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ON THE PROPAGATION OF DEPENDENCES

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An alternative proof of recent uniqueness theorems by Shanyu Ji is given. Ji's results are extended to the propagation of certain dependences from analytic subsets to the total space. Also these results are lifted from $\mathbb{C}^m$ to ramified covering spaces of $\mathbb{C}^m$. The first and second main theorems of value distribution are the essential tools in the proof.

Introduction. Let M be a connected, complex manifold of dimension m. Let $\pi: M \to \mathbb{C}^m$ be a proper, surjective, holomorphic map. Let $A_1, \ldots, A_q$ be pure $(m-1)$-dimensional analytic subsets of M with $\dim(A_i \cap A_j) \leq m-2$ whenever $i \neq j$. Define $A = A_1 \cap \cdots \cap A_q$. Let $E_1, \ldots, E_q$ be hyperplanes in general position in the projective space $\mathbb{P}_n$ with $n+1 < q$. Let p and k be integers with $2 \leq p \leq k \leq n+1$. For each $\lambda = 1, \ldots, k$ let $f_\lambda: M \to \mathbb{P}_n$ be a linearly nondegenerated meromorphic map. Assume that at least one of these maps f_λ grows quicker than the branching divisor of π. Assume that at least one of these maps f_λ has transcendental growth. For each $j = 1, \ldots, q$ assume that $f_\lambda^{-1}(E_j) = A_j$ does not depend on $\lambda = 1, \ldots, k$. Assume that for each collection of integers $1 \leq \lambda_1 < \lambda_2 < \cdots < \lambda_p \leq k$ the restricted maps $f_{\lambda_1}|A, \ldots, f_{\lambda_p}|A$ are not in general position. If

$$kn < (k-p+1)(q-n-1) \tag{0.1}$$

then $f_1, \ldots, f_k$ are not in general position (Theorem 4.2). This extends Theorem B of Shanyu Ji **[J1]** to parabolic covering spaces. He considers the case $M = \mathbb{C}^m, p = 2, k = 3$ and $q = 3n+1$ only. He concludes that f_1, f_2, f_3 satisfy a certain Property (P), which is perhaps a bit stronger but rather incomprehensible. Either condition implies algebraic dependence.

If each map $f_\lambda: M \to \mathbb{P}_n$ has rank n, condition (0.1) can be replaced by

$$k < (k-p+1)(q-n-1) \tag{0.2}$$

and we obtain the generalization of Ji's Theorem A (Theorem 6.2). Also Ji's Theorem C is extended (Theorem 6.1). Ji uses a special differential operator on $\mathbb{C}^m$ while we use the First Main Theorem for maps in general position. Then the proof becomes much shorter and clearer. The paper is self contained. The necessary concepts and results are explained to facilitate the reading of the paper. For the general theory of value distribution consult Stoll [S5], Stoll [S6], Stoll [S7], Stoll [S8] and Shabat [S1].

Historically the theory of uniqueness theorems began about 60 years ago. Many contributed. Some of the relevant papers are listed under "References". Smiley [S3], [S4] first considered the propagation of dependences from an analytic subset to the whole space.

1. General position. Let V be a complex vector space of dimension $n+1>1$. The vectors $\mathfrak{a}_1,\dots,\mathfrak{a}_k$ are said to be in *general position* if and only if for each selection of integers $1\leq j_0<j_1<\cdots<j_p\leq k$ with $p\leq n$, the vectors $\mathfrak{a}_{j_0},\dots,\mathfrak{a}_{j_p}$ are linearly independent, that is, if and only if

$$\mathfrak{a}_{j_0}\wedge\cdots\wedge\mathfrak{a}_{j_p}\neq 0. \tag{1.1}$$

If $k\leq n+1$, the vectors $\mathfrak{a}_1,\dots,\mathfrak{a}_k$ are in general position if and only if they are linearly dependent.

The vectors $\mathfrak{a}_1,\dots,\mathfrak{a}_k$ are said to be in *special position* if and only if they are not in general position. Take $p\in\mathbb{N}[1,k]$. Then $\mathfrak{a}_1,\dots,\mathfrak{a}_k$ are said to be in *p-special position* if and only if for each selection $1\leq j_1<\cdots<j_p\leq k$, the vectors $\mathfrak{a}_{j_1},\dots,\mathfrak{a}_{j_p}$ are in special position. If $p=1$, this means $\mathfrak{a}_j=0$ for $j=1,\dots,k$. If $p\leq n+1$, this means $\mathfrak{a}_{j_1},\dots,\mathfrak{a}_{j_p}$ are linearly dependent. If $1\leq q<p\leq k$ and if $\mathfrak{a}_1,\dots,\mathfrak{a}_k$ are in q-special position, then they are in p-special position. Also k-special position is the same as special position.

Put $V_*=V-\{0\}$. Let $\mathbb{P}(V)=V_*/\mathbb{C}_*$ be the complex projective space associated to V. Let $\mathbb{P}\colon V_*\to\mathbb{P}(V)$ be the residual map. For $A\subseteq V$ define $\mathbb{P}(A)=\{\mathbb{P}(\mathfrak{x})|0\neq\mathfrak{x}\in A\}$. Take $a_1,\dots,a_k$ in $\mathbb{P}(V)$. Then $a_j=\mathbb{P}(\mathfrak{a}_j)$ with $\mathfrak{a}_j\in V_*$ for $j=1,\dots,k$. The points $a_1,\dots,a_k$ are said to be in *general position* (respectively *special position*, respectively *p-special position*) if and only if $\mathfrak{a}_1,\dots,\mathfrak{a}_k$ are in general position (respectively special position, respectively p-special position). If $a_1,\dots,a_k$ are in p-special position, then $2\leq p\leq k$. Obviously $a_1,\dots,a_k$ are in 2-special position if and only if $a_1=a_2=\cdots=a_k$. Take $a_1,\dots,a_k$ in general position in $\mathbb{P}(V)$ with $1\leq k\leq n+1$. Take $\mathfrak{a}_j\in V_*$ with $\mathbb{P}(\mathfrak{a}_j)=a_j$ for $j=1,\dots,k$. Then $\mathfrak{a}_1\wedge\cdots\wedge\mathfrak{a}_k\neq 0$. Define

$$a_1\wedge\cdots\wedge a_k=\mathbb{P}(\mathfrak{a}_1\wedge\cdots\wedge\mathfrak{a}_k)\in\mathbb{P}\left(\bigwedge_k V\right). \tag{1.2}$$

These definitions do not depend on the choice of the representatives $\mathfrak{a}_j$.

Let V^* be the dual vector space of V. Take $a \in \mathbb{P}(V^*)$. Then $\mathfrak{a} \in V_*^*$ exists with $\mathbb{P}(\mathfrak{a}) = a$. Here $\mathfrak{a}: V \to \mathbb{C}$ is a linear map. The kernel $\ker \mathfrak{a}$ depends on a only and is a n-dimensional linear subspace of V. Then $E[a] = \mathbb{P}(\ker \mathfrak{a})$ is a hyperplane in $\mathbb{P}(V)$. Thus $\mathbb{P}(V^*)$ provides a bijective enumeration of all the hyperplanes in $\mathbb{P}(V)$. Take $a_1, \dots, a_k$ in $\mathbb{P}(V^*)$. Then $E[a_1], \dots, E[a_k]$ are said to be in *general position* if and only if $a_1, \dots, a_k$ are in general position.

