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SMALL ISOMORPHISMS OF $C(X, E)$ SPACES

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A linear map T between two Banach spaces A and B is called ε-isometry if $(1-\varepsilon)\|f\| \leq \|Tf\| \leq (1+\varepsilon)\|f\|$, for any $f \in A$. In the paper we investigate injective and surjective ε-isometries between Banach spaces of continuous E-valued functions.

We prove that, under some geometrical assumptions on the Banach space E, any such ε-isometry is induced by a continuous function between the corresponding compact Hausdorff spaces. We discuss also the question whether such an ε-isometry has to be just a small perturbation of an isometry.

1. Introduction. The classical Banach-Stone theorem says that the Banach spaces $C(X)$ and $C(Y)$ are linearly isometric if and only if X and Y are homeomorphic. During the last 25 years this theorem has been directly generalized in four different ways. It was proven that the Banach-Stone theorem:

1° is stable, this means: if there is an isomorphism T from $C(X)$ into $C(Y)$ with $\|T\|\|T^{-1}\| < 1+\varepsilon$ and $\varepsilon \leq 1$ then $C(X)$ and $C(Y)$ are actually isometric and so X and Y homeomorphic [**9**].

2° holds for function algebras, this means: if there is a linear isometry T from a function algebra A onto a function algebra B then A and B are isomorphic in the category of Banach algebras, and so Shilov boundaries as well as maximal ideal spaces of A and B are homeomorphic [**25**].

3° holds for some spaces of the vector valued functions, this is: if the Banach spaces $C(X,E)$ and $C(Y,E)$ are isometric, where the Banach space E satisfies certain geometric conditions, then spaces X and Y are homeomorphic [**2**, **5**, **21**, **23**].

4° holds for "into" isometries, this is: if there is an isometry from $C(X)$ into $C(Y)$ then there is a continuous function from a subset of Y onto X [**15**, **16**, **18**, **19**].

Then in the last 10 years a number of common and much more far reaching generalizations were discovered. It was proven that:

(a) the vector-valued Banach-Stone theorem is stable (with some $\varepsilon > 0$) [**3**, **4**, **10**, **12**, **17**],

(b) the Banach-Stone theorem for function algebras is stable [**19**],

(c) the "into" Banach-Stone theorem is stable **[7]** and holds for function algebras, too **[18]**,

(d) the Banach-Stone theorem holds for a fairly large class of non-function algebras and spaces **[13, 20, 21, 26, 27]**.

The most far reaching result in the direction (a) was that of E. Behrends and M. Cambern **[3, 4]**. They proved that the stable Banach-Stone theorem holds for some sufficiently small $\varepsilon > 0$ for any Banach space E such that

$$\lambda_{B-C}(E^*) := \inf\{d_{B-M}(l_2^1, E') : E' \subseteq E^*, \dim E' = 2\} > 1,$$

where l_2^1 is the two-dimensional l^1-space and $d_{B-M}(\cdot,\cdot)$ the Banach-Mazur distance.

The ideology of the above assumption is that the Banach space E^* is far from having any l^1-structure, which in turn means that E is far from having any $C(X)$ structure. A condition like this seems to be the most natural since for $E = C(X)$ even the non-stable vector-valued Banach-Stone theorem fails **[2]**.

To address the $C(X)$ structure of a Banach space E more directly let us define λ_0 by

$$\lambda_0(E) := \inf\{d_{B-M}(l_2^\infty, E') : E' \subseteq E, \dim E' = 2\}$$

where l_2^∞ is the two-dimensional l^∞-space. It is easy to observe that for any Banach space we have $\lambda_0(E) \geq \lambda_{B-M}(E^*)$ and for any real Banach space E, $\lambda_{B-C}(E) = \lambda_{B-C}(E^*) = \lambda_0(E) = \lambda_0(E^*)$. In the next section we introduce two more parameters, $\lambda(E)$ and $\mu(E)$, and discuss simple relations between all four.

In this paper we get a common generalization of the Banach-Stone theorem which covers (a), (b) and (c), for any Banach space E with $\lambda_0(E) > 1$. The constant $\varepsilon > 0$ we get is the best one at least in the metric case; at the last section of the paper we discuss the problems which arise when spaces X, Y are non-metrizable. We get $\varepsilon = 1$ for the scalars and $\varepsilon = \sqrt{2} - 1$ for the Hilbert space. We also get some information about the general form of small isomorphisms and isometries from $C(X, E)$ onto $C(Y, F)$.

Many of the results presented here grew out of the author's discussions with Professor M. Cambern. The author is greatly indebted to him.

2. Definitions and notation. We use the standard Banach space terminology. For a Banach space E we denote by E_1 the closed unit ball of E and by E^* the dual space. For a locally compact set X we denote by $C_0(X)$ ($C_0(X,E)$) the Banach space of all scalar-valued (E-valued) continuous functions vanishing at infinity provided with the usual sup-norm. By X_* we denote the one-point compactification $X \cup \{\infty\}$ of X if X is non-compact and just X if X is compact. We often consider $C_0(X)$ ($C_0(X,E)$) to be a subspace of $C(X_*)$ ($C(X_*,E)$). By an extremely regular subspace A of $C_0(X)$ we mean a closed subspace such that

$$\forall x_0 \in X\ \forall \varepsilon > 0\ \forall U^{\text{open}} \ni x_0\ \exists f \in A\ \forall x \notin U\ \|f\| = 1 = f(x_0) \quad \text{and } |f(x)| < \varepsilon.$$

A basic example of an extremely regular subspace of $C_0(X)$ is a function algebra (i.e. closed subalgebra of $C_0(X)$ which separates points of X) such that the Choquet boundary of A is equal to its Shilov boundary.

We denote by $A\check{\otimes}E$ the complete injective tensor product of A and E and we observe that $C_0(X)\check{\otimes}E$ can be naturally identified with $C_0(X,E)$. For isomorphic Banach spaces E, F we set

$$d_{B-M}(E,F) := \inf\{\|T\|\|T^{-1}\| \colon T \text{ is an isomorphism from } E \text{ onto } F\}.$$

We put

$$\lambda(E) := \inf\{\max\{\|e_1 + \lambda e_2\| \colon |\lambda| = 1\} \colon e_1, e_2 \in E,\ \|e_1\| = \|e_2\| = 1\}$$

and

$$\mu(E) := \sup\{\inf\{\|e_1 + \lambda e_2\| \colon |\lambda| = 1\} \colon e_1, e_2 \in E,\ \|e_1\| = \|e_2\| = 1\}.$$

The value $\lambda(E)$ measures how far from $(l_2^\infty)_1$ are the two dimensional sections of the unit ball of E, and $\mu(E)$ measures how far from $(l_2^1)_1$ are the two dimensional sections of the unit ball of E. If such sections are arbitrarily close to $(l_2^1)_1$ then $\mu(E) = 2$; otherwise $\mu(E) < 2$.

