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ON THE HARDY SPACE H^1 ON PRODUCTS OF HALF-SPACES

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We show that the Hardy space $H^1_{\text{anal}}(\mathbf{R}^2_+ \times \mathbf{R}^2_+)$ can be identified with the class of functions f such that f and all its double and partial Hilbert transforms $H_k f$ belong to $L^1(\mathbf{R}^2)$. A basic tool used in the proof is the bisubharmonicity of $|F|^q$, where F is a vector field that satisfies a generalized conjugate system of Cauchy-Riemann type.

Introduction. The interest of a theory for the H^p spaces on products of half-spaces was first raised by C. Fefferman and E. M. Stein in the now classic paper "H^p spaces of several variables" [**6**]. Afterward several authors have contributed on this subject. It is worth mentioning the survey paper by C. Y. A. Chang and R. Fefferman [**4**], and the references quoted there. In particular, the H^1 spaces on products of half-spaces was studied by H. Sato [**8**] giving definitions via maximal functions and via the multiple Hilbert transform. On the other hand Merryfield [**7**] proves the equivalence of the definitions given via the area integrals and via the multiple Hilbert transforms. More recently S. Sato [**9**] proved the equivalence between the Lusin area integral and the nontangential maximal function.

The purpose of this paper is to derive directly the equivalence of the definitions of the H^1 space given via the multiple Hilbert transforms and via an L^1 condition on a biharmonic vector field $F = (u_1, u_2, u_3, u_4)$ which is a solution of a generalized Cauchy-Riemann system introduced by Bordin-Fernandez [**3**]. The main tool we shall use is the bi-subharmonicity of $|F|^q, 0 < q < 1$. But the proof of this fact here is different from the classical one given by Stein-Weiss [**10**]. We rely on ideas of A. P. Calderón, R. Coiffman and G. Weiss (see [**5**]). We shall confine ourselves to the bidimensional case.

This paper is part of the author's doctoral thesis presented to UNICAMP in 1982, and the results are announced in [**1**] and [**2**].

NOTATION. We shall use the following notations throughout:

$$\square = \{k = (k_1, k_2),\ k_j = 0, 1,\ j = 1, 2\}$$

i.e.

$$\square = \{(0,0),(1,0),(0,1),(1,1)\}$$

and

$$\mathbf{R}_+^2 \times \mathbf{R}_+^2 = \{(x,s;y,t); x,y \in \mathbf{R},\ s,t>0\}.$$

1. The Hardy spaces H^1_{anal} and H^1_{Hb}.

1.1. DEFINITION. A generalized conjugate vector field or simply a conjugate vector field is a vector field $F(x,s;y,t) = (u_k(x,s;y,t); k \in \square)$, in $\mathbf{R}_+^2 \times \mathbf{R}_+^2$, such that each u_k is biharmonic and satisfies the generalized Cauchy-Riemann system

$$(1)\qquad \frac{\partial u_k}{\partial x_j} + (-1)^{k_j+1}\frac{\partial u_{k'}}{\partial t_j} = 0,\quad \begin{array}{l} j=1,2, \\ k=(k_1,k_2)\in\square, \\ k' = k+(-1)^{k_1}(1,0) \text{ if } j=1, \\ k' = k+(-1)^{k_2}(0,1) \text{ if } j=2, \end{array}$$

where $x_1 = x, x_2 = y, t_1 = s, t_2 = t$.

The generalized Cauchy-Riemann system was introduced by Bordin-Fernandez **[3]**.

Let $P_r(x)$ and $Q_r(x)$ denote the Poisson and conjugate Poisson kernels in $\mathbf{R}_+^2$, i.e.,

$$P_r(x) = cr/(r^2+x^2) \quad \text{and} \quad Q_r(x) = cx/(r^2+x^2)$$

the vector field $(P_sP_t * f, Q_sP_t * f, P_sQ_t * f, Q_sQ_t * f)$, where $f \in L^p(\mathbf{R}^2)$, $1 \le p < \infty$, is a generalized conjugate vector field.

