfrak{g}(q_0')$$

for every $\mathfrak{g}$ in $I_1$. Let $J$ be a bounded linear transformation of $I_1$ into $c_0$, where $c_0$ denotes the Banach space of all convergent sequences of complex numbers with limit 0, such that

$$J(\mathfrak{g}) = \{\mathfrak{g}(y_{m(n)})\}_{n=1}^{\infty}.$$

We show that $J$ is onto. Let $\{\alpha_n\} \in c_0$ with $\sup_n |\alpha_n| \le 1$. Then

$$f = \sum_{n=1}^{\infty} \alpha_n f_{m(n)}$$

is in $I_1$ and $\|f\|_\infty \le 4$ and

$$|f(y_{m(n)}) - \alpha_n| \le 1/2.$$

Thus we see that $J$ is onto by Lemma 8. It follows by the open mapping theorem that Lemma 9 holds.

Sequel of proof of Theorem 2. Since $z_0$ is a point of discontinuity for $h$, there is an $\varepsilon_0 > 0$ and a sequence $\{z_n\}$ in $\{z\colon |z| < \delta\}$ such that $z_n \to z_0$ and

$$|h(z_0) - h(z_n)| > \varepsilon_0$$

for every positive integer $n$. Without loss of generality we may assume

$$\sup_n |z_n - z_0| < d\delta/(18M).$$

Let $\{q_n\}$ be a subsequence of $\{y_{m(n)}\}$ such that

$$\{q_n\} = \{y_{m(n)}\} \cap \{x \in \overline{Y}_0\colon |g(x) - z_0| < d\delta/(18M),\ |k(x) - d| < d/3\}.$$

Let $q_0''$ be an accumulation point of $\{q_n\}$. Then we have $g(q_0'') = g(q_0') = g(q_0)$. Let $\alpha_n = (z_n - g(q_n))/k(q_n)$. Then $|\alpha_n| \le \delta/(6M)$ and $\alpha_n \to 0$ as $n \to \infty$. So by Lemma 9 there is an $f$ in $\mathrm{cl}\, I$ with

$$f(q_n) = \alpha_n, \qquad \|f\|_\infty \le M\delta/(6M) = \delta/6.$$

We have $fk + g \in I$ since $\mathrm{cl}\, I \subset A$ and $I$ is an ideal of $A$. We also have

$$\|fk + g\|_\infty \leq 2\delta/3, \quad (fk + g)(q_n) = z_n, \quad (fk + g)(q_0'') = z_0$$

since $q_0''$ is an accumulation point of $\{q_n\}$. While

$$h \circ (fk + g) \in \mathrm{Re}\, A$$

since range of $fk + g$ is contained in $D$, we also have

$$h \circ (fk + g)(q_n) = h(z_n),$$
$$h \circ (fk + g)(q_0'') = h(z_0),$$

which is a contradiction since

$$|h(z_n) - h(z_0)| > \varepsilon_0$$

for every positive integer $n$, while $q_0''$ is an accumulation point of $\{q_n\}$. Thus we conclude that $h$ is continuous near the origin.

Now we need Lemma 10.

LEMMA 10. *Let $B$ be a uniformly closed subalgebra of $C_0(Y)$ for a locally compact Hausdorff space $Y$ which separates the points of $Y$. Let $D$ be a plane domain containing the origin. Suppose that $x$ is a point in $Y$ such that $B_x \neq B$. Suppose also that there is a function $f$ in $C_0(Y)$ with $f(x) \neq 0$ which satisfies that*

$$f \cdot B \subset B,$$

*where $f \cdot B = \{fg\colon g \in B\}$. If there is a function $h$ in $\mathrm{Op}((f \cdot B)_D, \mathrm{Re}\, B)$ which is continuous near the origin but is not harmonic on any neighborhood of the origin, then there is a compact neighborhood $G$ of $x$ with*

$$B|G = C(G).$$

Before we prove Lemma 10 we show the rest of the proof of Theorem 2 by using Lemma 10. By the definition of $\overline{Y}_1$ we may suppose that $A$ is a uniformly closed subalgebra of $C(\overline{Y}_1)$ which separates the points of $\overline{Y}_1$. Let $x$ be a point in $\overline{Y}_1 - \mathrm{Ker}\, I$. Then we have that $A \neq A_x$ and that there is a function $f$ in $I$ such that $f(x) \neq 0$. Since $I$ is an ideal of $A$, $f \cdot A$ is contained in $I$, so we have

$$\mathrm{Op}(I_D, \mathrm{Re}\, A) \subset \mathrm{Op}((f \cdot A)_D, \mathrm{Re}\, A).$$