Obviously for any Banach space E we have $\lambda(E) \leq \lambda_0(E)$ and $\mu(E) \cdot \lambda_{B-C}(E) \geq 2$, as a matter of fact for most classical Banach spaces we have even equations above. If the norm on E is more pathological, then $\lambda(E)$ may be strictly smaller than $\lambda_0(E)$; nevertheless $\lambda(E) > 1$ if and only if $\lambda_0(E) > 1$. One can even check that for any non-one dimensional Banach space E we have:

(i) $2\lambda_0(E)/(1 + \lambda_0(E)) \leq \lambda(E) \leq \lambda_0(E)$.

Similarly $\mu(E) < 2$ if and only if $\lambda_{B-C}(E) > 1$.

We also have

(ii) $\lambda_0(E) > 1$ if and only if $\lambda_{B-C}(E^*) > 1$.

There is a great number of Banach spaces with the above property.

For example:

(iii) if E is uniformly convex or uniformly smooth then $\lambda(E) > 1$ (and consequently $\mu(E^*) < 2$, $\lambda_{B-C}(E^*) > 1$ and $\lambda_0(E) > 1$).

Let us also mention some other simple properties of these parameters:

(iv) if E is a Hilbert space with $\dim E > 1$ then

$$\lambda(E) = \mu(E) = \lambda_0(E) = \lambda_{B-C}(E) = \sqrt{2}.$$

(v) if $\dim E = 1$ then $\lambda(E) = 2$, $\mu(E) = 1$ (in this case it is customary to assume that $\lambda_0(E) = \lambda_{B-C}(E) = \infty$, since E has no subspace isomorphic with l_2^∞, nor with l_2^1).

(vi) $\lambda(C(X)) = \lambda_0(C(X)) = 1$ provided $\mathrm{card}(X) \geq 2$.

3. The results.

THEOREM 1. *Let X be a locally compact metric space, Y a locally compact Hausdorff space and A, B extremely regular subspaces of $C_0(X)$ and $C_0(Y)$, respectively. Let E, F be Banach spaces and let $T: A\check{\otimes}E \to B\check{\otimes}F$ be a linear map such that $\|f\| \leq \|Tf\|$ for $f \in A\check{\otimes}E$ and $\|T\| < 1+\varepsilon < \lambda(F)$. Then for any $e \in E$ with $\|e\| = 1$ there is a subset Y_e of Y and a continuous surjective map $\varphi_e: Y_e \to X$ such that*

$$(*) \qquad |f_y^*(T(f\otimes e)(y)) - f\circ\varphi_e(y)| \leq \varepsilon\|f\| \qquad f \in A,$$

where $Y_e \ni y \mapsto f_y^*$ is a map from Y_e into ∂F_1^*. □

In the above theorem we have the assumption that $\|T\| < 1+\varepsilon < \lambda(F)$. Such ε is not the biggest possible for arbitrary Banach space F, and it seems to be unlikely we could have in general a simple formula for the best ε. However the constant $\lambda(F)$ is the best one for a number of classical Banach spaces, including Hilbert space and scalars, even for surjective isomorphisms.

In the scalar case we have $\lambda(F) = 2$ and it was proven by H. B. Cohen **[14]** that there are two non-homeomorphic, metric, compact sets X and Y such that there exists a $T: C(X) \overset{\text{onto}}{\to} C(Y)$ with $\|T\|\|T^{-1}\| = 2$.

If $F = E = H =$ infinite dimensional Hilbert space, the situation is even simpler. Let X be a two-point set and Y a one-point set so we have $C(X,H) \cong H\oplus_\infty H$ and $C(Y,H) \cong H$. We define $T: H\oplus_\infty H \to H$ by

$$T((\alpha_1,\alpha_2,\ldots),(\beta_1,\beta_2,\ldots)) = (\alpha_1,\beta_1,\alpha_2,\beta_2,\ldots).$$

It is easy to check that $\|T^{-1}\| = 1$, $\|T\| = \sqrt{2} = \lambda(H)$. Exactly the same method works for $F = E = l^p$ or L^p, $2 \leq p < \infty$.

In general we cannot assume above that Y_e is closed, even if $\|T\| = 1$ and $\dim F = 1$ [**18**, p. 377]. However if X is compact and $\|T\|$ is assumed to be a little closer to 1, then Y_e is closed.

THEOREM 2. *Let A, B, X, Y, E, F be as in Theorem 1, and let $T\colon A\check{\otimes}E \to B\check{\otimes}F$ be a linear map such that $\|f\| \leq \|Tf\|$ for $f \in A\check{\otimes}E$ and $\|T\| < 1+\varepsilon < 4/(1+\mu(F^*))$. Then for any $e \in E$, with $\|e\| = 1$ there is a subset $\tilde{Y}_e$ of Y and a continuous map φ_e from $\tilde{Y}_e$ onto X such that $(*)$ holds. Moreover if X is compact then so is $\tilde{Y}_e$.* □

Theorems 1 and 2 concern "into" isomorphisms. We can get much more information if we assume T is "onto".

THEOREM 3. *Let X, Y be locally compact metric spaces, A, B extremely regular subspaces of $C_0(X)$ and $C_0(Y)$, respectively, E, F Banach spaces and $T\colon A\check{\otimes}E \to B\check{\otimes}F$ a surjective isomorphism such that $\|T^{-1}\| \leq 1$ and $\|T\| < 1+\varepsilon < \min(\lambda(E), \lambda(F))$. Then there is a homeomorphism φ from Y onto X and for any $y \in Y$ there exists a surjective linear isomorphism $T_y\colon E \to F$ with $\|T_y\| \leq 1+\varepsilon$, $\|T_y^{-1}\| \leq 1/(1-\varepsilon)$ such that*

$$(**) \quad \|Tf(y) - T_y(f\circ\varphi(y))\| \leq 2\varepsilon(1+\varepsilon)\|f\|, \qquad f \in A\check{\otimes}E,\ y \in Y.$$

Moreover if E is finite dimensional then we may have $\|T_y\| \leq 1+\varepsilon$, $\|T_y^{-1}\| \leq 1$ so in this case $d_{B-M}(A\check{\otimes}E, B\check{\otimes}F) = d_{B-M}(E, F)$ and

$$\|Tf(y) - T_y(f\circ\varphi(y))\| \leq \varepsilon\|Tf\| \leq (1+\varepsilon)\varepsilon\|f\|, \qquad f \in A\check{\otimes}E,\ y \in Y.$$ □

From this theorem it follows immediately that, if $\lambda(F)$, $\lambda(E) > 1$ then any isometry T from $A\check{\otimes}E$ onto $B\check{\otimes}F$ is canonical, that means, is of the form

$$T(f)(y) = T_y(f(\varphi(y))), \qquad f \in A\check{\otimes}E,\ y \in Y,$$

where $\varphi\colon Y \to X$ is a homeomorphism and $Y \ni y \mapsto T_y$ is automatically a norm continuous map into the set of all isometries from E onto F.

It is natural to ask whether the same holds for small isomorphisms. This means whether any small isomorphism $T\colon A\check{\otimes}E \to B\check{\otimes}F$ is close to

1° a canonical isomorphism

and/or

2° a canonical isometry.