1.2. DEFINITION. Let $F = (u_k; k \in \square)$ be a conjugate vector field. We say that F belongs to $H^1_{\text{anal}}(\mathbf{R}_+^2 \times \mathbf{R}_+^2)$ if

$$\|F\|_{H^1_{\text{anal}}} = \sup_{s,t>0} \int\int |F(x,s;y,t)|\,dx\,dy < \infty.$$

1.3. DEFINITION. The partial and double Hilbert transforms of $f \in L^1(\mathbf{R}^2)$ are the tempered distributions, H_kf, defined by

$$(1)\qquad \begin{aligned} \mathscr{F}(H_{10}F)(x,y) &= i(\operatorname{sign} x)\hat{f}(x,y), \\ \mathscr{F}(H_{01}f)(x,y) &= i(\operatorname{sign} y)\hat{f}(x,y), \\ \mathscr{F}(H_{11}f)(x,y) &= i(\operatorname{sign} x)i(\operatorname{sign} y)\hat{f}(x,y), \end{aligned}$$

and

$$(H_{00}f)(x,y) = f(x,y)$$

or shortly by

(2) $\mathscr{F}(H_k f)(x,y)$
$$= (i \operatorname{sign} x)^{k_1}(i \operatorname{sign} y)^{k_2}\hat{f}(x,y), \qquad k=(k_1,k_2)\in\Box,$$

where $\mathscr{F}$ denotes the Fourier transformation and $\hat{f}$ the Fourier transform of f.

1.4. Definition. By $H^1_{Hb}(\mathbf{R}\times\mathbf{R})$ we mean all $f\in L^1(\mathbf{R}^2)$ such that $H_k f\in L^1(\mathbf{R}^2)$, for each $k\in\Box$. The norm of $f\in H^1_{Hb}$ is defined by

$$\|f\|_{Hb}=\sum_{k\in\Box}\|H_k f\|_1.$$

2. The subharmonicity of $|F|^q$. The basic fact which enables us to develop the theory of H^p-spaces on the product of half spaces is the existence of a positive $q<1$ such that $|F|^q$ is bisubharmonic. We shall show that every conjugate vector field has this property.

2.1. Definition. Let (A_j), $j=1,\dots,n$, be a family of matrices $d\times m$. We say that (A_j) is an elliptic family provided that for an m-dimensional vector v and an n-tuple $\lambda=(\lambda_1,\lambda_2,\dots,\lambda_n)$ we have

$$\sum_{j=1}^{n}\lambda_j A_j v=0$$

only if either v or λ is zero.

2.2. Lemma (*Calderón*). *Let (A_j), $j=1,\dots,n$, be an elliptic family; v and $u^1,\dots,u^n$ vectors of $\mathbf{R}^m$ and $\lambda=(\lambda_1,\lambda_2,\dots,\lambda_n)$. Suppose that*

$$\sum_{j=1}^{n}A_j u^j=0 \quad \textit{and} \quad \sum_{j=1}^{n}\lambda_j A_j v=0.$$

Then, there exists a positive $\alpha<1$, depending only on $A_1,\dots,A_n$, such that

(1) $$\max\sum_{j=1}^{n}(u^j\cdot v)^2\le\alpha\sum_{j=1}^{n}|u^j|^2.$$

Proof. See **[5]**.

2.3. Proposition. *The generalized Cauchy-Riemann system* 1.1(1) *can be put in the form*

(1) $$A_1\frac{\partial F}{\partial s}+A_2\frac{\partial F}{\partial x}+A_3\frac{\partial F}{\partial t}+A_4\frac{\partial F}{\partial y}=0,$$

where

$$\frac{\partial F}{\partial z} = \left(\frac{\partial u_k}{\partial z}; k \in \square\right), \quad z = s, x, t, y$$

and A_j, $1 \le j \le 4$, *are the* 8×4 *matrices given by*

$$A_1 = \begin{pmatrix} B_1 \\ N \end{pmatrix}, \quad A_2 = \begin{pmatrix} B_2 \\ N \end{pmatrix}, \quad A_3 = \begin{pmatrix} N \\ B_3 \end{pmatrix}, \quad A_4 = \begin{pmatrix} N \\ B_4 \end{pmatrix}$$

where

$$B_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad B_2 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix},$$

$$B_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \end{pmatrix}, \quad B_4 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix},$$

and N *is the* 4×4 *null matrix. Moreover the families* (B_1, B_2) *and* (B_3, B_4) *are elliptic.*

Proof. We will first show that (B_3, B_4) is an elliptic family. The proof that (B_1, B_2) is an elliptic family is exactly the same.