Thus $h$ is a function in $\mathrm{Op}((f \cdot A)_D, \mathrm{Re}\, A)$ which is continuous near the origin but is not harmonic on any neighborhood of the origin. It follows by Lemma 10 that there is a compact neighborhood $G$ in $\overline{Y}_1$ of $x$ such that

$$A|G = C(G).$$

Without loss of generality we may suppose that $G \subset \overline{Y}_1 - \operatorname{Ker} I$, so we have

$$I|G = C(G)$$

since $I$ is an ideal of $A$. The same conclusion holds for every point in $\overline{Y}_1 - \operatorname{Ker} I$. Since we may suppose that $\overline{Y}_1 - \operatorname{Ker} I = \overline{Y}_0 - \{p\}$ we see that

$$I' = C(\overline{Y}_0)$$

by Corollary 2.13 in [3]. We conclude that $\operatorname{cl} I$ is selfadjoint. Since $A$ is uniformly closed and $I$ is an ideal of $A$. We see that $I \cdot C(\overline{Y}_0) \subset I$. Thus we conclude that $I|K = C(K)$ for every compact subset $K$ of $\overline{Y}_0 - \{p\}$, in short, $I|K$ is uniformly closed and self-adjoint for every compact subset $K$ of $Y - \operatorname{Ker} I$.

*Proof of Lemma* 10. Without loss of generality we may assume that $h$ is continuous on $\{z\colon |z| \le 1\}$ since $f \cdot B$ is closed under constant multiplication. We may also suppose that $\|f\|_\infty = 1$. We denote $f \cdot B_x = \{fg\colon g \in B_x\}$ by $\mathfrak{B}$. We see that $\mathfrak{B}$ is a Banach space with respect to the norm defined by

$$\|u\|_{\mathfrak{B}} = \inf\{\|g\|_\infty \colon g \in B_x,\ u = fg\}$$

for $u$ in $\mathfrak{B}$. It is trivial that $\|u\|_\infty \le \|u\|_{\mathfrak{B}}$ for every $u$ in $\mathfrak{B}$. Now we need Lemma 11, which can be proven in the same way as the proof of Lemma 1.2 in [7].

LEMMA 11. *Let* $\mathfrak{B}_1 = \{u \in \mathfrak{B}\colon \|u\|_{\mathfrak{B}} \le 1/2\}$. *Then there are a positive integer* $n_0$ *and a real number* $\varepsilon$ *with* $0 < \varepsilon < 1/2$ *and a function* $\psi$ *in* $\mathfrak{B}_1$ *such that*

$$\{g \in \mathfrak{B}\colon \|g - \psi\|_{\mathfrak{B}} < \varepsilon\} \subset \mathfrak{B}_1$$

*and there is a dense subset* $U$ *in* $\{g \in \mathfrak{B}\colon \|g - \psi\|_{\mathfrak{B}} < \varepsilon\}$ *with* $\psi$ *in* $U$ *which satisfies the following:*

*For every* $g$ *in* $U$ *we have*

$$h \circ g \in \operatorname{Re} B \quad \textit{and} \quad \|h \circ g\|_{\operatorname{Re} B} < n_0.$$

*Sequel of the proof of Lemma* 10. First we show that $B$ is ultraseparating near $x$. Let $\sigma_\eta(\cdot)$ be a smoothing operator of class $C^\infty$ supported in $\{z\colon |z| < \eta\}$ for a small $\eta > 0$. Put

$$h_\eta(z_1, z_2) = \iint h(z_1 - z_2 w)\sigma_\eta(w)\,dx\,dy \qquad (w = x + iy)$$

and

$$L_\eta(z_1, z_2, \alpha) = |\alpha|^2 \Delta_1(h_\eta(z_1, z_2)|z_2|^4),$$

where $\Delta_1$ is the Laplacian with respect to $x_1 = \operatorname{Re} z_1$ and $y_1 = \operatorname{Im} z_1$. By Lemma 5 in [7] we see that