The formula (∗∗) suggests that the answer, at least to the first question, is positive. The problem is that it is not clear whether we can make the map $Y \ni y \mapsto T_y$ continuous. It seems to be strongly related with the topological properties of X, Y, and unknown, in general (this is for non-paracompact sets), even if $A = C_0(X)$, $B = C_0(Y)$, $E = F =$ scalars.

The next theorem says that the answer is positive if X is metric, $A = C_0(X)$ and $B = C_0(Y)$. The answer to the second question is, of course, negative in general; even in the simplest situation when both X and Y are one-point sets. The reason is, that by standard arguments **[6]**, we can construct for any $\delta > 0$ a separable, reflexive Banach space such that

(i) $\lambda \operatorname{Id}_E$, $|\lambda| = 1$ are the only isometries of E,

(ii) $\forall \varepsilon > 0\ \exists T\colon E \overset{\text{onto}}{\rightarrow} E$ $\|T\|$, $\|T^{-1}\| \leq 1 + \varepsilon$ and $\forall \alpha$ $\|T - \alpha \operatorname{Id}\| \geq 2 - \varepsilon$,

(iii) $d_{B-M}(E, H) < 1 + \delta$, H—a separable Hilbert space.

The answer is also negative when $E = F =$ scalars. To observe this put $A = B = H^\infty =$ algebra of all bounded analytic functions on the unit disc. The algebra H^∞ can be considered as an extremely regular subspace of its Shilov boundary. Exactly as in the proof of Theorem 17.4 of **[19]**, it can be shown, that for any $\delta > 0$ there is a surjective isomorphism $T\colon H^\infty \to H^\infty$ with $\|T\| \leq 1 + \delta$, $\|T^{-1}\| < 1 + \delta$ and such that for any isometry $S\colon H^\infty \to H^\infty$ we have $\|S - T\| \geq 2 - \delta$.

The next theorem also gives some information when the answer to the question 2° is positive.

THEOREM 4. *Let X, Y be metric, locally compact spaces, let E be a Banach (Hilbert) space and let T be a linear isomorphism from $C_0(X,E)$ onto $C_0(Y,E)$ such that $\|T^{-1}\| \leq 1$ and $\|T\| < 1 + \varepsilon <$* $\min(\lambda(E), 1.25)$. *Then there is a canonical isomorphism (respectively isometry) S from $C_0(X,E)$ onto $C_0(Y,E)$ such that $\|S - T\| < 2\varepsilon\|T\|$ (respectively $\|S - T\| < 4\varepsilon/(1 - 4\varepsilon)$).* □

4. Proof of the results. Before proving the results we need some more notation and auxiliary observations. We usually consider $A \check{\otimes} E$ as a subspace of $C_0(X,E)$ and $C_0(X,E)$ as a subspace of $C_0(X \times E_1^*)$ or $C_0(X \otimes E_1^*)$, where E_1^* is taken with the weak * topology. For an

$f \in C_0(X, E)$ the obvious element of $C_0(X \times E_1^*)$ which corresponds to f is denoted by the same symbol and so

$$e^*(f(x)) = f(x \otimes e^*) \quad \text{for } x \in X,\ e^* \in E_1^*.$$

We also consider X both as the domain, where the functions from A are defined, and as a subset of A^*, where

$$x(f) = f(x) \quad \text{for } f \in A, x \in X.$$

To emphasize this we also write δ_x for $x \in X$.

We say that a net $(f_\gamma)_{\gamma\in\Gamma}$ from $A\check{\otimes}E$ (from A) peaks at a point $(x, e) \in X \times E_1$ (at $x \in X$) if

$$\|f_\gamma\| \underset{\gamma\in\Gamma}{\to} 1, \quad \|f_\gamma(x) - e\| \underset{\gamma\in\Gamma}{\to} 0 \qquad \left(\text{respectively } |f_\gamma(x) - 1| \underset{\gamma\in\Gamma}{\to} 0\right)$$

and

$$\|f_\gamma(\cdot)\| \underset{\gamma\in\Gamma}{\to} 0 \quad \text{uniformly off any neighbourhood of } x.$$

For $F \in (A\check{\otimes}E)^*$ by $\mu \sim F$ we mean that μ is a Borel measure on $X \times E_1^*$ which is a norm preserving extension of F to $C_0(X \times E_1^*)$. We have the following easy observations.

PROPOSITION 1. *Assume $\mu \sim F \in (A\check{\otimes}E)^*$. Then for any $x \in X$ there is an $e^* \in E^*$ such that $\mu|_{\{x\}\times E_1^*} \sim \delta_x \otimes e^*$.* □

Evidently the functional

$$A\check{\otimes}E \ni f \mapsto \int_{\{x\}\times E_1^*} f \, d\mu$$

depends only on the value of f at the point x and so is of the form $\delta_x \otimes e^*$.

PROPOSITION 2. *Assume $F \in (A\check{\otimes}E)^*$, $\mu_i \sim F$ and $\mu_i|_{\{x\}\times E_1^*} \sim \delta_x \otimes e_i^*$, $i = 1, 2$. Then $e_1^* = e_2^*$.* □

The proposition follows from the assumption that A is extremely regular by considering a suitable net peaking at the point x. By this proposition it makes sense to write $F_x := F|_{\{x\}\times E_1^*}$ for any $F \in (A\check{\otimes}E)^*$. If $\dim E = 1$ then, to simplify the notation, we just identify $A\check{\otimes}E$ with the space $A \subseteq C_0(X)$ though formally, according to the previous notation, we should consider $A\check{\otimes}E$ as a subspace of $C_0(X \otimes E_1^*)$ where E_1^* is now the segment $[-1, +1]$ in the real case or the unit disc in the complex case. Hence in this case F_x is of the form $\lambda\delta_x$ where $\lambda = \mu(\{x\})$ and μ is a measure on X such that $\mu \sim F$. We will just write $\lambda = F(\{x\})$.

Proposition 3. *Assume that $\mu \sim F$ and that X_0 is a Borel subset of X. Then*

$$\operatorname{var}(\mu|_{X_0\times E_1^*}) = \|\mu|_{X_0\times E_1^*}\|$$

where the norm is taken in $(A\check{\otimes}E)^*$. □

Put $\mu_1 = \mu|_{X_0\times E_1^*}$ and $\mu_2 = \mu|_{(X\setminus X_0)\times E_1^*}$. We have

$$\begin{aligned}\|\mu_1\| + \|\mu_2\| \geq \|\mu_1 + \mu_2\| = \|\mu\| = \operatorname{var}(\mu) = \operatorname{var}(\mu_1) + \operatorname{var}(\mu_2)\\ \geq \|\mu_1\| + \|\mu_2\|.\end{aligned}$$

Hence $\|\mu_1\| = \operatorname{var}(\mu_1)$.