Let $(|): V \times V \to \mathbb{C}$ be a positive definite hermitian form on V. It is called a *hermitian metric* on V. Also V together with such a hermitian metric is called a *hermitian vector space.* The associated *norm* is defined by $\|\mathfrak{x}\| = (\mathfrak{x}|\mathfrak{x})^{1/2}$ for all $\mathfrak{x} \in V$. The given hermitian metric on V defines associated hermitian metrics on V^* and $\bigwedge_k V$. If $a_j = \mathbb{P}(\mathfrak{a}_j) \in \mathbb{P}(V)$ for $j = 1, \dots, k$, then

$$(1.3) \qquad \Box a_1 \dot{\wedge} \cdots \dot{\wedge} a_k \Box = \frac{\|\mathfrak{a}_1 \wedge \cdots \wedge \mathfrak{a}_k\|}{\|\mathfrak{a}_1\| \cdots \|\mathfrak{a}_k\|}$$

depends on $a_1, \dots, a_k$ only with $0 \le \Box a_1 \dot{\wedge} \cdots \dot{\wedge} a_k \Box \le 1$. The dots over $\wedge$ indicate that $\Box \cdots \Box$ is not a function of $a_1 \wedge \cdots \wedge a_k$ as defined in (1.2). In fact $\Box a_1 \dot{\wedge} \cdots \dot{\wedge} a_k \Box \ne 0$ for $k \le n+1$, if and only if $a_1, \dots, a_k$ are in general position.

An inner product $\langle \mathfrak{x}, \mathfrak{a} \rangle$ between $\mathfrak{x} \in V$ and $\mathfrak{a} \in V^*$ is defined by $\langle \mathfrak{x}, \mathfrak{a} \rangle = \mathfrak{a}(\mathfrak{x}) \in \mathbb{C}$. If $x = \mathbb{P}(\mathfrak{x}) \in \mathbb{P}(V)$ and $a = \mathbb{P}(\mathfrak{a}) \in \mathbb{P}(V^*)$, the *distance from x to $E[a]$* is defined by

$$(1.4) \qquad \Box x, a \Box = \frac{|\langle \mathfrak{x}, \mathfrak{a} \rangle|}{\|\mathfrak{x}\| \|\mathfrak{a}\|}$$

where $0 \le \Box x, a \Box < 1$. The distance $\Box x, a \Box$ depends on x and a only. Here $\Box x, a \Box = 0$ if and only if $x \in E[a]$.

These concepts shall be extended to meromorphic maps. Let M and N be connected, complex manifolds of dimension m and n respectively. Let S be an analytic subset of M with $S \ne M$. Let $f: M - S \to N$ be holomorphic. The closure $\Gamma(f)$ of $\{(x, f(x)) | x \in M - S\}$ in $M \times N$ is called the *closed graph* of f. Let $\pi: \Gamma(f) \to M$ and $\tilde{f}: \Gamma(f) \to N$ be the projections defined by $\pi(x, y) = x$ and $\tilde{f}(x, y) = y$ for all $(x, y) \in \Gamma(f)$. Then f is said to be *meromorphic* on M if and only if π is proper and $\Gamma(f)$ is analytic. Assume that f is meromorphic. Define $m = \dim M$. Then the *indeterminacy.*

$$(1.5) \qquad I(f) = \{x \in M | \# \pi^{-1}(x) > 1\}$$

is analytic with $\dim I(f) \leq m-2$. The map f extends to a holomorphic map $f\colon M - I(f) \to N$, but does not continue holomorphically to any larger open subset of M. If $A \subseteq M$ and $B \subseteq N$ define

$$(1.6) \qquad f(A) = \tilde{f}(\pi^{-1}(A)), \quad f^{-1}(B) = \pi(\tilde{f}^{-1}(B)).$$

Let V be a complex vector space of dimension $n+1$. If $N = \mathbb{P}(V)$, an alternative definition for a holomorphic map $f\colon M - S \to N$ to be meromorphic on M is available: Let $U \neq \varnothing$ be an open, connected subset of M. A holomorphic map $0 \neq \mathfrak{v}\colon U \to V$ is called a *representation* (at $p \in M$ if $p \in U$) of f if and only if $f(x) = \mathbb{P}(\mathfrak{v}(x))$ for all $x \in U - S$ with $\mathfrak{v}(x) \neq 0$. The map f is meromorphic, if and only if there is a representation of f at every point of M. A representation $\mathfrak{v}\colon U \to V$ is said to be *reduced* if and only if $\dim \mathfrak{v}^{-1}(0) \leq m-2$, which is equivalent to $I(f) \cap U = \mathfrak{v}^{-1}(0)$ if f is meromorphic. If $\mathfrak{v}_j\colon U_j \to V$ are representations of f for $j = 1, 2$ with $U_1 \cap U_2 \neq \varnothing_1$ then there is a meromorphic function $h\colon U_1 \cap U_2 \to \mathbb{C}$ such that $\mathfrak{v}_2 = h\mathfrak{v}_1$ on $U_1 \cap U_2$. If $\mathfrak{v}_1$ is reduced, h is holomorphic; if $\mathfrak{v}_1$ and $\mathfrak{v}_2$ are reduced, h is holomorphic and without zeroes.

For $j = 1, \ldots, k$ let $f_j\colon M \to \mathbb{P}(V)$ be a meromorphic map. Put $I = I(f_1) \cup \cdots \cup I(f_k)$. Then $f_1, \ldots, f_k$ are said to be in *general position* if and only if there is a point $x \in M - I$ such that $f_1(x), \ldots, f_k(x)$ are in general position. If so, this is true for all $x \in M - S$, where S is analytic with $I \subseteq S \neq M$. Let $\mathfrak{v}_j\colon U \to V$ be a representation of f_j for $j = 1, \ldots, k$. If $k \leq n+1$, then $f_1, \ldots, f_k$ are in general position if and only if $\mathfrak{v}_1 \wedge \cdots \wedge \mathfrak{v}_k \neq 0$. If so, one and only one meromorphic map

$$(1.7) \qquad f_1 \wedge \cdots \wedge f_k\colon M \to \mathbb{P}\left(\bigwedge_k V\right)$$

is defined by

$$(1.8) \qquad (f_1 \wedge \cdots \wedge f_k)(x) = f_1(x) \wedge \cdots \wedge f_k(x) \quad \text{for all } x \in M - S.$$

Take $k \in \mathbb{N}$ and $p \in \mathbb{N}[1, k]$. Let $f_j\colon M \to \mathbb{P}(V)$ be meromorphic maps for $j = 1, \ldots, k$. Take $x \in M$. Let $\mathfrak{v}_j\colon U \to V$ be a reduced representation of f_j at x for $j = 1, \ldots, k$. Then $f_1, \ldots, f_k$ are said to be in *p-special position at x* if and only if $\mathfrak{v}_1(x), \ldots, \mathfrak{v}_k(x)$ are in p-special position at x. This definition does not depend on the choice of the reduced representations $\mathfrak{v}_j$. If $Q \neq \varnothing$ is a subset of M, then $f_1, \ldots, f_k$ are in *p-special position on Q*, if and only if they are in *p-special position* at every point of Q. If $Q = M$ omit "on Q". Also "*special position*" means "k-special position". Obviously, $f_1, \dots, f_k$ are in special position if and only if they are not in general position.

If S is a set $S^k = S \times \cdots \times S$ (k-times). Let N be a connected, complex manifold. For $j = 1, \dots, k$ let $f_j \colon M \to N$ be meromorphic maps. Put $I = I(f_1) \cup \cdots \cup I(f_k)$. Then $f_1, \dots, f_k$ are said to be *algebraically dependent* if there exists an analytic subset G of N^k with $G \neq N^k$ such that $(f_1(x), \dots, f_k(x)) \in G$ for all $x \in M - I$.