$$L_\eta(fg_2, fg_3, g_1) \in \operatorname{cl}\operatorname{Re} B$$

for every $g_1$, $g_2$ and $g_3$ in $B$ with $\|g_i\|_\infty < 1/2$ for $i = 2$ and 3 and a small $\eta > 0$. Thus we see that

$$C_{0,R}(Y)\Delta_1(h_\eta(fg_2, fg_3)|fg_3|^4) \in \operatorname{cl}\operatorname{Re} B$$

by the Stone-Weierstrass theorem. Since $h$ is not harmonic near the origin we see that

$$|L_\eta(z, w, 1)| \geq (1/2)|L_\eta(0, z_2, 1)| \neq 0$$

on $\{(z, w) \in C^2\colon |z| \leq \varepsilon',\ |w - z_2| \leq \varepsilon'\}$ for a small $\eta > 0$ and a smoothing operator $\sigma_\eta$ and an $\varepsilon' > 0$ and a $z_2$ with sufficiently small non-zero absolute value. Choose $g_2$ and $g_3$ in $B$ with $\|g_i\|_\infty < 1/2$ for $i = 2$ and 3 such that

$$g_2(x) = 0, \qquad fg_3(x) = z_2.$$

Let

$$G' = \{y \in Y\colon |f(y)| \geq |f(x)|/2, |fg_2(y)| \leq \varepsilon',\ |fg_3(y) - fg_3(x)| \leq \varepsilon'\}.$$

So $G'$ is a compact neighborhood of $x$ with

$$L_\eta(fg_2(y), fg_3(y), 1) \neq 0$$

for every $y$ in $G'$. We show that $B|G'$ is ultraseparating. Let $Y_1$ and $Y_2$ be compact subsets of $G'$. By the definition of $G'$ there is a function $u$ in $\operatorname{cl}\operatorname{Re} B$ such that

$$\|u\|_\infty \leq 2,$$
$$u(y) > 1 \quad \text{for } \forall y \in Y_1,$$
$$u(y) < -1 \quad \text{for } \forall y \in Y_2$$

since $C_{0,R}(Y)\cdot L_\eta(fg_2, fg_3, 1) \subset \operatorname{cl}\operatorname{Re} B$ and since $L_\eta(fg_2(y), fg_3(y), 1) \neq 0$ for $\forall y \in G'$. We see that there are functions $u'$ and $v$ in $\operatorname{Re} B$ with

$$\|u'\|_\infty \leq 3,$$

$$u'(y) > 1/2 \quad \text{for } \forall y \in Y_1,$$
$$u'(y) < -1/2 \quad \text{for } \forall y \in Y_2,$$

and $u' + iv \in B$. Then we have $\exp(u' + iv) \in B$ since $B$ is uniformly closed and we have

$$\|\exp(u' + iv)\|_\infty \leq \exp 3,$$
$$|\exp(u' + iv)(y)| > \exp(1/2) \quad \text{for } \forall y \in Y_1,$$
$$|\exp(u' + iv)(y)| < \exp(-1/2) \quad \text{for } \forall y \in Y_2.$$

Let $a$ and $b$ be different points in $\tilde{G}'$ and $U_a$ and $U_b$ be disjoint compact neighborhoods of $a$ and $b$ respectively. Put $U_a^k = U_a \cap (G' \times \{k\})$ and $U_b^k = U_b \cap (G' \times \{k\})$ for every positive integer $k$. Then we see that $U_a^k \cap U_b^k = \varnothing$ for every $k$ and $a \in \bigcup_{n=1}^\infty U_a^n$, $b \in \bigcup_{n=1}^\infty U_b^n$. Let $t$ be the map

$$t \colon \tilde{G}' \to G'$$

which satisfies $\langle g \rangle(p) = g(t(p))$ for every $f$ in $C(G')$ and for every $p$ in $\tilde{G}'$, since $t(U_a^k)$ and $t(U_b^k)$ are disjoint compact subsets of $Y$ for every $k$. For every positive integer $k$ choose a function $f_k$ in $B$ such that

$$\|f_k\|_\infty \leq \exp 3,$$
$$|f_k(y)| > \exp(1/2) \quad \text{for } \forall y \in t(U_a^k),$$
$$|f_k(y)| < \exp(-1/2) \quad \text{for } \forall y \in t(U_b^k).$$