Proposition 4. *Let μ be a finite, positive measure defined on a set Ω and let $(A_n)_{n=1}^{\infty}$ be a sequence of μ-measurable subsets of Ω where $\mu(A_n) \geq \lambda_0$ for all $n \in \mathbf{N}$. Then*

$$\mu\left(\bigcap_{n=1}^{\infty}\bigcup_{j=n}^{\infty} A_j\right) \geq \lambda_0. \qquad \square$$

Proposition 5. *Let H be a Hilbert space and let $I_\varepsilon(H)$ be the set of all surjective isomorphisms $S\colon H \to H$ with $\|S\|$, $\|S^{-1}\| \leq 1+\varepsilon$. Assume that $\varepsilon < \frac{1}{4}$. Then there is a norm continuous retraction $\Phi\colon I_\varepsilon(H) \to I_0(H)$ such that for all S, S' in $I_\varepsilon(H)$ we have*

$$\|\Phi(S) - S\| \leq \frac{2(1+\varepsilon)\varepsilon}{1-4\varepsilon}. \square \tag{1}$$

Proof. Let $S \in I_\varepsilon(H)$, $\varepsilon < \frac{1}{4}$. We define $\Phi(S)$ to be just the unitary part of the polar decomposition of S. This means we put $\Phi(S) = S \circ P^{-1}$, where $P = (S^* \circ S)^{1/2}$. The map $S \circ P^{-1}$ is an isometry and we check (1) by a direct computation. We have

$$\begin{aligned}\|\Phi(S) - S\| &\leq \|S\|\|\operatorname{Id} - (S^*\circ S)^{-1/2}\|\\ &\leq (1+\varepsilon)\left\|\sum_{n=1}^{\infty}\binom{-\frac{1}{2}}{n}(\operatorname{Id} - S^*\circ S)^n\right\|\\ &\leq (1+\varepsilon)\sum_{n=1}^{\infty}\left|\binom{-\frac{1}{2}}{n}\right|\|\operatorname{Id} - S^*\circ S\|^n.\end{aligned}$$

We have also

$$\begin{aligned}\|\operatorname{Id} - S^* \circ S\| &\le \sup_{\|x\|=1} ((\operatorname{Id} - S^* \circ S)x \mid (\operatorname{Id} - S^* \circ S)x) \\ &= \sup_{\|x\|=1} (\|x\|^2 + \|S^* \circ Sx\|^2 - 2\|Sx\|^2) \\ &\le \sup_{\|y\| \ge (1+\varepsilon)^{-1}} (1 + \|S^* y\|^2 - 2\|y\|^2) \\ &\le \sup_{\|y\| \ge (1+\varepsilon)^{-1}} (1 + ((1+\varepsilon)^2 - 2)\|y\|^2) \\ &= 1 + \frac{(1+\varepsilon)^2 - 2}{(1+\varepsilon)^2} \le 4\varepsilon.\end{aligned}$$

Hence

$$\begin{aligned}\|\Phi(S) - S\| &\le (1+\varepsilon) \sum_{n=1}^{\infty} \left| \binom{-\frac{1}{2}}{n} \right| (4\varepsilon)^n = (1+\varepsilon) \sum_{n=1}^{\infty} \binom{-\frac{1}{2}}{n} (-4\varepsilon)^n \\ &= (1+\varepsilon)((1-4\varepsilon)^{-1/2} - 1) \le (1+\varepsilon)\frac{2\varepsilon}{1-4\varepsilon}.\end{aligned}$$

Now we are ready to prove Theorem 1. To this end we fix $e \in E$ with $\|e\| = 1$ and denote by T_e the map from A into $B\check{\otimes}F$ defined by $T_e(f) = T(f \otimes e)$ for $f \in A$. Evidently T_e is norm non-decreasing and $\|T_e\| \le \|T\|$. In the remaining part of the proof we deal only with the map T_e.

Let M be such that

$$\max\left(\frac{1 - \varepsilon + \|T_e\|}{2}, \frac{1+\varepsilon}{\lambda(F)}\right) < M < 1.$$

For any $x \in X$ we put

$$S_x = \{y \in Y : \exists f^* \in \partial F_1^* \ |T_e^*(\delta_y \otimes f^*)(\{x\})| \ge M\}.$$

For any sequence $(f_n)_{n=1}^{\infty}$ from A which peaks at x we put

$$\begin{gathered} A_n = \{y \otimes f^* \in Y \otimes F_1^* : |T_e f_n(y \otimes f^*)| \ge M\}, \\ A_\infty = \bigcap_{n=1}^{\infty} \bigcup_{j=n}^{\infty} A_j, \\ A_x = \{y \in Y : \exists f^* \in F_1^* \ \exists (f_n)_{n=1}^{\infty} \subset A \text{ peaking at } x \\ \text{with } y \otimes f^* \in A_\infty\}. \end{gathered}$$

LEMMA 1. *For any $x \in X$ the set A_x is non-empty.* □

Proof. Let $(f_n)_{n=1}^{\infty}$ be a sequence peaking at the point $x \in X$. Let $\nu \sim (T_e^{-1})^*(\delta_x) \in (B\check{\otimes}F)^*$, where by T_e^{-1} we mean a map defined on the range of the map T_e. We have

$$\operatorname{var}\nu \leq \|T_e^{-1}\| \leq 1,$$

$$\limsup_n \|T_e f_n\| \leq \|T_e\| \quad \text{and} \quad \lim_n \int T f_n \, d\nu = \lim_n f_n(x) = 1.$$

Hence

$$\limsup_n |\nu|(A_n) \geq \frac{1-M}{\|T\|-M} := \lambda_0 > 0$$

and the lemma follows from Proposition 4.

LEMMA 2. *For any $x \in X$, $S_x = A_x$.* □

Proof. Let $y \in A_x$. Then there is an $f^* \in F_1^*$ and a sequence $(f_n)_{n=1}^{\infty}$ in A peaking at x such that $|T_e f_n(y \otimes f^*)| \geq M$ for infinitely many $n \in \mathbf{N}$, so without loss of generality we can assume that this equation holds for all $n \in \mathbf{N}$. Let

$$C_0(X)^* \ni \mu \sim T_e^*(y \otimes f^*) \in A^*.$$

We have $|\int_X f_n \, d\mu| = |T_e f_n(y \otimes f^*)| \geq M$ for $n \in \mathbf{N}$. Hence by the definition of a peaking sequence we get $|\mu(\{x\})| \geq M$. This proves that $A_x \subseteq S_x$.

Let now $f^* \in F_1^*$, $\mu \sim T_e^*(y \otimes f^*)$ be such that $|\mu(\{x\})| \geq M$. We have

$$\mu = \lambda\delta_x + \Delta\mu \quad \text{where } |\lambda| \geq M \text{ and } |\Delta\mu|(\{x\}) = 0.$$

Hence there is a sequence $(f_n)_{n=1}^{\infty}$ in A, peaking at x and such that

$$|T f_n(y \otimes f^*)| = \left| \int_X f_n \, d\mu \right| \geq M \quad \text{for } n \in \mathbf{N}.$$

Observe that in general it may not be possible to find the functions f_n as above such that $\|f_n\| = 1 = f_n(x)$. This is the reason that at the definition of a peaking sequence we required only $f_n(x) \to 1$ and $\|f_n\| \to 1$ as $n \to \infty$.