PROPOSITION 1.1. *Let M be a connected, complex manifold of dimension m. Let V be a complex vector space of dimension $n+1$. For each $j \in \mathbb{N}[1,k]$ let $f_j \colon M \to \mathbb{P}(V)$ be a meromorphic map where $k \leq n+1$. If $f_1, \dots, f_k$ are in special position, then $f_1, \dots, f_k$ are algebraically dependent.*

Proof. Since $k \leq n+1$, an analytic subset $\tilde{G}_k$ of V^k with $\tilde{G}_k \neq V^k$ is defined by

$$\tilde{G}_k = \{(\mathfrak{x}_1, \dots, \mathfrak{x}_k) \in V^k \mid \mathfrak{x}_1 \wedge \cdots \wedge \mathfrak{x}_k = 0\}. \tag{1.9}$$

A surjective holomorphic map $\mathbb{P}^k \colon (V_*)^k \to \mathbb{P}(V)^k$ is defined by

$$\mathbb{P}^k(\mathfrak{x}_1, \dots, \mathfrak{x}_k) = (\mathbb{P}(\mathfrak{x}_1), \dots, \mathbb{P}(\mathfrak{x}_k)) \tag{1.10}$$

for all $(\mathfrak{x}_1, \dots, \mathfrak{x}_k) \in (V_*)^k$. If $0 \neq \lambda_j \in \mathbb{C}$ and $(\mathfrak{x}_1, \dots, \mathfrak{x}_k) \in \tilde{G}_k \cap (V_*)^k$, then $(\lambda_1 \mathfrak{x}_1, \dots, \lambda_k \mathfrak{x}_k) \in \tilde{G}_k \cap (V_*)^k$. Hence $G_k = \mathbb{P}(\tilde{G}_k)$ is an analytic subset of $\mathbb{P}(V)^k$ with $G_k \neq \mathbb{P}(V)^k$.

Define $I = I(f_1) \cup \cdots \cup I(f_k)$. Take $x \in M - I$. Let $\mathfrak{v}_j \colon U \to V$ be a reduced representation of f_j at x for $j = 1, \dots, k$. Since $k \leq n+1$ and since $f_1, \dots, f_k$ are in special position $\mathfrak{v}_1 \wedge \cdots \wedge \mathfrak{v}_k \equiv 0$. Hence $\mathfrak{v}_1(x) \wedge \cdots \wedge \mathfrak{v}_k(x) = 0$. Thus $(\mathfrak{v}_1(x), \dots, \mathfrak{v}_k(x)) \in \tilde{G}_k \cap (V_*)^k$. Hence $(f_1(x), \dots, f_k(x)) \in G_k$. Thus $f_1, \dots, f_k$ are algebraically dependent. □

The *rank* of a holomorphic map is explained in **[A1]**. Let M and N be connected, complex manifolds. Let $f \colon M \to N$ be a meromorphic map. Let $\pi \colon \Gamma(f) \to M$ and $\tilde{f} \colon \Gamma(f) \to N$ be the projections. Define $\operatorname{rank} f = \operatorname{rank} \tilde{f}$. Then $\operatorname{rank} f = \dim N$ if and only if $f(M - I(f))$ contains an interior point.

PROPOSITION 1.2. *Let V be a finite dimensional complex vector space. Let N be a connected, n-dimensional, compact, complex submanifold of $\mathbb{P}(V)$ such that N is not contained in any hyperplane of $\mathbb{P}(V)$. Let M be a connected, complex manifold of dimension m. Take $k \in \mathbb{N}$*

with $k \leq n+1$. For each $j = 1, \ldots, k$, let $f_j\colon M \to N$ be a meromorphic map. Let $\imath\colon N \to \mathbb{P}(W)$ be the inclusion map. Assume that $f_1, \ldots, f_k$ are algebraically independent. Then $\imath \circ f_1, \ldots, \imath \circ f_k$ are in general position.

Proof. If $s \in \mathbb{N}[1,k]$, the set $\tilde{G}_s$ is analytic in V^s and $G_s = \mathbb{P}^s(\tilde{G}_s)$ is analytic in $\mathbb{P}(V)^s$. Hence $D_s = N^s \cap G_s$ is analytic in N^s. Abbreviate $g_j = \imath \circ f_j$ for $j = 1, \ldots, k$. Assume that $g_1, \ldots, g_k$ are in special position. A smallest integer p exists such that $g_1, \ldots, g_k$ are in p-special position. Then $2 \leq p \leq k$. We re-enumerate such that $g_1, \ldots, g_{p-1}$ are in general position. If $p < k$, put $A = D_p \times N^{k-p}$; if $p = k$, put $A = D_p$. Then A is analytic in N^k. The set $I = I(f_1) \cup \cdots \cup I(f_k)$ is analytic in M with $\dim I \leq m-2$. Take $x \in M - I$. We claim

$$(f_1(x), \ldots, f_k(x)) \in A \neq N^k. \tag{1.11}$$

There is an open, connected neighborhood U of x in $M-I$ such that there is a reduced representation $\mathfrak{v}_j\colon U \to V_*$ of g_j for $j = 1, \ldots, k$. Because $g_1, \ldots, g_{p-1}$ are in general position, $z \in U$ exists with $\mathfrak{v}_1(z) \wedge \cdots \wedge \mathfrak{v}_{p-1}(z) \neq 0$. The linearly independent vectors $\mathfrak{v}_1(z), \ldots, \mathfrak{v}_{p-1}(z)$ span a complex linear subspace L of V with

$$\dim L = p - 1 < p \leq k < n+1 < \dim V. \tag{1.12}$$

Thus $N \subsetneq \mathbb{P}(L)$. Take $w = \mathbb{P}(\mathfrak{w}) \in N - \mathbb{P}(L)$ with $\mathfrak{w} \in V - L$, which implies $\mathfrak{v}_1(x) \wedge \cdots \wedge \mathfrak{v}_{p-1}(z) \wedge \mathfrak{w} \neq 0$. Thus $(f_1(z), \ldots, f_{p-1}, w) \in N^p - D_p$. Therefore

$$(f_1(z), \ldots, f_{p-1}(z), w, f_{p+1}(z), \ldots, f_k(z)) \in N^k - A. \tag{1.13}$$

Therefore $A \neq N^k$.

Because $g_1, \ldots, g_p$ are not in general position, $\mathfrak{v}_1(x) \wedge \cdots \wedge \mathfrak{v}_p(x) = 0$. Hence $(f_1(x), \ldots, f_p(x)) \in D_p$ and $(f_1(x), \ldots, f_k(x)) \in A$. The claim is proved. Thus $f_1, \ldots, f_k$ are algebraically dependent contrary to the assumption. Consequently, $g_1, \ldots, g_k$ are in general position. □

2. Divisors. Let M be a connected, complex manifold of dimension m. Let $\mathfrak{O}$ be the sheaf of germs of holomorphic functions on M. For each $a \in M$, the stalk $\mathfrak{O}_a$ of $\mathfrak{O}$ over a is an integral domain with unique prime factorization and with a unique maximal ideal $\mathfrak{m}_a$. For $p \in \mathbb{N}$, let $\mathfrak{m}_a^p$ be the pth power of $\mathfrak{m}_a$. Put $\mathfrak{m}_a^0 = \mathfrak{O}_a$. Take $0 \neq f \in \mathfrak{O}_a$. One and only one non-negative integer $\mu(f)$ exists with

$$f \in \mathfrak{m}_a^{\mu(f)} - \mathfrak{m}_a^{\mu(f)+1}. \tag{2.1}$$

The number $\mu(f)$ is called the *zero-multiplicity of* f. If $0 \neq f \in \mathfrak{O}_a$, and $0 \neq g \in \mathfrak{O}_a$, then

$$\mu(fg) = \mu(f) + \mu(g), \tag{2.2}$$

$$\mu(f+g) \geq \operatorname{Min}(\mu(f), \mu(g)) \quad \text{if } f+g \neq 0. \tag{2.3}$$

If $\mu(f) \neq \mu(g)$, equality holds in (2.3). If $g \in \mathfrak{O}_a - \mathfrak{m}_a$, then (2.2) implies

$$\mu(fg) = \mu(f). \tag{2.4}$$

Let $U \neq \emptyset$ be an open, connected subset of M. Let $f \neq 0$ be a holomorphic function on U. For each $z \in U$, the germ $f_z \neq 0$ of f at z is defined. A function $\mu_f^0 \colon U \to \mathbb{Z}_+$ called the *zero divisor of* f is defined by $\mu_f^0(z) = \mu(f_z)$ for $z \in U$.