It follows that $\tilde{f}$ separates $a$ and $b$, where $\tilde{f}$ is a function in $(B|G')^\sim$ such that $\tilde{f}(n) = f_n|G'$ for every $n$. Thus we conclude that $B|G'$ is ultraseparating on $G'$. Let $f \cdot B|G' = \{fg|G' \colon g \in B\}$. Then $f \cdot B|G'$ is a Banach space included in $C(G')$ with the norm defined by

$$\|u\|_{f \cdot B|G'} = \inf\{\|g\|_\infty \colon g \in B,\ fg|G' = u\}$$

for $u \in f \cdot B|G'$. Since $f$ never equals zero on $G'$, $(f \cdot B|G')^\sim|F_y = (B|G')^\sim|F_y$ for every $y$ in $G'$ by Lemma 4, so $f \cdot B|G'$ is ultraseparating by (3) of Lemma 6.

Let $\Lambda$ be a discrete space whose cardinality coincides with that of an open base for $x$. We will show that

$$\mathrm{cl}(\tilde{B}^\Lambda | F_x^\Lambda) = C(F_x^\Lambda).$$

Let ${}_0F_x^\Lambda$ be the quotient space of $F_x^\Lambda$ by $\tilde{B}_x^\Lambda$, that is, the space defined by identifying the points of $F_x^\Lambda$ which cannot be separated by $\tilde{B}_x^\Lambda$. Since $B$ is ultraseparating near $x$ we see that $Q^\Lambda(B_x) = [\{x\} \times \Lambda]$ by Proposition 2 and $Q^\Lambda(B_x)$ is the only subset of $F_x^\Lambda$ with more than one point which corresponds to a point in ${}_0F_x^\Lambda$. Let $\tilde{q}$ be the point in ${}_0F_x^\Lambda$ which corresponds to $Q^\Lambda(B_x)$. Let $B'$ be the function algebra on ${}_0F_x^\Lambda$ generated by $\tilde{B}_x^\Lambda|{}_0F_x^\Lambda$ and the constant functions. Let $\tilde{x}$ be a point in ${}_0F_x^\Lambda - \{\tilde{q}\}$. There is an $\tilde{f}$ in $\tilde{B}_x^\Lambda$ with

$$\langle f\rangle \tilde{f}(\tilde{x}) = s \neq 0,$$

where $s$ is a complex number with small absolute value. Without loss of generality we may suppose that

$$\Delta_1(h_\eta(0, s)) \neq 0,$$

where $\eta$ is a small positive number such that $\eta < \varepsilon/(2\|\tilde{f}\|_\infty)$ ($\varepsilon$ is the constant in Lemma 11) and

$$h_\eta(z_1, z_2) = \iint h(z_1 - z_2 w)\sigma_\eta(w)\,dx\,dy \qquad (w = x + iy)$$

for some smoothing operator $\sigma_\eta(\cdot)$ of class $C^\infty$ supported in $\{z\colon |z| < \eta\}$. We can choose an $\varepsilon' > 0$ such that

$$|\Delta_1(h_\eta(z, w))| \geq (1/2)|\Delta_1(h_\eta(0, s))|$$

on

$$\{(z, w) \in C^2\colon |z| \leq \varepsilon',\ |w - s| \leq \varepsilon'\}.$$

Put

$$Y' = \{\tilde{y} \in {}_0F_x^\Lambda\colon |\langle f\rangle\tilde{f}(\tilde{y}) - \langle f\rangle\tilde{f}(\tilde{x})| \leq \min\{\varepsilon'/2, |s|/2\}\}.$$

Then $Y'$ is a compact neighborhood of $\tilde{x}$ in ${}_0F_x^\Lambda$ with $\tilde{q} \notin Y'$, so we may suppose that $Y'$ is a compact subset of $F_x^\Lambda$. Let $\tilde{\tilde{g}}$ be a function in $(\tilde{B}^\Lambda)^\sim$. For a complex number $\beta$ with sufficiently small absolute value and a complex number $w$ with $|w| \leq \eta$ we have that

$$(\tilde{\tilde{g}}(n))(\alpha)(f\tilde{f}(\alpha))^2\beta - f\tilde{f}(\alpha)w$$

is in $\mathfrak{B}$ for every positive integer $n$ and every $\alpha$ in $\Lambda$ and

$$\|(\tilde{\tilde{g}}(n))(\alpha)(f\tilde{f}(\alpha))^2\beta - f\tilde{f}(\alpha)w\|_{\mathfrak{B}} < \varepsilon.$$