LEMMA 3. *If $x_1 \neq x_2 \in X$ then $S_{x_1} \cap S_{x_2} = \emptyset$.* □

Proof. Assume to the contrary that there are $y_0 \in Y$ and sequences $(f_n^1)_{n=1}^{\infty}$, $(f_n^2)_{n=1}^{\infty}$ in A peaking at x_1 and x_2, respectively, such that

$$\|T_e f_n^i(y_0)\| \geq M \quad \text{for } n \in \mathbf{N},\ i = 1, 2. \tag{2}$$

We have $\lim_n \|f_n^1 \pm f_n^2\| = 1$ for both $+$ and $-$, so

$$\limsup_n \|T_e f_n^1(y_0) \pm T_e f_n^2(y_0)\| \leq \|T_e\| \leq \|T\| \leq 1+\varepsilon. \tag{3}$$

By (2) and (3) we get $\lambda(F) \leq (1+\varepsilon)/M$ which contradicts the definition of M.

LEMMA 4. *For any $x \in X$ and $y \in S_x$ there is an $f^* \in F_1^*$ such that*

$$|f^*(T_e(f)(y)) - f(x)| \leq \varepsilon\|f\| \quad \text{for } f \in A. \tag{4}$$ □

Proof. Fix $y \in S_x$ and let $f^* \in \partial F_1^*$ be given by the definition of S_x. This means

$$T_e^*(\delta_y \otimes f^*) \sim \lambda\delta_x + \Delta\mu \in C_0(X)^*,$$
$$\text{where } |\lambda| \geq M \text{ and } |\Delta\mu|(\{x\}) = 0.$$

Multiplying f^* by a suitable scalar of modulus one we can assume that λ is a real, positive number. For any $f \in A$ with $\|f\| \leq 1$, by Proposition 3 and the definition of M we get

$$\begin{aligned}|f^*(T_e(f)(y)) - f(x)| &= |T_e^*(\delta_y \otimes f^*)(f) - f(x)| \\ &\leq |\lambda f(x) - f(x)| + |\Delta\mu(f)| \\ &\leq |1-\lambda| + \|\Delta\mu\| \leq |1-\lambda| + \|T_e\| - |\lambda| \\ &\leq \|T\| + (1-2M) = \|T\| - 1 + 2(1-M) \leq \varepsilon.\end{aligned}$$

To end the proof of Theorem 1 we define now a function $\varphi_e\colon \bigcup_{x\in X} S_x \to X$ by $\varphi_e(y) = x$ if $y \in S_x = A_x$. By Lemma 3 φ_e is well-defined, by Lemmas 1 and 2 φ_e is surjective and by Lemma 4 φ_e satisfies $(*)$. It remains to prove that φ_e is continuous. Assuming the contrary there are $x_n \in X$, $y_n \in Y$ and an open neighbourhood V of x_0 such that $y_n \in S_{x_n}$, $y_n \underset{n\to\infty}{\to} y_0 \in S_{x_0}$ and $x_n \in X\backslash\overline{V}$ for all $n \in \mathbf{N}$. Fix $\delta > 0$. Since $y_0 \in S_{x_0} = A_{x_0}$ there is an $f_1 \in A_1$ such that $\|f_1\| \leq 1+\delta$, $f_1(x_0) = 1$, $|f_1(x)| \leq \delta$ for $x \in X\backslash V$ and $\|T_e f_1(y_0)\| > M - \delta$. Next since $T_e f_1$ is norm continuous and $y_n \to y_0$ there is an $n_0 \in \mathbf{N}$ such that $\|T_e f_1(y_{n_0})\| > M - \delta$. Let $f_2 \in A$ be such that $\|f_2\| \leq 1+\delta$, $f_2(x_{n_0}) = 1$, $|f_2(x)| \leq \delta$ for $x \in V$ and $\|T_e f_2(y_{n_0})\| > M - \delta$. We have

$$\|T_e f_i(y_{n_0})\| > M - \delta \quad \text{for } i = 1, 2$$

and

$$\|T_e f_1(y_{n_0}) \pm T_e f_2(y_{n_0})\| \leq \|T_e\|\|f_1 \pm f_2\| \leq (1+\varepsilon)(1+2\delta).$$

Hence, since δ is an arbitrary positive number we get $\lambda(F) \leq (1+\varepsilon)/M$, which contradicts the definition of M and so ends the proof of Theorem 1.

Proof of Theorem 2. We use the same notation as in the proof of the previous theorem. For any $x \in X$ we let

$$\tilde{S}_x = \{y \in Y : \exists f^* \in F_1^* \ \forall f \in A \ |f^*(T_e(f)(y)) - f(x)| \leq \varepsilon \|f\|\}$$

and

$$\tilde{Y}_e = \bigcap_{x \in X} \tilde{S}_x.$$

Observe that by Theorem 1 $\tilde{S}_x \neq \emptyset$, for $x \in X$.

LEMMA. *If* $x_1 \neq x_2 \in X$ *then* $\tilde{S}_{x_1} \cap \tilde{S}_{x_2} = \emptyset$. □

Proof. Assume that $y \in \tilde{S}_{x_1} \cap \tilde{S}_{x_2}$ and let $f_1^*, f_2^* \in F_1^*$ be such that

$$\|f_i^*(T_e(f)(y)) - f(x_1)\| \leq \|f\|, \qquad f \in A, \ i = 1, 2.$$

The above inequality just means that

$$\|T_e^*(y \otimes f_i^*) - \delta_{x_i}\| \leq \varepsilon, \qquad i = 1, 2.$$

Since A is extremely regular $\|\delta_{x_1} + \lambda\delta_{x_2}\| = 2$, for $|\lambda| = 1$, so by the above inequality we get

$$\begin{aligned}\|f_1^* + \lambda f_2^*\| &= \|y \otimes f_1^* + \lambda y \otimes f_2^*\| \\ &\geq \frac{1}{\|T\|}\|T_e^*(y \otimes f_1^*) + \lambda T_e^*(y \otimes f_2^*)\| \\ &\geq \frac{1}{1+\varepsilon}(2 - 2\varepsilon) = 2\frac{1-\varepsilon}{1+\varepsilon}.\end{aligned}$$

Hence we get $\mu(F^*) \geq 2(1-\varepsilon)/(1+\varepsilon)$ which contradicts our assumption that

$$1 + \varepsilon > \frac{4}{2 + \mu(F^*)}.$$

To end the proof of Theorem 2 we define $\varphi_e : \tilde{Y}_e \to X$ by $\varphi_e(y) = x$ if $x \in \tilde{S}_x$. By the lemma φ_e is well-defined and since $\tilde{S}_x \neq \emptyset$, is surjective. The continuity of φ_e and the compactness of $\tilde{Y}_e$, if X is compact, follows by the standard arguments from weak * continuity of T_e^*, lower semi-continuity of the norm and the fact that $\varepsilon < 1$ (compare the proof of Lemma 2 of **[18]**). It is easy to observe that Y_e, given by Theorem 1, is a subset of the set $\tilde{Y}_e$ and the function φ_e, given by Theorem 2, is an extension of that given by Theorem 1.