Let $\nu \colon M \to \mathbb{Z}$ be an integral valued function. Then (U, g, h) is called a *Cousin definition of* ν (at a if $a \in U$) if and only if U is an open, connected subset of M and if $g \neq 0$ and $h \neq 0$ are holomorphic functions on U with $\nu | U = \mu_g^0 - \mu_h^0$ and with $\dim(g^{-1}(0) \cap h^{-1}(0)) \leq m-2$. The function $\nu \colon M \to \mathbb{Z}$ is said to be a *divisor* on M if and only if there is a Cousin definition of ν at every point of M. If (U_j, g_j, h_j) are Cousin definitions of ν for $j = 1, 2$ and if $U = U_1 \cap U_2$, then there exists a holomorphic function k without zeros on U such that $g_2 = kg_1$ and $h_2 = kh_1$ on U. The divisor ν is non-negative (as a function) if and only if there is a Cousin definition $(U, g, 1)$ at every point of M, that is for each $a \in M$, there is an open, connected neighborhood U of a and a holomorphic function $g \neq 0$ on U such that $\nu | U = \mu_g^0$.

If A is an analytic subset of M, the set $\mathfrak{R}(A)$ of regular points of A is open and dense in the topological space A. The set $\Sigma(A) = A - \mathfrak{R}(A)$ of singular points of A is analytic in M and nowhere dense in A. If A is a pure $(m-1)$-dimensional analytic subset of M, one and only one divisor ν_A on M exists with $\nu_A(x) = 1$ for all $x \in \mathfrak{R}(A)$ and $\nu_A(x) = 0$ for all $x \in M - A$. Then $\nu_A \geq 0$ on M.

The set $\mathfrak{D}_M$ of all divisors on M is a module under function addition. The zero element of $\mathfrak{D}_M$ is the *null-divisor* $\nu \equiv 0$. Take $\nu \in \mathfrak{D}_M$. The closure $\operatorname{supp} \nu$ of $\{x \in M | \nu(x) \neq 0\}$ is called the *support of* ν. Then $\operatorname{supp} \nu = \emptyset$ if and only if $\nu \equiv 0$. If $\nu \neq 0$, then $S = \operatorname{supp} \nu$ is a pure $(m-1)$-dimensional analytic subset of M. Here $\nu | \mathfrak{R}(S)$ is locally constant. Let $\mathfrak{B}$ be the set of branches of S. Then $\{\mathfrak{R}(S) \cap B\}_{B \in \mathfrak{B}}$ is the family of connectivity components of $\mathfrak{R}(S)$. Each $B \in \mathfrak{B}$ is the closure of $\mathfrak{R}(S) \cap B$. For each $B \in \mathfrak{B}$, there is a unique integer

$p(\nu, B) \neq 0$ such that

$$(2.5) \qquad \nu|(\mathfrak{R}(S) \cap B) = p(\nu, B).$$

The locally finite sum

$$(2.6) \qquad \nu = \sum_{B \in \mathfrak{B}} p(\nu, B)\nu_B$$

is called the *analytic chain representation* of ν. If $n \in \mathbb{Z}$, the divisor

$$(2.7) \qquad \nu^{(n)} = \sum_{B \in \mathfrak{B}} \operatorname{Min}(n, p(\nu, B))\nu_B$$

is called *the truncation of ν at level n*. Here $0 \neq \nu \geq 0$ if and only if $p(\nu, B) > 0$ for all $B \in \mathfrak{B} \neq \varnothing$ which is the case if and only if

$$(2.8) \qquad \operatorname{supp} \nu = \{x \in M | \nu(x) > 0\} \neq \varnothing.$$

Let E be an analytic subset of M with $\dim E \leq m - 2$. For each divisor $\nu\colon M - E \to \mathbb{Z}$ there is one and only one divisor $\hat{\nu}\colon M \to \mathbb{Z}$ with $\hat{\nu}|(M - E) = \nu$. If $\nu \geq 0$, then $\hat{\nu} \geq 0$. The map $\nu \to \hat{\nu}$ defines an isomorphism $\mathfrak{D}_{M-E} \to \mathfrak{D}_M$. Thus if ν_1 and ν_2 are divisors on M with $\nu_1|(M - E) = \nu_2|(M - E)$, then $\nu_1 = \nu_2$ on M.

Let N be a connected, complex manifold of dimension n. Let $f\colon M \to N$ be a meromorphic map. Let $\nu\colon N \to \mathbb{Z}$ be a divisor on N with $f(M - I(f)) \subsetneq \operatorname{supp} \nu$. Then there exists one and only one divisor $f^*(\nu)$ on M called the *pull back divisor* satisfying the following condition:

(C) Let $U \neq \varnothing$ be an open, connected subset of $M - I(f)$. Let (W, g, h) be a Cousin definition of ν with $f(U) \subseteq W$. Then $g \circ f|U \neq 0 \neq h \circ f|U$ and

$$(2.9) \qquad f^*(\nu)|U = \mu^0_{g\circ f} - \mu^0_{h\circ f}|U.$$

If $\nu \geq 0$, then $f^*(\nu) \geq 0$. If $f^*(\nu_j)$ exists for $j = 1, 2$, then $f^*(\nu_1 + \nu_2)$ exists with $f^*(\nu_1) + f^*(\nu_2) = f^*(\nu_1 + \nu_2)$. If A is a pure $(n-1)$-dimensional, analytic subset of N with $f(M - I(f)) \subsetneq A$, abbreviate $f^*(\nu_A) = f^*(A)$ and $f^*(a) = f^*(\{a\})$ if $A = \{a\}$.

Now, we will introduce various divisors which will be needed later on. Let $\mathbb{P}_1 = \mathbb{P}(\mathbb{C}^2) = \mathbb{C} \cup \{\infty\}$ be the Riemann sphere. A meromorphic function on M is a meromorphic map $f\colon M \to \mathbb{P}_1$ with $f \neq \infty$. Take $b \in \mathbb{P}_1$ with $f \neq b$. Then $f^*(b)$ is a non-negative divisor on M called the *b-divisor* of f. If f is a holomorphic function on M, then $\mu^0_f = f^*(0)$. Hence we denote $f^*(b) = \mu^b_f$.

Let V be a complex vector space of dimension $n+1$. Let $f: M \rightarrow \mathbb{P}(V)$ and $g: M \rightarrow \mathbb{P}(V^*)$ be meromorphic maps. Then f, g are said to be *free* if there exists a point $x \in M-(I(f) \cup I(g))$ such that $f(x) \notin E[g(x)]$. If f, g are free, then one and only one divisor $\mu_{f,g} \geq 0$ called the *intersection divisor* is defined by the following condition.

(I) Let $U \neq \varnothing$ be an open, connected subset of M with reduced representations $\mathfrak{v}: U \rightarrow V$ of f and $\mathfrak{w}: U \rightarrow V^*$ of g. Since f, g are free, $\langle \mathfrak{v}, \mathfrak{w} \rangle \not\equiv 0$. Then $\mu_{f,g}|U = \mu^0_{\langle \mathfrak{v}, \mathfrak{w} \rangle}$.

If $g \equiv a$ is constant, $\mu_{f,a}$ is also called the *intersection divisor of f with $E[a]$* and we have $\mu_{f,a} = f^*(E[a])$.

