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# $M$-IDEALS OF COMPACT OPERATORS

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**Suppose $X$ is a reflexive Banach space and $Y$ is a closed subspace of a $c_0$-sum of finite dimensional Banach spaces. If $K(X, Y)$, the space of the compact linear operators from $X$ to $Y$, is dense in $L(X, Y)$, the space of the bounded linear operators from $X$ to $Y$, in the strong operator topology, then $K(X, Y)$ is an $M$-ideal in $L(X, Y)$.**

**1. Introduction.** Harmand and Lima [7] proved that if $X$ is a Banach space for which $K(X)$, the space of compact operators on $X$, is an $M$-ideal in $L(X)$, the space of continuous linear operators on $X$, then there exists a net $\{T_\alpha\}$ in $K(X)$ such that

(i) $||T_\alpha|| \leq 1$ for all $\alpha$,
(ii) $T_\alpha \to I_X$ strongly,
(iii) $T_\alpha^* \to I_{X^*}$ strongly,
(iv) $||I_X - T_\alpha|| \to 1$.

Thus, if $K(X)$ is an $M$-ideal in $L(X)$ then $X$ satisfies the metric compact approximation property.

Later Cho and Johnson [3] proved that if $X$ is a closed subspace of $(\sum_{n=1}^{\infty} X_n)_p$ ($\dim X_n < \infty$, $1 < p < \infty$) which has the compact approximation property, then $K(X)$ is an $M$-ideal in $L(X)$.

Recently Werner [15] obtained the same conclusion for a closed subspace $X$ of a $c_0$-sum of finite dimensional Banach spaces which has the metric compact approximation property. More specifically, he proved the following.

THEOREM. *If $X$ is a closed subspace of a $c_0$-sum of finite dimensional spaces, then the following are equivalent:*

(a) *$X$ has the metric compact approximation property.*
(b) *For each Banach space $W$, $K(W, X)$ is an $M$-ideal in $L(W, X)$.*
(c) *$K(X)$ is an $M$-ideal in $L(X)$.*

*Werner's proof* [15] *of the implication* (c) $\Rightarrow$ (a) *above can be used in the case of a pair of Banach spaces $X$ and $Y$ to prove that if $K(X, Y)$ is an $M$-ideal in $L(X, Y)$, then the closed unit ball of $K(X, Y)$ is dense in the closed unit ball of $L(X, Y)$ in the topology of uniform convergence on compact sets in $X$.*

The main result of this paper is Theorem 3. In Theorem 3 we will prove that if $X$ is a reflexive Banach space and $Y$ is a closed subspace of $(\sum Z_i)_{c_0}$, the $c_0$-sum of a family $\{Z_i : i \in I\}$ of finite dimensional Banach spaces, for which $K(X, Y)$ is dense in $L(X, Y)$ in the strong operator topology, then $K(X, Y)$ is an $M$-ideal in $L(X, Y)$. Thus, if either $X$ or $Y$ has the compact approximation property then $K(X, Y)$ is an $M$-ideal in $L(X, Y)$.

**2. Notation and preliminaries.** A closed subspace $J$ of a Banach space $X$ is said to be an $L$-summand if there exists a closed subspace $J'$ of $X$ such that $X$ is an algebraic direct sum of $J$ and $J'$, and also satisfies a norm condition $\|j + j'\| = \|j\| + \|j'\|$ for all $j \in J$ and $j' \in J'$. In this case we write $X = J \oplus_1 J'$. A closed subspace $J$ of a Banach space $X$ is called an $M$-ideal in $X$ if $J^0$, the annihilator of $J$ in $X^*$, is an $L$-summand in $X^*$.

If $X$ and $Y$ are Banach spaces, $L(X, Y)$ (resp. $K(X, Y)$) will denote the space of all bounded linear operators (resp. compact operators) from $X$ to $Y$. If $X = Y$, then we simply write $L(X)$ (resp. $K(X)$). Unless otherwise specified, these spaces are understood to be Banach spaces with operator norm.

If $X$ is a Banach space, $B_X$ will denote the closed unit ball of $X$. A Banach space $X$ is said to have the compact approximation property (resp. metric compact approximation property) if the identity operator on $X$ is in the closure of $K(X)$ (resp. $B_{K(X)}$) with respect to the topology of uniform convergence on compact sets in $X$.

If $\{Z_i : i \in I\}$ is a family of Banach spaces, the $c_0$-sum $(\sum Z_i)_{c_0}$ of $\{Z_i\}$ is the Banach space of all functions $z$ on $I$ with the properties that for $i \in I$, $z(i) \in Z_i$ and for any $\varepsilon > 0$ there exists a finite set $A \subseteq I$ such that $|z(i)| < \varepsilon$ for $i \in I \backslash A$. The norm on $(\sum Z_i)_{c_0}$ is the supremum norm. For a subset $A$ of $I$, the projection $P_A$ in $(\sum Z_i)_{c_0}$ associated with $A$ is defined by

$$(P_A z)(i) = \begin{cases} z(i) & \text{if } i \in A, \\ 0 & \text{if } i \notin A \text{ for } z \in (\sum Z_i)_{c_0}. \end{cases}$$

**3. $M$-ideals.** Alfsen and Effros **[1]** and Lima **[9]** characterized $M$-ideals by the intersection properties of balls. In this paper we will use the following characterization of $M$-ideals due to Lima **[9]**: A closed subspace $J$ of a Banach space $X$ is an $M$-ideal in $X$ if and only if for any $\varepsilon > 0$, for any $x \in B_X$ and for any $y_i \in B_J$ $(i = 1, 2, 3)$, there exists $y \in J$ such that $\|x + y_i - y\| \leq 1 + \varepsilon$ for $i = 1, 2, 3$.