Proof of Theorem 3. We prove that the set Y_e and the function φ_e, given by Theorem 1, do not depend on $e \in E$ with $\|e\| = 1$. To this end it is enough to prove that Y_e and φ_e locally do not depend on e. Assuming the contrary, since φ_e is surjective, we get, for any $\delta > 0$, two elements e_1 and e_2 of E with norm one and in the distance less than δ such that there are two distinct points y_1 and y_2 in the domains respectively of φ_{e_1} and φ_{e_2} with $\varphi_{e_1}(y_1) = \varphi_{e_2}(y_2) := x_0 \in X$. Let f_1^*, f_2^* be given by $(*)$, that is such that

$$\|T_{e_i}^*(y_i \otimes f_i^*) - \delta_{x_0}\| \leq \varepsilon \quad \text{for } i = 1, 2.$$

Hence

$$\|T_{e_1}^*(y_1 \otimes f_1^*) - T_{e_2}^*(y_2 \otimes f_2^*)\| \leq 2\varepsilon < 2.$$

On the other hand

$$\|T_{e_1}^* - T_{e_2}^*\| = \|T_{e_1} - T_{e_2}\| \leq \|e_1 - e_2\|\|T\| \leq 2\delta,$$

and, since B is extremely regular

$$\|y_1 \otimes f_1^* - y_2 \otimes f_2^*\|_{(B\otimes F)^*} \geq \|\delta_{y_1} - \delta_{y_2}\|_{B^*} = 2$$

so

$$\|T_{e_1}^*(y_1 \otimes f_1^*) - T_{e_2}^*(y_2 \otimes f_2^*)\| \geq 2 - 2\delta.$$

Setting $\delta = (1 - \varepsilon)/4$ we get a contradiction which proves, by the definition of $Y_e = \bigcup_{x\in X} S_x$ and of S_x that there is a subset Y_0 of Y (equal to Y_e, for any $e \in \partial E_1$) and a continuous surjective map $\varphi\colon Y_0 \to X$ such that

$$\forall e \in \partial E_1 \ \forall y \in Y_0 \ \exists f^* \in F_1^* \ |T_e^*(\delta_y \otimes f^*)(\{x\})| \geq M. \tag{5}$$

Observe that we have proven even more, namely putting $e_1 = e_2$ we get $y_1 = y_2$, which means that φ is injective or equivalently that $S_x = A_x$ are one point sets.

Fix now $y \in Y_0$ and $f_0 \in (A\check{\otimes}E)_1$. Let $e_0 = f_0(x)$, where $x = \varphi(y)$, let $e \in \partial E_1$ be such that $e_0 = \|e_0\|e$ and let $f^* \in F_1^*$ be given by (5). By Propositions 1–3 we can write $T^*(y \otimes f^*)$ in the form

$$T^*(y \otimes f^*) = \delta_x \otimes e^* + \Delta\mu, \tag{6}$$

where

$$\|\Delta\mu\| + \|e^*\| = \|T^*(y \otimes f^*)\| \leq \|T\| < 1 + \varepsilon \tag{6'}$$

and by (5), and the definition of T_e

$$\|e^*\| \geq |e^*(e)| \geq M. \tag{7}$$

Evaluating (6) at f_0 we get

$$f^*(T(f_0)(y)) = e^*(e_0) + \Delta\mu(f_0),$$

and by the definition of e

$$(8) \qquad |\|e_0\|e^*(e) + \Delta\mu(f_0)| \leq \|T(f_0)(y)\|.$$

Direct computations, (6′), (7), (8) and the definition of M give

$$\|e_0\| \leq \frac{\|Tf_0(y)\|}{M} + \frac{1+\varepsilon-M}{M}.$$

So we have proven the following

$$(9) \quad \forall y \in Y_0 \ \forall f_0 \in (A\check{\otimes}E)_1 \quad \|f_0(\varphi(y))\| \leq \frac{\|Tf_0(y)\|}{M} + \frac{1+\varepsilon-M}{M}.$$

At the very beginning of our construction we defined M to be any number smaller than one and sufficiently close to one. If we take now M' in between M and 1 we get sets S'_x, $x \in X$ which, directly by the definition, are contained in the corresponding S_x sets. On the other hand we have just observed that now all sets S_x, and so S'_x are one point sets so $S'_x = S_x$ for any M' such that $M < M' < 1$. Consequently we also get the same function φ for each such M'. Hence from (9) we get

$$(10) \qquad \forall y \in Y_0 \ \forall f_0 \in (A\check{\otimes}E)_1 \quad \|f_0(\varphi(y))\| \leq \|Tf_0(y)\| + \varepsilon.$$

By symmetry, for $\tilde{T} = \|T\|T^{-1}$ in place of T, we get a subset X_0 of X and a continuous bijection $\psi\colon X_0 \to Y$ such that

$$(11) \quad \forall x \in X_0 \ \forall f_0 \in A\check{\otimes}E \quad \|Tf_0(\psi(x))\| \leq \|T\|\|f_0(x)\| + \varepsilon\|Tf_0\|.$$

Let x_0 be an arbitrary point of X, $y_0 \in Y_0$ be such that $\varphi(y_0) = x_0$ and $x_1 \in X_0$ be such that $\psi(x_1) = y_0$. By (10) and (11) we have

$$\|f_0(x_0)\| \leq \|T\|\|f_0(x_1)\| + \varepsilon\|Tf_0\| + \varepsilon, \quad f_0 \in (A\check{\otimes}E)_1.$$

Hence

$$(12) \qquad \|f_0(x_0)\| \leq (1+\varepsilon)\varepsilon + \varepsilon \quad \text{if } f_0(x_1) = 0.$$

We now consider two possibilities:

(i) $\max(\dim E, \dim F) > 1$,

(ii) $\dim E = \dim F = 1$.

Assuming (i) we have $1 + \varepsilon < \min(\lambda(E), \lambda(F)) \leq \sqrt{2}$; hence $(1+\varepsilon)\varepsilon+\varepsilon < 1$ and (12) together with the extreme regularity of A gives $x_1 = x_0$. By this and the symmetry arguments we have $\varphi \circ \psi = \mathrm{Id}_X$ and $\psi \circ \varphi = \mathrm{Id}_Y$, so φ is a surjective homeomorphism.