Let s be a holomorphic section of a holomorphic vector bundle W over M. The *zero set* $Z(s) = \{x \in M | s(x) = 0_x \in W_x\}$ is analytic. Here $Z(s) = M$ if and only if $s \equiv 0$. Assume that $s \not\equiv 0$. Let $\mathfrak{A} = \{(U_\lambda, t_\lambda, h_\lambda)\}_{\lambda \in \Lambda}$ be the family of all triples $(U_\lambda, t_\lambda, h_\lambda)$ where $U_\lambda \neq \varnothing$ is an open, connected subset of M, where $h_\lambda \not\equiv 0$ is a holomorphic function on U_λ and where t_λ is a holomorphic section of W over U_λ with $\dim Z(t_\lambda) \leq m-2$ such that $s|U_\lambda = h_\lambda t_\lambda$. Then $\{U_\mu\}_{\mu \in \Lambda}$ is a covering of M. If $\mu \in \Lambda$ and $\zeta \in \Lambda$ with $U = U_\mu \cap U_\zeta \neq \varnothing$, then $\mu^0_{h_\mu}|U = \mu^0_{h_\zeta}|U$. Hence one and only one divisor μ_s called the *zero divisor* of s exists on M such that $\mu_s|U_\lambda = \mu^0_{h_\lambda}$ for all $\lambda \in \Lambda$. Obviously $\mu_s \geq 0$ and $\operatorname{supp} \mu_s \subseteq Z(s)$ with $\dim_x Z(s) \leq m-2$ if $x \in Z(s) - \operatorname{supp} \mu_s$. If W is a line bundle, $\operatorname{supp} \mu_s = Z(s)$ and $Z(t_\lambda) = \varnothing$ for all $\lambda \in \Lambda$.

Let $\mathfrak{T}(M)$ be the holomorphic cotangent bundle on M. Take $p \in \mathbb{N}[1, m]$. A holomorphic form φ of degree p is nothing but a holomorphic section of $\bigwedge_p \mathfrak{T}(M)$. Hence $\mu_\varphi \geq 0$ is defined if $\varphi \not\equiv 0$. Recall that $K_M = \bigwedge_m \mathfrak{T}(M)$ is the *canonical bundle* of M. It is a holomorphic line bundle.

Let M and N be connected complex manifolds of dimension m. Let $f: M \rightarrow N$ be a holomorphic map of rank m. If $U \neq \varnothing$ is open and connected in M and W is open and connected in N with $f(U) \subseteq W$ and if $\varphi \not\equiv 0$ is any holomorphic form of degree m on N, then $f^*(\varphi) \not\equiv 0$. There exists one and only one divisor β called the *branching divisor of β* such that the following property is satisfied:

(B) Let $U \neq \varnothing$ be open and connected in M. Let W be open and connected in N with $f(U) \subseteq W$. Let φ be a holomorphic form of degree m on N with $Z(\varphi) = \varnothing$. Then $\beta|U = \mu_{f^*(\varphi)|U}$.

Obviously $\beta \geq 0$. If $B = \operatorname{supp} \beta$, then f is locally biholomorphic at $x \in M$ if and only if $x \in M - B$.

Let V be a complex vector space. Let $\mathfrak{v}: M \to V$ be a holomorphic vector function with $\mathfrak{v} \not\equiv 0$. Then $M \times V$ is a trivial line bundle. A holomorphic section $\tilde{\mathfrak{v}}$ of $M \times V$ is defined by $\tilde{\mathfrak{v}}(x) = (x, \mathfrak{v}(x))$ for all $x \in M$. Then $\tilde{\mathfrak{v}} \not\equiv 0$ if and only $\mathfrak{v} \not\equiv 0$. Assume that $\mathfrak{v} \not\equiv 0$. Then the *zero divisor* $\mu_{\mathfrak{v}} = \mu_{\tilde{\mathfrak{v}}} \geq 0$ is defined with $\operatorname{supp} \mu_{\mathfrak{v}} \subseteq \mathfrak{v}^{-1}(0)$. If $x \in \mathfrak{v}^{-1}(0) - \operatorname{supp} \mu_{\mathfrak{v}}$ then $\dim_x \mathfrak{v}^{-1}(0) \leq m - 2$.

Let V be a complex vector space of dimension $n + 1$. Take $k \in \mathbb{N}[1, n+1]$. Let $f_j: M \to \mathbb{P}(V)$ be a meromorphic map for $j = 1, \ldots, k$. Assume that $f_1, \ldots, f_k$ are in general position. Then there exists one and only one divisor $\mu(f_1 \dot\wedge \cdots \dot\wedge f_k)$ on M satisfying the following condition:

(G) Let $U \neq \varnothing$ be an open, connected subset of M. Let $\mathfrak{v}_j: U \to V$ be a reduced representation of f_j for $j = 1, \ldots, k$. Since $f_1, \ldots, f_k$ are in general position with $k \leq n + 1$, the vector function $\mathfrak{w} = \mathfrak{v}_1 \wedge \cdots \wedge \mathfrak{v}_k: U \to \bigwedge_k: V$ is not identically zero. Then

$$\mu(f_1 \dot\wedge \cdots \dot\wedge f_k) | U = \mu_{\mathfrak{w}}. \tag{2.10}$$

Obviously $\mu(f_1 \dot\wedge \cdots \dot\wedge f_k) \geq 0$. The dots indicate that μ is not a function of $f_1 \wedge \cdots \wedge f_k$ as defined in (1.7) and (1.8) but of the k-tuple $f_1, \ldots, f_k$.

The following theorem is the fundament of our proof of Ji's theorems.

THEOREM 2.1. *Let M be a connected complex manifold of dimension m. Let A be a pure $(m-1)$-dimensional, analytic subset of M. Let V be a complex vector space of dimension $n+1 > 1$. Let p and k be integers with $1 \leq p \leq k \leq n+1$. For each $j = 1, \ldots, k$, let $f_j: M \to \mathbb{P}(V)$ be a meromorphic map. Assume that $f_1, \ldots, f_k$ are in general position. Also assume that $f_1, \ldots, f_k$ are in p-special position on A. Then we have*

$$(k - p + 1)\nu_A \leq \mu(f_1 \dot\wedge \cdots \dot\wedge f_k). \tag{2.11}$$

REMARK TO THEOREM 2.1. The assumptions imply $p \geq 2$. If $p = 2$, then $f_1, \ldots, f_k$ are in 2-special position on A if and only if

$$f_1 | A = f_2 | A = \cdots = f_k | A. \tag{2.12}$$

In his paper **[J1]** Ji considers only the case $p = 2$.

Proof. Since $\nu_A | (M - A) = 0$ and $\mu(f_1 \dot\wedge \cdots \dot\wedge f_k) \geq 0$ it suffices to prove (2.11) on A. Again by the properties of divisors, it suffices to prove (2.11) on $A - I$ where I is an analytic subset of M with $\dim I \leq m-2$. Abbreviate $S = \operatorname{supp}\mu(f_1 \dot\wedge \cdots \dot\wedge f_k)$. Then

$$I = \Sigma(S) \cup \Sigma(A) \cup I(f_1 \wedge \cdots \wedge f_k) \cup \bigcup_{j=1}^{k} I(f_j) \tag{2.13}$$

is an analytic subset of M with $\dim I \leq m-2$.

Take any $x \in A - I$. Then there exists an open, connected neighborhood U of x with $U \cap I = \varnothing$ and reduced representations $\mathfrak{v}_j \colon U \to V$ of f_j for $j = 1, \dots, k$ and $\mathfrak{y} \colon U \to \bigwedge_k V$ of $f_1 \wedge \cdots \wedge f_k$. Since $U \cap I = \varnothing$, we have $\mathfrak{y}(z) \neq 0$ and $\mathfrak{v}_j(z) \neq 0$ for $j = 1, \dots, k$ for all $z \in U$. Also their exists a unique holomorphic function h on U such that $h\mathfrak{y} = \mathfrak{v}_1 \wedge \cdots \wedge \mathfrak{v}_k$. Then $\mu(f_1 \dot\wedge \cdots \dot\wedge f_k)|U = \mu_h^0$ and $S \cap U = h^{-1}(0)$. Since $f_1, \dots, f_k$ are in p-special position on A, we have $\mathfrak{v}_1(z) \wedge \cdots \wedge \mathfrak{v}_p(z) = 0$ and $\mathfrak{v}_1(z) \wedge \cdots \wedge \mathfrak{v}_k(z) = 0$ for all $z \in U \cap A$. Since $\mathfrak{y}(z) \neq 0$ for $z \in U \cap A$, we obtain $h(z) = 0$ for all $z \in U \cap A$. Thus $A \cap U \subseteq S \cap U$. Consequently $x \in S$. Thus $A - I \subseteq S$. Then $A = \overline{A - I} \subseteq \overline{S} = S$.