The following theorem is essentially due to Werner [15] although he restricted attention to the case $X = Y$ and the identity map on $X$.

THEOREM 1. *Let $X$ and $Y$ be Banach spaces. If $K(X,Y)$ is an $M$-ideal in $L(X,Y)$, then $B_{K(X,Y)}$ is dense in $B_{L(X,Y)}$ in the topology of uniform convergence on compact sets in $X$.*

*Proof.* Suppose $K(X,Y)$ is an $M$-ideal in $L(X,Y)$ and suppose $L(X,Y)^* = K(X,Y)^\circ \oplus_1 J$ for a subspace $J$ of $L(X,Y)^*$. Then the map $\phi \to \phi + K(X,Y)^\circ$ defines an isometry from $J$ onto $L(X,Y)^*/K(X,Y)^\circ$ and hence the map $\phi \to \phi|_{K(X,Y)}$ defines an isometry from $J$ onto $K(X,Y)^*$ via $L(X,Y)^*/K(X,Y)^\circ$.

Let $Q$ be the projection from $L(X,Y)^*$ onto $J$. Then $\phi \in L(X,Y)^*$ is in the range of $Q$ if and only if the restriction of $\phi$ to $K(X,Y)$ has the same norm as $\phi$.

If $T \in L(X,Y) \subseteq L(X,Y)^{**}$ with $||T|| \leq 1$, then for $\phi = \phi_1 + \phi_2$ in $L(X,Y)^*$ with $\phi_1 \in K(X,Y)^\circ$ and $\phi_2 \in J$ we have

$$(Q^*T)\phi = TQ(\phi_1 + \phi_2) = T\phi_2.$$

Thus $Q^*T \in K(X,Y)^{\circ\circ} = J^* = K(X,Y)^{**}$.

Since $Q^*T \in K(X,Y)^{**}$ and $||Q^*T|| \leq 1$, by Goldstine's theorem there is a net $\{K_\alpha\}$ in $B_{K(X,Y)}$ such that

$$K_\alpha \to Q^*T \quad \text{in the weak}^*\text{-topology induced by } K(X,Y)^*.$$

Since for each $x \in X$ and each $y^* \in Y^*$, $y^* \otimes x$ is in the range of $Q$, we have

$$y^*(K_\alpha x) = K_\alpha(y^* \otimes x) \to (Q^*T)(y^* \otimes x) = y^*(Tx).$$

This shows that $T$ is in the closure of $B_{K(X,Y)}$ in the weak operator topology and hence in the strong operator topology.

The following theorem plays a key role in the proof of the main theorem.

THEOREM 2. *Let $X$ be a reflexive Banach space and let $Y$ be a closed subspace of $Z = (\sum Z_i)_{c_0}$, the $c_0$-sum of a family $\{Z_i : i \in I\}$ of finite dimensional Banach spaces. If $K(X,Y)$ is dense in $L(X,Y)$ in the strong operator topology, then for any $T \in B_{L(X,Y)}$ there exist nets $\{K_\alpha\}$ in $K(X,Y)$ and $\{Q_\alpha\}$ in $B_{K(X,Z)}$ such that $||T - Q_\alpha|| \leq ||T||$,*

$\|Q_\alpha - K_\alpha\| \to 0$ *and for any finite subset* $A$ *of* $I$ *there exists* $\alpha_0$ *such that* $P_A(T - Q_\alpha) = 0$ *for* $\alpha \geq \alpha_0$.

*Proof*. Let $T \in B_{L(X,Y)}$ and let $\{T_\beta\}$ be a net in $K(X,Y)$ such that $T_\beta \to T$ strongly. View $T$ and $T_\beta$'s as operators in $L(X,Z)$. Since $P_A T \to T$ strongly ($A$ ranges over the finite subsets of $I$), by replacing the index sets of $\{T_\beta\}$ and $\{P_A\}$ by the product directed set we have $T_\gamma - P_\gamma T \to 0$ strongly and there exists $r > 0$ such that $\|T_\gamma - P_\gamma T\| \leq r$ for all $\gamma$.

We claim that $T_\gamma - P_\gamma T \to 0$ weakly in $K(X,Z)$. $Z^* = (\sum Z_i^*)_{l^1}$ has the metric approximation property and the Radon Nikodym property [5, p. 219]. Since $Z$ is an $M$-ideal in $Z^{**}$ [7], $Z^*$ is complemented in $Z^{***}$ by norm one projection [7].