Assuming (ii) we have $\varepsilon < 1$ and by $(*)$ of Theorem 1

$$(13) \qquad |\varepsilon(y) \cdot T(f)(y) - f \circ \varphi(y)| \leq \varepsilon \|f\| \quad \text{for } f \in A,$$

where $|\varepsilon(y)| = 1$ for $y \in Y_0$. By the symmetry arguments we have also

$$(14) \qquad |\varepsilon'(x)\|T\|T^{-1}(g)(x) - g \circ \psi(x)| \leq \varepsilon \|g\| \quad \text{for } g \in B.$$

Let x_0, y_0, x_1 be such that $\varphi(y_0) = x_0$ and $\psi(x_1) = y_0$. By (13) and (14) we get

$$(15) \qquad |f(x_0) - \varepsilon(y_0)\varepsilon'(x_1)\|T\|f(x_1)| \leq \varepsilon(\|f\| + \|Tf\|).$$

Assuming $x_0 \neq x_1$, since A is extremely regular, the norm of the functional $A \ni f \mapsto f(x_0) - \varepsilon(y_0)\varepsilon'(x_1)\|T\|f(x_1)$ is equal to $1 + \|T\|$ and by (15) we have

$$1 + \|T\| \leq \varepsilon(1 + \|T\|),$$

which contradicts the assumption that $\varepsilon < 1$. Hence as at the point (i) we get that φ is a surjective homeomorphism.

Now, to construct isomorphisms $T_y \colon E \to F$, $y \in Y$, we let $f_y \in A$ be such that $\|f_y\| = 1 = f_y(\varphi(y))$ and define $T_y \colon E \to F$ by

$$T_y(e) = T(f_y \otimes e)(y), \qquad e \in E.$$

We have $\|T_y\| \leq \|T\| < 1 + \varepsilon$ and, by (10),

$$(16) \qquad \|T_y e\| \geq (1 - \varepsilon)\|e\|, \qquad e \in E.$$

By (11), for any $f \in A\check{\otimes}E$ and $y \in Y$ we also have

$$\begin{aligned}(17) \qquad &\|Tf(y) - T_y(f \circ \varphi(y))\| \\ &\leq \|T\|\|(f - f_y \otimes (f \circ \varphi(y)))(\varphi(y))\| \\ &\quad + \varepsilon\|T(f - f_y \otimes (f \circ \varphi(y)))\| \\ &\leq 2\varepsilon\|T\|\|f\|,\end{aligned}$$

which proves $(**)$. It remains to show that T_y are surjective. We prove this under the additional assumption that $\dim F > 1$ which gives $1 + \varepsilon < \lambda(F) \leq \sqrt{2}$. These T_y are surjective also when $\dim E = 1$ but this case is covered by the next step, where we get an even better result.

By (16) $T_y(E)$ is a closed subspace of F, so assuming T_y is not surjective we get $k_0 \in F_1$ such that for any $e \in E$ we have $\|k_0 - T_y(e)\| \geq 1$. Let $g_0 \in (B\check{\otimes}F)_1$ be such that $g_0(y) = k_0$ and put

$f_0 = T^{-1}(g_0)$. By (17) we have

$$\begin{aligned}\|k_0 - T_y(f_0 \circ \varphi(y))\| &= \|Tf_0(y) - T_y(f_0 \circ \varphi(y))\| \\ &\le \varepsilon(\|Tf_0\| + \|T(f_y \otimes (f_0 \circ \varphi(y)))\|) \\ &\le \varepsilon(1 + \|T\|) < \varepsilon(2+\varepsilon) < (\sqrt{2}-1)(\sqrt{2}+1) = 1.\end{aligned}$$

The contradiction proves surjectivity of T_y.

To end the proof we now assume that F is finite dimensional. By what we have proven E is isomorphic with a subspace of F (as a matter of fact with F) so E is also finite dimensional. For $y \in Y$ we let $(f_y^n)_{n=1}^{\infty}$ be a sequence in A peaking at y and such that

$$|f_y^n(x) - \max(0, \operatorname{Re} f_y^n(x))| \to 0 \tag{18}$$

uniformly off any neighbourhood of $\varphi(y)$ ([**18**] Lemma 1). We define $T_y^n \colon E \to F$ by

$$T_y^n(e) = T(f_y^n \otimes e)(y), \qquad e \in E$$

and we let $T_y^0 \colon E \to F$ be such that there is a subsequence of (T_y^n) convergent to T_y^0 in the norm topology.

Observe that by (18) for any $f \in A\check{\otimes}E$ we have

$$\lim_n \|f - f_y^n \otimes (f \circ \varphi(y))\| = \|f\|.$$

Hence, by (17)

$$\|Tf(y) - T_y^0(f \circ \varphi(y))\| \le \varepsilon\|Tf\|, \qquad f \in A\check{\otimes}E.$$

By the definition of T_y^0 we evidently have $\|T_y^0\| \le \|T\| \le 1 + \varepsilon$. Let $e \in \partial E_1$, $f \in (A\check{\otimes}E)_1$ be such that $f(\varphi(y)) = e$ and $f^* \in F_1^*$ be given by (5). By the definition of T_y^0 we have

$$|f^*(T_y^0(e))| = |T^*(y \otimes f^*)(f_y^n \otimes e)| = |e^*(e)|$$

where e^* is given by (6) and (7). Hence

$$|f^*(T_y^0(e))| \ge M\|f(\varphi(y))\| = M\|e\|.$$

Thus, since M was arbitrary number sufficiently close to one and $\|f^*\| \le 1$ we have $\|T_y^0(e)\| \ge \|e\|$, $e \in E$.

Proof of Theorem 4. Theorem 4 is an easy consequence of Theorem 3 and Proposition 5. The only problem is to define the map $Y \ni y \mapsto T_y$ in such a way that the functions $Y \ni y \mapsto T_y(f \circ \varphi(y)) \in E$ are continuous for any $f \in A$.

To this end we define $\Phi \colon X \to 2^{C_0(X)}$ by

$$\Phi(x) = \{f \in C_0(X) \colon \|f\| = |f(x)| = 1\}.$$

The map Φ is a norm lower-semicontinuous, convex, complete selection so by the Michael Selection Theorem **[8]** there is a norm continuous map $\chi: X \to (C_0(x))^*$ such that $\chi(x)(x) = 1$ for $x \in X$. We define $T_y: E \to E$ by

$$T_y(e) = T(\chi(\varphi(y)) \otimes e)(y).$$

The map $Y \ni y \mapsto T_y$ we get is norm continuous so the operator

$$C_0(X)\check{\otimes}E \ni f \overset{S}{\mapsto} T_y(f \circ \varphi(y)) \in C_0(Y)\check{\otimes}E$$

is well defined and exactly as in the proof of Theorem 3 we have $\|S - T\| \leq 2\varepsilon\|T\|$.