Again consider the local situation constructed above. Since $U \cap I = \varnothing$, we have $x \in \mathfrak{R}(A) \cap \mathfrak{R}(S)$ with $A \subseteq S$ and $\dim_x A = m-1 = \dim_x S$. Therefore we can take U such that $U \cap A = U \cap S = U \cap \mathfrak{R}(A) = U \cap \mathfrak{R}(S)$ is a connected, $(m-1)$-dimensional complex submanifold of U and such that there is a biholomorphic map $\alpha = (\beta, \chi) \colon U \to P \times Q$. Here P is a ball centered at $\beta(x) = 0 \in \mathbb{C}^{n-1}$ and Q is a disc centered at $\chi(x) = 0 \in \mathbb{C}$. The restriction $\beta \colon U \cap A \to P$ is biholomorphic. Let $\delta = (\beta|U \cap A)^{-1} \colon P \to U \cap A$ be the inverse of β. We have $\chi^{-1}(0) = A \cap U$ with $\chi(z) \neq 0$ for all $z \in U$. Hence $\nu_A|U = \mu_\chi^0$. The Hartogs series development of $\mathfrak{v}_j$ delivers holomorphic vector functions $\mathfrak{w}_j \colon P \to V$ and $\mathfrak{z}_j \colon U \to V$ such that

$$\mathfrak{v}_j = \mathfrak{w}_j \circ \beta + \chi \cdot \mathfrak{z}_j. \tag{2.14}$$

Since $\delta \colon P \to U \cap A$ is biholomorphic and $\chi \circ \delta = 0$, we obtain $\mathfrak{w}_j = \mathfrak{v}_j \circ \delta$.

Take any $q \in \mathbb{N}[1,k]$. Let T_q be the set of all increasing, injective maps $\tau \colon \mathbb{N}[1,q] \to \mathbb{N}[1,k]$. If $1 \leq q < k$ and if $\tau \in T_q$, then there exists one and only one $\hat\tau \in T_{k-q}$ such that $(Jm\tau) \cap (Jm\hat\tau) = \varnothing$. Obviously, we have $(Jm\tau) \cup (Jm\hat\tau) = \mathbb{N}[1,k]$. One and only one permutation $\pi_\tau \colon \mathbb{N}[1,k] \to \mathbb{N}[1,k]$ is defined by $\pi_\tau(j) = \tau(j)$ for $j = 1, \dots, q$ and $\pi_\tau(j) = \hat\tau(j-q)$ for all $j = q+1, \dots, k$. If $q = k$, define $\pi_\tau = \tau$. If

$\tau \in Tq$ with $q \in \mathbb{N}[1, k]$ define $\varepsilon_\tau = \operatorname{sign} \pi_\tau$ and

$$\mathfrak{w}_\tau = \mathfrak{w}_{\tau(1)} \wedge \cdots \wedge \mathfrak{w}_{\tau(q)} : P \to \bigwedge_q V, \tag{2.15}$$

$$\mathfrak{z}_\tau = \mathfrak{z}_{\tau(1)} \wedge \cdots \wedge \mathfrak{z}_{\tau(q)} : U \to \bigwedge_q V. \tag{2.16}$$

The identity $\iota : \mathbb{N}[1, k] \to \mathbb{N}[1, k]$ is the only element of T_k. Define

$$\mathfrak{z} = \mathfrak{z}_\iota = \mathfrak{z}_1 \wedge \cdots \wedge \mathfrak{z}_k : U \to \bigwedge_k V. \tag{2.17}$$

If $q \in \mathbb{N}[1, k-1]$, define

$$\mathfrak{y}_q = \sum_{\tau \in T_q} \varepsilon_\tau (\mathfrak{w}_\tau \circ \beta) \wedge \mathfrak{z}_{\hat{\tau}} : U \to \bigwedge_k V. \tag{2.18}$$

The vector functions $\mathfrak{w}_\tau, \mathfrak{z}_\tau, \mathfrak{y}_q$ and $\mathfrak{z}$ are holomorphic.

Because $f_1, \ldots, f_k$ are in p-special position on A with $2 \leq p \leq k \leq n + 1$, we have $\mathfrak{w}_\tau = 0$ for all $\tau \in T_q$ with $p \leq q \leq k$. Therefore we obtain

$$h \cdot \mathfrak{y} = \mathfrak{v}_1 \wedge \cdots \wedge \mathfrak{v}_k = \sum_{q=1}^{p-1} \chi^{k-q} \mathfrak{y}_q + \chi^k \cdot \mathfrak{z}, \tag{2.19}$$

$$h \cdot \mathfrak{y} = \chi^{k-p+1} \left(\sum_{q=1}^{p-1} \chi^{p-q-1} \mathfrak{y}_q + \chi^{p-1} \mathfrak{z} \right). \tag{2.20}$$

Since $\mathfrak{y}(z) \neq 0$ for all $z \in U$, (2.20) implies

$$\mu(f_1 \wedge \cdots \wedge f_k) | U = \mu_h^0 \geq (k - p + 1) \mu_\chi^0 = (k - p + 1) \nu_A | U. \tag{2.21}$$

Thus (2.11) holds on $M - I$. Since I is analytic with $\dim I \leq m - 2$, the inequality (2.11) holds on M. □

3. Value distribution theory on parabolic manifolds.

(a) *Parabolic manifolds.* Let M be a connected, complex manifold of dimension m. Let τ be a non-negative function of class C^∞ on M. For $0 \leq r \in \mathbb{R}$ define

$$M[r] = \{x \in M | \tau(x) \leq r^2\}, \quad M(r) = \{x \in M | \tau(x) < r^2\}, \tag{3.1}$$

$$M\langle r \rangle = \{x \in M | \tau(x) = r^2\}, \quad M_* = \{x \in M | \tau(x) > 0\}. \tag{3.2}$$

The exterior derivative d splits into $d = \partial + \overline{\partial}$ and twists to $d^c = (i/4\pi)(\overline{\partial} - \partial)$. Define

$$\upsilon = dd^c \tau \quad \text{on } M, \qquad \omega = dd^c \log \tau \quad \text{on } M_*, \tag{3.3}$$

$$\sigma = d^c \log \tau \wedge \omega^{m-1} \quad \text{on } M_*. \tag{3.4}$$

Then τ is said to be a *parabolic exhaustion* and (M, τ) a *parabolic manifold* if and only if τ is unbounded, $M[r]$ is compact for all $0 \leq r \in \mathbb{R}$ and

$$\omega \geq 0, \quad d\sigma = \omega^m \equiv 0 \not\equiv \upsilon^m \quad \text{on } M_*. \tag{3.5}$$

Then $\upsilon \geq 0$ on M. Define $\mathfrak{C}_\tau = \{r \in \mathbb{R}^+ | d\tau(x) \neq 0 \text{ for all } x \in M\langle r\rangle\}$. Then $\mathbb{R}_+ - \mathfrak{C}_\tau$ has measure zero. If $r \in \mathfrak{C}_\tau$, then $M\langle \tau\rangle$ is the boundary of $M(r)$ and $M\langle r\rangle$ is a differentiable, $(2m-1)$-dimensional submanifold of class C^∞ which we orient to the exterior of $M(r)$. A constant $\varsigma > 0$ is defined by

$$\varsigma = \int_{M\langle r\rangle} \sigma \quad \text{if } r \in \mathfrak{C}_\tau, \qquad \int_{M[r]} \upsilon^m = \varsigma r^{2m} \quad \text{if } 0 \leq r \in \mathbb{R}. \tag{3.6}$$

For example $(\mathbb{C}^m, \tau_0)$ is a parabolic manifold where

$$\tau_0(z_1, \ldots, z_m) = |z_1|^2 + \cdots + |z_m|^2 \quad \text{if } (z_1, \ldots, z_m) \in \mathbb{C}^m. \tag{3.7}$$

Here $\mathfrak{C}_{\tau_0} = \mathbb{R}^+$ and $\varsigma = 1$.