Let $\lambda = (\lambda_3, \lambda_4)$, $v = (v_1, v_2, v_3, v_4)$ denote elements of $\mathbf{R}^2$ and $\mathbf{R}^4$, respectively, such that

$$\sum_{j=3}^{4} \lambda_j B_j v = 0; \tag{2}$$

we will show that $\lambda = 0$ or $v = 0$. Suppose $v \neq 0$ with $v_1 \neq 0$, for example; then we will show that $\lambda = 0$. Indeed, from (2) we have

$$\lambda_3 v_1 + \lambda_4 v_3 = 0 \quad \textit{and} \quad -\lambda_3 v_3 + \lambda_4 v_1 = 0.$$

Since $v_1 \neq 0$, then $\lambda_3 = \lambda_4 = 0$; therefore $\lambda = 0$. In the same way, if $v_j \neq 0$, $j \neq 1$, we have that $\lambda = 0$. This proves the proposition.

2.4. Theorem. *Let* $F = (u_k, k \in \square)$ *be a generalized conjugate vector field. Then, there exists a positive* $q < 1$ *such that* $|F|^q$ *is bisubharmonic.*

Proof. We shall use the following notation:

$$F \cdot G = \sum_{k \in \square} u_k \cdot v_k, \quad \text{where } F = (u_k; k \in \square)$$

$$\text{and } G = (v_k; k \in \square).$$

We shall prove that there exists $0 < q_1 < 1$ such that

$$\Delta_{01}|F|^{q_1} = \frac{\partial^2 F}{\partial y^2} + \frac{\partial^2 F}{\partial t^2} \geq 0. \tag{1}$$

Since $F = (u_k; k \in \square)$ is a conjugate vector field, then

$$B_3 \frac{\partial F}{\partial t} + B_4 \frac{\partial F}{\partial y} = 0, \tag{2}$$

where B_2 and B_4 are defined in Proposition 2.3.

The system (2) is elliptic, by Proposition 2.3. Therefore, by Lemma 2.2, there exists $0 < \alpha_1 < 1$ such that

$$\left(\frac{F}{|F|} \cdot \frac{\partial F}{\partial t}\right)^2 + \left(\frac{F}{|F|} \cdot \frac{\partial F}{\partial y}\right)^2 \leq \alpha_1 \left(\left|\frac{\partial F}{\partial t}\right|^2 + \left|\frac{\partial F}{\partial y}\right|^2\right). \tag{3}$$

Hence, since

$$\Delta_{01}|F|^{q_1} = q_1|F|^{q_1-2}\left\{\left|\frac{\partial F}{\partial t}\right|^2 + \left|\frac{\partial F}{\partial y}\right|^2 + (q_1 - 2)\left[\left(\frac{F}{|F|} \cdot \frac{\partial F}{\partial t}\right)^2 + \left(\frac{F}{|F|} \cdot \frac{\partial F}{\partial y}\right)^2\right]\right\}$$

we have, by (3),

$$\Delta_{01}|F|^{q_1} \geq 0 \tag{4}$$

with $q_1 \geq 2 - 1/\alpha_1$. In this way, for $q_2 \geq 2 - 1/\alpha_2$, with α_2 given as in Lemma 2.2 we have

$$\Delta_{10}|F|^{q_2} \geq 0. \tag{5}$$

Hence, by (4) and (5) we have that there exists $0 < q < 1$ such that $|F|^q$ is bisubharmonic and therefore subharmonic.