Assume finally that E is a Hilbert space and let Φ be as at Proposition 5. We define an isometry

$$\tilde{S}: C_0(X)\check{\otimes}E \to C_0(Y)\check{\otimes}E \quad \text{by}$$
$$\tilde{S}(f)(y) = \Phi(T_y)(e), \qquad f \in C_0(X)\check{\otimes}E,\ y \in Y.$$

By Proposition 5 we have

$$\begin{aligned}\|\tilde{S} - T\| &\leq \|\tilde{S} - S\| + \|S - T\| \\ &\leq 2(1+\varepsilon)\varepsilon + \frac{2(1+\varepsilon)\varepsilon}{1-4\varepsilon} \leq \frac{4\varepsilon}{1-4\varepsilon}.\end{aligned}$$

5. Non-metrizable case. Up to now in all the results we discuss here we have the metrizability assumption. All but one step of the proofs of Theorems 1, 2 and 3 works in the exactly same way also without this assumption. The only change we would have to adopt is simply put the nets in place of the sequences. However this one spot where we do need the metrizability is very essential. This is Lemma 1. The author does not know whether $A_x = S_x$ has to be non-empty in general, certainly the simple method we used to prove this lemma does not work without the assumption that any point of X is a G_δ set. So as a matter of fact Theorems 1, 2 and 3 work for any compact set such that any of its points is a G_δ set. We did not put them this way because such an assumption is only very slightly weaker than the metrizability.

There is one more point where we used the metrizability, namely Theorem 4, where in the proof we applied the Michael Selection Theorem. The Michael Theorem does not work for any locally compact set but does for any paracompact space. So if one could prove Lemma 1 in general we would get Theorem 4 for any paracompact set.

Hence it is still an open problem whether we can omit the metrizability assumption in our theorems. However we show that using what we already know and some "general nonsense" we can easily get some information about the non-metrizable case, too. We get almost the same results, as in metric case only with worse ε, which is probably no longer the best possible.

THEOREM 5. *Let X, Y be locally compact spaces, E, F Banach spaces and $T\colon C_0(X,E) \to C_0(Y,F)$ a linear map such that $\|f\| \leq \|Tf\|$ for $f \in C_0(X,E)$ and $\|T\| < 1+\varepsilon < 4/(2+\mu(F^*))$. Then for any $e \in E$, with $\|e\| = 1$ there is a subset $\tilde{Y}_e$ of Y and a continuous map φ_e from $\tilde{Y}_e$ onto X such that*

$$|f_y^*(T(f \otimes e)(y)) - f \circ \varphi_e(y)| \leq \varepsilon\|f\|, \qquad f \in A,$$

where $Y_e \ni y \mapsto f_y^*$ is a map from Y_e into F_1^*.

Moreover if X is compact so is $\tilde{Y}_e$. □

Proof. Let $\tilde{S}_x$, $\tilde{Y}_e$ and φ_e be defined exactly as in the proof of Theorem 2. Observe that in that proof, in particular in the lemma there, we do not use directly the metrizability assumption, so φ_e is now still well-defined and by the same standard arguments continuous, as well as $\tilde{Y}_e$ is compact if X is so. The only point we use the metrizability assumption in the proof of Theorem 2 is when we claim that "by Theorem 1, $\tilde{S}_x \neq \varnothing$ for all $x \in X$". Now we can no longer use Theorem 1 directly, but we use it indirectly to prove that $\tilde{S}_x$ is still non-empty.

To this end fix $x \in X$ and let A' be a separable subset of A such that there is an $f \in A'$ with $f(x) \neq 0$. Let $[A']$ be the C^*-subalgebra of $C_0(X)$ generated by A'. $[A']$ is separable and as a commutative C^*-function algebra is of the form $C_0(X_{A'})$ for some locally compact metric set $X_{A'}$. It is easy to observe that $X_{A'}$ is a quotient space of $X_* = X \cup \{\infty\}$ by the relation $\sim$ defined as $x_1 \sim x_2$ if and only if $f(x_1) = f(x_2)$ for $f \in A'$.

We denote by $T_{A'}$ the restriction of T_e to A' and set

$$\Omega(x,A') = \{(y,f^*) \in Y_* \times F_1^*\colon \|T_{A'}^*(y \otimes f^*) - \delta_x\| \leq \varepsilon\}.$$

By Theorem 1 or 2 the set $\Omega(x,A')$ is not empty. By the weak * continuity of $T_{A'}^*$ and by the lower semi-continuity of the norm this is also a closed subset of the compact set $Y_* \times F_1^*$. Let now $A^1, \dots, A^n$ be separable subspaces of A with the property that for any $j = 1, \dots, n$ there is an $f_j \in A_j$ with $f_j(x) \neq 0$. Let $[A^1, \dots, A^n]$ be the C^*-subalgebra of

$C_0(X)$. This subalgebra is still separable so we have

$$\bigcap_{j=1}^{n} \Omega(x, A^j) \supseteq \Omega(x, [A^1, \dots, A^n]) \neq \emptyset.$$

Hence by standard compactness arguments

$$\Omega(x, A) = \bigcap \Omega(x, A') \neq \emptyset,$$

where the intersection is over the set of all separable subspaces of $C_0(X)$.

Let $(y_0, f_0^*) \in \Omega(x, A)$. Since $\varepsilon < 1$ we have $y_0 \neq \infty$ and so $y_0 \in \tilde{S}_x$ and we are done.

Now, since Theorems 3 and 4 are only corollaries of Theorem 1, by exactly the same arguments, using just Theorem 5 in place of Theorem 1, we get a few more results in the non-metric case.

THEOREM 6. *Let X, Y be locally compact spaces, E, F Banach spaces and $T\colon C_0(X, E) \to C_0(Y, F)$ be a surjective isomorphism with $\|T^{-1}\| \leq 1$ and*

$$\|T\| < 1 + \varepsilon < \min\left(\frac{4}{2 + \mu(E^*)}, \frac{4}{2 + \mu(F^*)}\right).$$

Then

(i) *there is a homeomorphism φ from Y onto X and for any $y \in Y$ there exists a surjective linear isomorphism $T_y\colon E \to F$ with $\|T_y\| \leq 1 + \varepsilon$, $\|T_y^{-1}\| \leq 1/(1 - \varepsilon)$ such that*

$$\|Tf(y) - T_y(f \circ \varphi(y))\| \leq 2\varepsilon(1 + \varepsilon)\|f\|, \qquad f \in C_0(X, E),\ y \in Y;$$

(ii) *if E is finite dimensional then we have $\|T_y\| \leq 1+\varepsilon$ and $\|T_y^{-1}\| \leq 1$ so in this case $d_{B-M}(C_0(X,E), C_0(Y,F)) = d_{B-M}(F,F)$ and*

$$\|Tf(y) - T_y(f \circ \varphi(y))\| \leq \varepsilon\|Tf\| \leq (1 + \varepsilon)\varepsilon\|f\|, \qquad f \in C_0(X, E),\ y \in Y;$$

(iii) *if Y is paracompact and $E = F$ then there is a canonical isomorphism $S\colon C_0(X, E) \to C_0(Y, E)$ with $\|S - T\| \leq 2\varepsilon\|T\|$,*

(iv) *if Y is paracompact and $E = F = H =$ Hilbert space then there is a canonical isometry $S\colon C_0(X, H) \to C_0(Y, H)$ with $\|S - T\| \leq 4\varepsilon/(1 - 4\varepsilon)$ (for $\varepsilon < \frac{1}{4}$).*

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