Let M be a connected complex manifold of dimension m. Let $\pi\colon M \to \mathbb{C}^m$ be a surjective, proper holomorphic map. Then $\tau = \tau_0 \circ \pi$ is a parabolic exhaustion of M. Then (M, τ) is called a *parabolic covering space of* $\mathbb{C}^m$. Let β be the branching divisor of M. Then (M, τ) is said to be *affinely branched* if and only if the $(m-1)$-dimensional component of $\pi(\operatorname{supp}\beta)$ is affine algebraic.

The disjoint union $\mathbb{P}^m = \mathbb{C}^m \cup \mathbb{P}_{m-1}$ is the projective compactification of $\mathbb{C}^m$. The parabolic covering space (M, τ) is said to be *affine algebraic* if and only if the following conditions are met:

(1) M is an affine algebraic manifold with projective closure $\overline{\overline{M}}$.

(2) $\pi\colon M \to \mathbb{C}^m$ extends to a holomorphic map $\overline{\overline{\pi}}\colon \overline{\overline{M}} \to \mathbb{P}_m = \mathbb{C}^m \cup \mathbb{P}_{m-1}$.

(3) $\overline{\overline{M}} - M = \overline{\overline{\pi}}^{-1}(\mathbb{P}_{m-1})$.

If so, $\pi(\operatorname{supp}\beta)$ is an affine algebraic variety in $\mathbb{C}^m$ of pure dimension $m-1$ if $\beta \equiv 0$. In particular, (M, τ) is affinely branched. Every connected m-dimensional affine algebraic manifold M can be represented as an affine algebraic, parabolic covering space (M, τ) of $\mathbb{C}^m$.

(b) *Divisors on parabolic manifolds.* Let (M, τ) be a parabolic manifold of dimension m. Let ν be a divisor on M. Put $S = \operatorname{supp}\nu$ and $S[t] = S \cap M[t]$ for $0 \leq t \in \mathbb{R}$. The *counting function* $n_\nu\colon \mathbb{R}^+ \to \mathbb{R}$ of ν is defined by

$$n_\nu(t) = t^{2-2m} \int_{S[t]} \nu \upsilon^{m-1} \quad \text{for all } t \in \mathbb{R}^+. \tag{3.8}$$

Then $n_\nu(t) \to n_\nu(0)$ for $t \to 0$ with $t > 0$. We have

$$n_\nu(t) = \int_{S[t]} \nu\omega^{m-1} + n_\nu(0) \quad \text{for all } t \in \mathbb{R}^+. \tag{3.9}$$

If $\nu \geq 0$, then $n_\nu \geq 0$ increases, define

$$0 \leq n_\nu(\infty) = \lim_{t\to\infty} n_\nu(t) \leq \infty. \tag{3.10}$$

A divisor $\nu \geq 0$ is said to have *affine growth* if and only if $n_\nu(\infty) < \infty$. A divisor $\nu \geq 0$ on an affine algebraic parabolic covering space has affine growth if and only if $\pi(\operatorname{supp}\nu)$ is affine algebraic in $\mathbb{C}^m$.

If ν_1 and ν_2 are divisors on M, then $n_{\nu_1+\nu_2} = n_{\nu_1} + n_{\nu_2}$. If $(M, \tau) = (\mathbb{C}^m, \tau_0)$ and if ν is a divisor on $\mathbb{C}^m$, then $n_\nu(0) = \nu(0)$.

For all $0 < s < r$ the *valence function* N_ν of ν is defined by

$$N_\nu(r, s) = \int_s^r n_\nu(t)\frac{dt}{t}. \tag{3.11}$$

If ν_1, ν_2 are divisors on M, then $N_{\nu_1+\nu_2} = N_{\nu_1} + N_{\nu_2}$. If $\nu \geq 0$, then $N_\nu \geq 0$ increases with r and decreases with s. We have

$$\lim_{r\to\infty} \frac{N_\nu(r,s)}{\log r} = n_\nu(\infty) \leq \infty. \tag{3.12}$$

(c) *The First Main Theorem.* Let (M, τ) be a parabolic manifold of dimension m. Let V be a hermitian vector space of dimension $n + 1 > 1$. Define $\tau_V \colon V \to \mathbb{R}^+$ by $\tau_V(\mathfrak{x}) = \|\mathfrak{x}\|^2$ for all $\mathfrak{x} \in V$. Then there exists one and only one form Ω of bidegree $(1, 1)$ on $\mathbb{P}(V)$ with

$$\mathbb{P}^*(\Omega) = dd^c \log \tau_V \quad \text{on } V_*. \tag{3.13}$$

The form Ω is positive and of class C^∞. It is called the *Fubini-Study form* on $\mathbb{P}(V)$.

Let $f \colon M \to \mathbb{P}(V)$ be a meromorphic map. For all $t > 0$ the *spherical image function* A_f of f is defined by

$$A_f(t) = t^{2-2m} \int_{M[t]} f^*(\Omega) \wedge \upsilon^{m-1}. \tag{3.15}$$

Then $A_f \geq 0$ increases. Define $A_f(0) = \lim_{0<t\to 0} A_f(t) \in \mathbb{R}_+$ and $A_f(\infty) = \lim_{t\to\infty} A_f(t)$. For $0 < t \in \mathbb{R}$ we have

$$A_f(t) = \int_{M[t]} f^*(\Omega) \wedge \omega^{m-1} + A_f(0) \tag{3.16}$$

for all $t > 0$. The map f is said to have *rational growth* if $A_f(\infty) < \infty$ and *transcendental growth* if $A_f(\infty) = \infty$. If (M, τ) is an affine algebraic parabolic covering space of $\mathbb{C}^m$, then f has rational growth if and only if f is rational. In the general case f is constant if and only if $A_f(\infty) = 0$.

For all $0 < s < r \in \mathbb{R}$ the *characteristic function* T_f of f is defined by

$$T_f(r, s) = \int_s^r A_f(t) \frac{dt}{t}. \tag{3.17}$$

Here $T_f \geq 0$ increases in r and decreases in s. Also f is constant if and only if $T_f \equiv 0$. If f is not constant, then $T_f(r, s) \to \infty$ for $r \to \infty$. Also we have

$$\lim_{r\to\infty} \frac{T_f(r, s)}{\log r} = A_f(\infty). \tag{3.18}$$

The non-constant meromorphic map $f\colon M \to \mathbb{P}(V)$ is said to *grow quicker* than the non-negative divisor $\nu\colon M \to \mathbb{Z}_+$ if and only if

$$\frac{N_\nu(r, s)}{T_f(r, s)} \to 0, \quad r \to \infty. \tag{3.19}$$

Let $g\colon M \to \mathbb{P}(V^*)$ be a meromorphic map such that f, g is free. Then $n_{f,g} \geq 0$ denotes the *counting function* and $N_{f,g} \geq 0$ the *valence function* of the intersection divisor $\mu_{f,g} \geq 0$. Also $\square f, g \square \not\equiv 0$. A continuous function $m_{f,g}$ on $\mathbb{R}^+$ called the *compensation function* of f and g is uniquely defined by

$$m_{f,g}(r) = \int_{M\langle r\rangle} \log \frac{1}{\square f, g \square} \sigma \geq 0 \quad \text{for all } r \in \mathfrak{E}_\tau. \tag{3.20}$$

For $0 < s < r \in \mathbb{R}$, the *First Main Theorem* holds

$$T_f(r, s) + T_g(r, s) = N_{f,g}(r, s) + m_{f,g}(r) - m_{f,g}(s). \tag{3.21}$$

If $g \equiv a \in \mathbb{P}(V^*)$ is constant, then $T_a(r, s) \equiv 0$. The meromorphic map f is said to be *linearly non-degenerated* if and only if (f, a) is free for all $a \in \mathbb{P}(V^*)$. If so, then

$$T_f(r, s) = \int_{a\in\mathbb{P}(V^*)} N_{f,a}(r, s)\Omega_* \quad \text{for } 0 < s < r \in \mathbb{R} \tag{3.22}$$

where Ω^* is the Fubini-Study form on $\mathbb{P}(V^*)$.