3. The equivalence of $H^1_{Hb}(\mathbf{R} \times \mathbf{R})$ and $H^1_{\text{anal}}(\mathbf{R}^2_+ \times \mathbf{R}^2_+)$.

3.1. THEOREM. (i) *If $F = (u_k; k \in \square)$ belongs to $H^1_{\text{anal}}(R^2_+ \times \mathbf{R}^2_+)$, there exists an $f \in L^1(\mathbf{R}^2)$ such that $H_k f \in L^1(\mathbf{R}^2)$ and $u_k = (P_s P_t) * H_k f$ for each $k \in \square$. Moreover, there is a positive constant C, independent of F, such that*

$$\sum_{k \in \square} \|H_k f\|_1 \leq C\|F\|_{H^1_{\text{anal}}}. \tag{1}$$

(ii) *Let* $f \in L^1(\mathbf{R}^2)$. *If* $H_k f \in L^1(\mathbf{R}^2)$, *for each* $k \in \square$, *then the conjugate vector field*

$$F = ((P_s P_t) * H_k f; k \in \square)$$

belongs to $H^1_{\text{anal}}(\mathbf{R}^2_+ \times \mathbf{R}^2_+)$ *and there exists a positive constant* C, *independent of* f, *such that*

$$\|F\|_{H^1_{\text{anal}}} \leq C \sum_{k \in \square} \|H_k f\|_1. \tag{2}$$

Thus, $H^1_{\text{anal}}(\mathbf{R}^2_+ \times \mathbf{R}^2_+)$ can be identified with $H^1_{Hb}(\mathbf{R} \times \mathbf{R})$ with equivalence of norms. In the proof of this theorem we will use the result stated in the next lemma.

3.2. LEMMA. *If* $F = (u_k; k \in \square)$ *belongs to* $H^1_{\text{anal}}(\mathbf{R}^2_+ \times \mathbf{R}^2_+)$, *there exists a positive constant* C, *independent of* F, *such that*

$$\int\int \sup_{s,t>0} |F(x,s;y,t)|\, dx\, dy \leq C \sup_{s,t>0} \int\int |F(x,s;y,t)|\, dx\, dy. \tag{1}$$

Moreover,

$$\lim_{s,t \to 0} F(x,s;y,t) = F(x,y)$$

exists almost everywhere and in $L^1(\mathbf{R}^2)$ *norm.*

Proof. Suppose that each u_k takes values in a fixed finite-dimensional Hilbert space, V_1. We take a conjugate vector field $\varphi = (v_k; k \in \square)$, where each v_k takes its values in V_2 (V_2 is another finite-dimensional Hilbert space and we consider $V = V_1 \oplus V_2$), satisfying:

$$|\varphi(x,s;y,t)|^2 = 2/[x^2 + (1+s)^2]^2[y^2 + (1+t)^2]^2, \tag{2}$$

$$\lim_{|(x,s)| \to \infty, (x,s) \in \overline{\mathbf{R}}^2_+} |v_k(x,s;y,t)| = 0, \quad \text{for each pair } (y,t) \in R^2_+, \tag{3}$$

and

$$\lim_{|(y,t)| \to \infty, (y,t) \in \overline{\mathbf{R}}^2_+} |v_k(x,s;y,t)| = 0, \quad \text{for each pair } (x,s) \in \mathbf{R}^2_+. \tag{4}$$

We define

$$v_{00} = \left(\frac{\partial^2 H}{\partial s^2}\frac{\partial^2 H}{\partial t^2}, \frac{\partial^2 H}{\partial s\,\partial x}\frac{\partial^2 H}{\partial t\,\partial y}\right),$$
$$v_{10} = \left(\frac{\partial^2 H}{\partial s\,\partial x}\frac{\partial^2 H}{\partial t^2}, \frac{\partial^2 H}{\partial x^2}\frac{\partial^2 H}{\partial t\,\partial y}\right),$$
$$v_{01} = \left(\frac{\partial^2 H}{\partial s^2}\frac{\partial^2 H}{\partial t\,\partial y}, \frac{\partial^2 H}{\partial s\,\partial x}\frac{\partial^2 H}{\partial y^2}\right),$$
$$v_{11} = \left(\frac{\partial^2 H}{\partial s\,\partial x}\frac{\partial^2 H}{\partial t\,\partial y}, \frac{\partial^2 H}{\partial x^2}\frac{\partial^2 H}{\partial y^2}\right),$$