So for every small positive $\varepsilon''$, positive integer $n$ and $\alpha$ in $\Lambda$ there is a function $g_{\varepsilon'',\alpha,n}$ in $U$ which satisfies the condition that

$$\|\psi + (\tilde{\tilde{g}}(n))(\alpha)(f\tilde{f}(\alpha))^2\beta - f\tilde{f}(\alpha)w - g_{\varepsilon'',\alpha,n}\|_{\mathfrak{B}} < \varepsilon'',$$

where $\psi$ is the function in Lemma 11. We see that

$$h \circ g_{\varepsilon'',\alpha,n} \in \operatorname{Re} B$$

and

$$\|h \circ g_{\varepsilon'',\alpha,n}\|_{\operatorname{Re} B} < n_0.$$

Thus we see that

$$h \circ \tilde{\tilde{g}}_{\varepsilon''} \in \operatorname{Re}(\tilde{B}^{\Lambda})^{\sim},$$

where $\tilde{\tilde{g}}_{\varepsilon''}$ is a function in $(\tilde{B}^{\Lambda})^{\sim}$ such that $(\tilde{\tilde{g}}_{\varepsilon''}(n))(\alpha) = g_{\varepsilon'',\alpha,n}$ for every $n$ and $\alpha$. Since the inequality $\|u\|_{\infty} \leq \|u\|_B$ holds for every $u$ in $B$ and since $h$ is continuous we see that

$$h(\langle\langle\psi\rangle\rangle + \tilde{\tilde{g}}\langle\langle f\rangle\tilde{f}\rangle^2\beta - \langle\langle f\rangle\tilde{f}\rangle w)$$

is in $\operatorname{cl}(\operatorname{Re}(\tilde{B}^{\Lambda})^{\sim})$, where $\langle\psi\rangle$ (resp. $\langle f\rangle$) is the constant function in $\tilde{B}^{\Lambda}$ with constant value $\psi$ (resp. $f$) and $\langle\langle\psi\rangle\rangle$ (resp. $\langle\langle f\rangle\tilde{f}\rangle$) is the constant function in $(\tilde{B}^{\Lambda})^{\sim}$ with constant value $\langle\psi\rangle$ (resp. $\langle f\rangle\tilde{f}$). Thus we have that

$$h_{\eta}(\langle\langle\psi\rangle\rangle + \tilde{\tilde{g}}\langle\langle f\rangle\tilde{f}\rangle^2\beta, \langle\langle f\rangle\tilde{f}\rangle)$$

is in $\operatorname{cl}(\operatorname{Re}(\tilde{B}^{\Lambda})^{\sim})$ for a complex number $\beta$ with sufficiently small absolute value. It follows by Lemma 5 in [7] that

$$L_{\eta}(\langle\langle\psi\rangle\rangle, \langle\langle f\rangle\tilde{f}\rangle, \tilde{\tilde{g}}) = |\tilde{\tilde{g}}|^2 L_{\eta}(\langle\langle\psi\rangle\rangle, \langle\langle f\rangle\tilde{f}\rangle, 1)$$

is in $\operatorname{cl}(\operatorname{Re}(\tilde{B}^{\Lambda})^{\sim})$. Since $\langle\psi\rangle = 0$ and $\langle f\rangle = f(x)$ on $F_x^{\Lambda}$ we see that

$$|\tilde{\tilde{g}}|^2 L_{\eta}(0, f(x)\langle\tilde{f}\rangle, 1)|\tilde{Y}'$$

is in $\operatorname{cl}(\operatorname{Re}(\tilde{B}^{\Lambda})^{\sim})|\tilde{Y}'$. Since $\tilde{f}$ is in $\tilde{B}_x^{\Lambda}$ we see that