Thus $K(X,Z)^* = X^{**} \hat{\oplus} Z^* = X \hat{\otimes} Z^*$ [5, p. 247].

If $x \otimes z^* \in X \hat{\otimes} Z^*$ and $T = \sum_{i=1}^n x_i^* \otimes z_i$ is a finite rank operator from $X$ to $Z$,

$$(x \otimes z^*)(T) = \sum_{i=1}^n x(x_i^*) z^*(z_i) = z^*(Tx).$$

Since the finite rank operators are dense in $K(X,Z)$,

$$(x^{**} \otimes z^*)(T_\gamma - P_\gamma T) = x^{**}(T_\gamma - P_\gamma T)^* z^* \to 0.$$

Since $X \otimes Z^*$ is dense in $X \hat{\otimes} Z^* = K(X,Z)^*$, $T_\gamma - P_\gamma T \to 0$ weakly in $K(X,Z)$.

Since $T_\gamma - P_\gamma T \to 0$ weakly in $K(X,Z)$, there exists a net $\{K_\alpha - Q_\alpha\}$ of convex combinations of $\{T_\gamma - P_\gamma T\}$ which converges to zero in norm, where $K_\alpha$ is a convex combination of $T_\gamma$'s and $Q_\alpha$ is a convex combination of $P_\gamma T$'s. Moreover, we can choose the net $\{K_\alpha - Q_\alpha\}$ so that for any finite set $A$ of $I$, there exists $\alpha_0$ such that

$$P_A Q_\alpha = P_A T \quad \text{for all } \alpha \geq \alpha_0.$$

From the construction of $Q_\alpha$, it is obvious that $\|(T - Q_\alpha)x\| \leq \|Tx\|$ for all $\alpha$ and all $x \in X$.

THEOREM 3. *Let* $X, Y$ *and* $Z$ *be as in Theorem* 2. *If* $K(X,Y)$ *is dense in* $L(X,Y)$ *in the strong operator topology, then* $K(X,Y)$ *is an* $M$*-ideal in* $L(X,Y)$.

*Proof*. Let $S_1, S_2, S_3 \in B_{K(X,Y)}$ and $T \in B_{L(X,Y)}$. It suffices to show that for any $\varepsilon > 0$ there exists $K \in K(X,Y)$ such that

$$\|S_i + T - K\| \leq 1 + \varepsilon \qquad (i = 1, 2, 3).$$

Since $\bigcup_{i=1}^{3} S_i(B_X)$ has the compact closure in $Y$, there exists a finite subset $A$ of $I$ such that

$$\|S_i x - P_A S_i x\| < \tfrac{1}{2}\varepsilon \quad \text{for } x \in B_X \text{ and } i = 1, 2, 3.$$

By Theorem 2, choose a net $\{Q_\alpha\}$ in $B_{L(X,Y)}$, a net $\{K_\alpha\}$ in $K(X, Y)$ and $\alpha_0$ such that

$$P_A(T - Q_\alpha) = 0 \quad \text{and} \quad \|Q_\alpha - K_\alpha\| < \tfrac{1}{2}\varepsilon \quad \text{for } \alpha \geq \alpha_0$$

and

$$\|(T - Q_\alpha)x\| \leq \|Tx\| \quad \text{for all } x \in X \text{ and all } \alpha.$$

Then for $x \in B_X$, $\alpha \geq \alpha_0$ and $i = 1, 2, 3$, we have

$$\begin{aligned} \|S_i x + Tx - K_\alpha x\| &\leq \|P_A S_i x + (T - Q_\alpha)x\| + \tfrac{1}{2}\varepsilon + \tfrac{1}{2}\varepsilon \\ &= \max\{\|P_A S_i x\|, \|(T - Q_\alpha)x\|\} + \varepsilon \\ &\leq \max\{\|S_i x\|, \|Tx\|\} + \varepsilon \leq 1 + \varepsilon. \end{aligned}$$

Thus $\|S_i + T - K_\alpha\| \leq 1 + \varepsilon$ for $\alpha \geq \alpha_0$ and $i = 1, 2, 3$.

COROLLARY 4. *Let $X$ be a reflexive Banach space and let $Y$ be a closed subspace of $(\sum Z_i)_{c_0}$ $(\dim Z_i < \infty)$. If either $X$ or $Y$ has the compact approximation property, then $K(X, Y)$ is an $M$-ideal in $L(X, Y)$.*

*Proof.* Suppose $X$ has the compact approximation property. Let $0 \neq T \in L(X, Y)$ and let $K$ be a compact set in $X$. Then for any $\varepsilon > 0$ there exists a compact operator $T_1$ on $X$ such that $\|T_1 x - x\| \leq \varepsilon / \|T\|$ for all $x \in K$. Now $TT_1 \in K(X, Y)$ and $\|TT_1 x - Tx\| \leq \varepsilon$ for all $x \in K$. This shows that $K(X, Y)$ is dense in $L(X, Y)$ in the topology of uniform convergence on compact sets in $X$. By Theorem 3, $K(X, Y)$ is an $M$-ideal in $L(X, Y)$. The proof of the other case is similar.

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