where $H\colon \overline{\mathbf{R}}^2_2 \times \overline{\mathbf{R}}^2_+ \to \mathbf{R}$ for

$$H(x,s;y,t) = \frac{1}{2}\log[(x^2+(1+s)^2)^{-1}(y^2+(1+t)^2)^{-1}].$$

(2), (3) and (4) follow easily.

Now, we define for every $\varepsilon > 0$

$$F_\varepsilon(x,s;y,t) = F(x,s+\varepsilon;y,t+\varepsilon) + \varepsilon\varphi(x,s;y,t).$$

We can verify that F_ε is continuous in $(x,s) \in \overline{\mathbf{R}}^2_+$ $((y,t) \in \overline{\mathbf{R}}^2_+)$ for each pair (y,t) $((x,s))$; F_ε tends to zero as $|(x,s)|$ or $|(y,t)|$ tends to ∞, and $|F_\varepsilon| > 0$. Then by Theorem 2.4, there exists a q, $0 < q < 1$, such that $|F_\varepsilon|^q$ is bisubharmonic.

Next, we define $g_\varepsilon(x,y) = |F_\varepsilon(x,0;y,0)|^q$. By (2) and from our assumptions on F, for $p = 1/q$, we have

$$\|g_\varepsilon\|_p^p \le \|F\|_{H^1_{\text{anal}}} + \varepsilon\|\varphi\|_1.$$

Now let $G_\varepsilon(x,s;y,t)$ be the iterated Poisson integral of g_ε. By the properties of F_ε, the properties of the iterated Poisson integral and the maximum principle, we get

$$|F_\varepsilon(x,s;y,t)|^q \le G_\varepsilon(x,s;y,t).$$

Hence, we can select a subsequence g_ε which converges weakly to a function $g \in L^p(\mathbf{R}^2)$ and such that

$$\|g\|_p^p \le \|F\|_{H^1_{\text{anal}}}.$$

Hence, this yields

$$|F(x,s;y,t)|^q \le G(x,s;y,t),$$

where $G(x,s;y,t)$ is the iterated Poisson integral of g. By the properties of the partial Hardy-Littlewood maximal functions M^{01} and M^{10}

[10], and of the maximal functions $u^*(G)(x,y) = \sup_{s,t>0} G(x,s;y,t)$, we have

$$\begin{aligned}\int\int \sup_{s,t>0} |F(x,s;y,t)|\,dx\,dy \\ &\leq C\int\int M^{01}(M^{10}(u^*(G))(x,y)\,dx\,dy \\ &\leq C\|u^*(G)\|_p^p \leq C\|g\|_p^p \leq C\|F\|_{H^1_{\text{anal}}}.\end{aligned}$$

Hence, we have

$$\int\int \sup_{s,t>0} |F(x,s;y,t)|\,dx\,dy \leq C\|F\|_{H^1_{\text{anal}}}.$$

This proves (1).

Next, we shall prove that $\lim_{s,t\to 0} F(x,s;y,t)$ exists almost everywhere and in the $L^1(\mathbf{R}^2)$ norm. We have that

$$|u_k(x,s;y,t)| \leq G(x,s;y,t)^p, \qquad k \in \Box.$$

Since G is nontangentially bounded, each u_k is nontangentially bounded, $\lim_{s,t\to 0} u_k(x,s;y,t)$ exists almost everywhere. On the other hand, the dominated convergence theorem implies the convergence in the $L^1(\mathbf{R}^2)$ norm.