$$L_{\eta}(0, f(x)\langle\tilde{f}\rangle, 1) = 0$$

on $\{x\} \times \Lambda \times N$ by Lemma 5 in [7]. Thus we conclude that

$$|\tilde{\tilde{g}}|^2 L_{\eta}(0, f(x)\langle\tilde{f}\rangle, 1)|\tilde{Y}'$$

is in $\operatorname{cl}(\operatorname{Re}(\tilde{B}_x^{\Lambda})^{\sim})|\tilde{Y}'$. Since $B$ is ultraseparating near $x$, $(\tilde{B}^{\Lambda})^{\sim}$ separates the points in $[F_x^{\Lambda} \times N]$, in particular, in $\tilde{Y}'$ by Corollary 1. By the definition of $Y'$ we see that $L_{\eta}(0, f(x)\langle\tilde{f}\rangle, 1)$ never equals zero on $\tilde{Y}'$. It follows by the Stone-Weierstrass theorem that

$$C_R(\tilde{Y}') \subset \operatorname{cl}(\operatorname{Re}(\tilde{B}_x^{\Lambda}|Y')^{\sim})$$

since $(\operatorname{cl}(\operatorname{Re}(\tilde{B}_x^{\Lambda})^{\sim}))|\tilde{Y}' \subset \operatorname{cl}(\operatorname{Re}(\tilde{B}_x^{\Lambda}|Y')^{\sim})$, so by Bernard's lemma we have

$$C_R(Y') = \operatorname{Re}(\tilde{B}_x^{\Lambda}|Y')$$

so

$$C(Y') = \tilde{B}_x^{\Lambda}|Y'$$

by a theorem of Hoffman-Wermer-Bernard [**2**, **8**]. We conclude that

$$B' = C({}_0F_x^{\Lambda})$$

by Corollary 2.13 in [**3**]. It follows that

$$\mathrm{cl}(\tilde{B}^{\Lambda}|F_x^{\Lambda}) = C(F_x^{\Lambda}).$$

We conclude by (2) of Theorem 1 that there is a compact neighborhood $G$ of $x$ such that $G' \supset G$ and

$$B|G = C(G).$$

REMARK 1. Let $A$ be a function algebra on a compact Hausdorff space $X$ which contains an infinite number of points and $B$ be a Banach function algebra on $X$. Then every function in $\mathrm{Op}(A_D, \mathrm{Re}\, B)$ for a plane domain $D$ is continuous on $D$ (cf. Remark 2 in [**7**]). This is not the case for a point separating closed subalgebra of $C(X)$ which does not contain the constant functions. Let $X = \{0, 1, 1/2, 1/3, \dots\}$. Let $A = \{f \in C(X) \colon f(0) = 0\}$ and $D = \{z \colon |z| < 1\}$. Take any sequence $\{\lambda_n\}$ in $D$ with $\lambda_n \neq 0$ but $\lambda_n \to 0$ and let

$$h(z) = \begin{cases} z & \text{if } z \in \{\lambda_n\}, \\ 0 & \text{if } z \notin \{\lambda_n\}. \end{cases}$$

Then we see that discontinuous function $h$ is in $\mathrm{Op}(A_D, \mathrm{Re}\, A)$. (This example was corrected by the referee.)

REMARK 2. The condition that $I$ is an ideal is necessary in Theorem 2, that is, if $I$ is merely a subalgebra of $A$ or even if $I$ is a closed subalgebra of $A$, there may be a continuous function $h$ in $\mathrm{Op}(I_D, \mathrm{Re}\, A)$ which is not harmonic near the origin (cf. Remark 1 in [**7**]).

COROLLARY 2. *Let $A$ be a function algebra on a compact Hausdorff space $X$ and $I$ be an ideal of $A$. Let $D$ be a plane domain. Suppose that* $\mathrm{cl}\, I$ *is not self-adjoint. Then every function in* $\mathrm{Op}((I + C)_D, \mathrm{Re}\, A)$ *is harmonic on $D$.*

COROLLARY 3 (*cf.* [**13**]). *Let $A$ be a uniformly closed subalgebra of $C_0(Y)$. Suppose that $I$ is an ideal of $A$ or the sum of an ideal of $A$ and the space of the constant functions. If*

$$\mathrm{Re}\, I \cdot \mathrm{Re}\, I \subset \mathrm{Re}\, A,$$

*then* $\mathrm{cl}\, I$ *is self-adjoint.*

*Proof*. If $\operatorname{Re} I \cdot \operatorname{Re} I \subset \operatorname{Re} A$, then we see that

$$z \mapsto (\operatorname{Re} z)^2$$

is in $\operatorname{Op}(I_C, \operatorname{Re} A)$, but is not harmonic.

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