Proof of Theorem 3.1. *Step* 1. Let $F = (u_k; k \in \Box)$ in H^1_{anal}. Then, there exists finite Borel measures μ_k such that

$$u_k(x,s;y,t) = (P_sP_t) * \mu_k(x,y).$$

Now, by the Lemma 3.2, the limits

$$\lim_{s,t\to 0} u_k(x,s;y,t) = f_k(x,y), \qquad k \in \Box, \tag{1}$$

exists in the $L^1(\mathbf{R}^2)$ norm, and by the Fourier transform we have

$$u_k(x,s;y,t) = \int\int ((P_sP_t)_* f)\hat{}(x',y')e^{-2\pi i(xx'+yy')}\,dx'\,dy'. \tag{2}$$

Then, as $F = (u_k; k \in \Box)$ is a conjugate vector field, we have from (1) and (2) that

$$f_k(x,y) = (H_k f_{00})(x,y), \qquad k \in \Box. \tag{3}$$

Since $f_k \in L^1(\mathbf{R}^2)$, from (3) we have $f_{00} \in H^1_{Hb}$ and

$$\begin{aligned}\|f_{00}\|_{H^1_{Hb}} &= \sum_{k\in\Box} \|H_k f_{00}\|_1 \\ &\leq 4 \sup_{s,t>0} \int\int |F(x,s;y,t)|\,dx\,dy.\end{aligned}$$

Therefore $f = f_{00} \in H^1_{Hb}$ and $\|f\|_{H^1_{Hb}} \leq C\|F\|_{H^1_{\mathrm{anal}}}$. This proves (i).

Step 2. Let f be a function in $L^1(\mathbf{R}^2)$ such that $H_k f \in L^1(\mathbf{R}^2)$, $k \in \Box$. We will show that the vector field given by

$$F = ((P_s P_t)_* H_k f; k \in \Box) \tag{4}$$

belongs to H^1_{anal}.

By Fourier transform we see that F is a conjugate vector field. Indeed,

$$\hat{u}_k(x, s; y, t) = (P_s P_t)\hat{\ }(x, y)(H_k f)\hat{\ }(x, y).$$

Since $H_k f \in L^1$, we have $(H_k f)\hat{\ }$in L^∞ and

$$\int\int |\hat{u}_k(x, s; y, t)|\, dx\, dy \leq \|(H_k f)\hat{\ }\|_\infty \int\int (P_s P_t)\hat{\ }(x, y)\, dx\, dy.$$

Then, for each $k \in \Box$, we have,

$$u_k(x, s; y, t) = \int\int \hat{u}_k(x', s; y', t) e^{-2\pi i(xx' + yy')}\, dx'\, dy'$$

and consequently

$$u_{00}(x, s; y, t) = \int\int e^{-2\pi|x'|s} e^{-2\pi|y'|t} \hat{f}(x', y') e^{-2\pi i(xx'+yy')}\, dx'\, dy',$$

$$u_{10}(x, s; y, t) = \int\int e^{-2\pi|x'|s} e^{-2\pi|y'|t} (i \operatorname{sign} x') \hat{f}(x', y') \cdot e^{-2\pi i(xx'+yy')}\, dx'\, dy',$$

$$u_{01}(x, s; y, t) = \int\int e^{-2\pi|x'|s} e^{-2\pi|y'|t} (i \operatorname{sign} y') \hat{f}(x', y') \cdot e^{-2\pi i(xx'+yy')}\, dx'\, dy',$$

$$u_{11}(x, s; y, t) = \int\int e^{-2\pi|x'|s} e^{-2\pi|y'|t} (i \operatorname{sign} x')(i \operatorname{sign} y') \hat{f}(x', y') \cdot e^{-2\pi i(xx'+yy')}\, dx'\, dy'.$$

Henceforth $(u_k; k \in \Box)$ is a conjugate vector field. Moreover, from (4) and Young's inequality we get

$$\int\int |F(x, s; y, t)|\, dx\, dy \leq \sum_{k \in \Box} \|(P_s P_t)_* H_k f\|_1 \leq \sum_{k \in \Box} \|H_k f\|_1 = \|f\|_{H^2_{Hb}}.$$

This proves that $F \in H^1_{\mathrm{anal}}(\mathbf{R}^2_+ \times \mathbf{R}^2_+)$ and (2).

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