(d) *The Second Main Theorem.* Take $0 < s \in \mathbb{R}$. Let g and h be real valued functions on $\mathbb{R}(s, +\infty)$. We write $g \;\dot{\leq}\; h$ if and only if there exists a set E of finite measure in $\mathbb{R}(s, +\infty)$ such that $g(r) \leq h(r)$ for all $r \in \mathbb{R}(s, +\infty) - E$. Since our functions may depend on several parameters we write $g(r) \underset{\cdot}{\leq} h(r)$ instead of $g \underset{\cdot}{\leq} h$ where the function variable is always denoted by r and E may depend on the other variables.

THEOREM 3.1. *Second Main Theorem. Let (M, τ) be a parabolic covering manifold of $\mathbb{C}^m$ with branching divisor β. Let V be a hermitian vector space of dimension $n+1$. Let $f\colon M \to \mathbb{P}(V)$ be a linearly non-degenerated meromorphic map. Let $a_1, \dots, a_q$ be in general position in $\mathbb{P}(V^*)$ with $q \geq n+1$. For $j = 1, \dots, q$, let $N_{f,a_j}^{(n)} \geq 0$ be the valence function of the truncation $\mu_{f,a_j}^{(n)}$ of the intersection divisor μ_{f,a_j}. Take $s > 0$. Then there is a constant $c > 0$ such that*

$$
\begin{aligned}
(3.23)\qquad & (q-n-1)T_f(r,s) \\
& \underset{\cdot}{\leq} \sum_{j=1}^{q} N_{f,a_j}^{(n)}(r,s) + \frac{1}{2}n(n+1)N_\beta(r,s) \\
& \quad + c(\log T_f(r,s) + \log^+ N_\beta(r,s) + \log(r/s)).
\end{aligned}
$$

Proof. We refer to Stoll [S7, pp. 169–180]. The assumptions (B1)–(B5) on p. 171 are satisfied with $\mathrm{Ric}_\tau(r,s) = N_\beta(r,s)$ by (11.27). There exists a holomorphic form B of bidegree $(m-1, 0)$ on M such that τ majorized B with majorant $Y(r) \leq 1 + r^{2n-2}$. Thus assumptions (A1)–(A8) are satisfied. Therefore (11.23) holds. Thus a constant $c > 0$ holds such that

$$
\begin{aligned}
(3.24)\qquad & (q-n-1)T_f(r,s) + N_{d_n}(r,s) \\
& \underset{\cdot}{\leq} \sum_{j=1}^{n} N_{f,a_j}(r,s) + \frac{1}{2}n(n+1)N_\beta(r,s) \\
& \quad + c(\log T_f(r,s) + \log^+ N_\beta(r,s) + \log(r/s)).
\end{aligned}
$$

By [S7, Lemma 13.3, p. 180, estimate 13.21] or by [S8, Theorem 8.7, p. 260, estimate (8.25)] we have

$$
(3.25)\qquad \sum_{j=1}^{q} N_{f,a_j}(r,s) \leq N_{d_n}(r,s) + \sum_{j=1}^{q} N_{f,a_j}^{(n)}(r,s).
$$

Now (3.24) and (3.25) imply (3.23) immediately. □

(e) *The First Main Theorem for general position.* Let (M, τ) be a parabolic manifold of dimension m. Let V be a hermitian vector space of dimension $n+1$. For $j = 1, \dots, k$ let $f_j\colon M \to \mathbb{P}(V)$ be a meromorphic map. Assume that $1 \leq k \leq n+1$. Assume that $f_1, \ldots, f_k$ are in general position. The divisor $\mu(f_1 \mathring{\wedge} \cdots \mathring{\wedge} f_k) \geq 0$ exists. Let $N_{f_1} \mathring{\wedge} \cdots \mathring{\wedge} f_k$ be the *valence function.* Also $\square f_1 \mathring{\wedge} \cdots \mathring{\wedge} f_k \square \equiv 0$. Hence the *compensation function*

$$(3.26) \qquad m_{f_1 \wedge \cdots \wedge f_k}(r) = \int_{M\langle r\rangle} \log \frac{1}{\square f_1 \mathring{\wedge} \cdots \mathring{\wedge} f_k \square} \sigma \geq 0$$

exists for all $r \in \mathfrak{E}_\tau$ and extends to a continuous function on $\mathbb{R}^+$. For $0 < s < r \in \mathbb{R}$ we have the *First Main Theorem for general position*

$$(3.27) \qquad \begin{aligned} \sum_{j=1}^{k} T_{f_j}(r,s) = N_{f_1 \wedge \cdots \wedge f_k}(r,s) &+ m_{f_1 \wedge \cdots \wedge f_k}(r) \\ &- m_{f_1 \wedge \cdots \wedge f_k}(s) + T_{f_1 \wedge \cdots \wedge f_k}(r,s) \end{aligned}$$

(see Stoll [**S8**, p. 146, equation (3.36)]. Now (3.27) yields the estimate

$$(3.28) \qquad N_{f_1 \wedge \cdots \wedge f_k}(r,s) \leq \sum_{j=1}^{k} T_{f_j}(r,s) + m_{f_1 \wedge \cdots \wedge f_k}(s)$$

for $0 < s < r \in \mathbb{R}$.

4. The propagation theorem for maps into projective space. In this section we consider the case of a meromorphic map $f\colon M \to \mathbb{P}(V)$. In §6 we shall consider the case of a dominant meromorphic map $f\colon M \to N$ where M and N are connected complex manifolds and N is a compact projective variety. *Dominant* means $\dim N = \operatorname{rank} f$.

THEOREM 4.1. *Let (M, τ) be a parabolic manifold of dimension m. Let A be a pure $(m-1)$-dimensional, analytic subset of M. Let N_A be the valence function of the divisor ν_A. Let V be a hermitian vector space of dimension $n+1$. Let p and k be integers with $2 \leq p \leq k \leq n+1$. For $\lambda = 1, \ldots, k$ let $f_\lambda\colon M \to \mathbb{P}(V)$ be a meromorphic map. Assume that $f_1, \ldots, f_k$ are in general position on M. Assume that $f_1, \ldots, f_k$ are in p-special position on A. Then for $0 < s < r \in \mathbb{R}$ we have*

$$(4.1) \qquad (k-p+1)N_A(r,s) \leq \sum_{\lambda=1}^{k} T_{f_\lambda}(r,s) + m_{f_1 \wedge \cdots \wedge f_k}(s).$$

Proof. Theorem 3.1 implies

$$(4.2) \qquad (k-p+1)N_A(r,s) \leq N_{f_1 \wedge \cdots \wedge f_k}(r,s).$$

Now (3.28) and (4.2) imply (4.1). □

THEOREM 4.2. *First Propagation Theorem. Let (M, τ) be a parabolic covering manifold of $\mathbb{C}^m$ with branching divisor β. Let V be a hermitian vector space of dimension $n+1 > 1$. Let p and k be integers with $2 \leq p \leq k \leq n+1$. For $\lambda = 1, \dots, k$ let $f_\lambda : M \to \mathbb{P}(V)$ be a linearly non-degenerated, meromorphic map. Assume that at least one of these maps f_λ grows quicker than the branching divisor β. Assume that at least one of these maps f_λ has transcendental growth. Let $a_1, \dots, a_q$ be in general position in $\mathbb{P}(V^*)$ with $q \geq n+1$. Assume that for each $j = 1, \dots, q$ the analytic set $A_j = \operatorname{supp} \mu_{f_\lambda, a_j}$ does not depend on $\lambda = 1, \dots, k$. Assume that $\dim(A_{j_1} \cap A_{j_2}) < m-2$ whenever $1 \leq j_1 \leq j_2 \leq q$. Define $A = A_1 \cup \dots \cup A_q$. Assume that $f_1, \dots, f_k$ are in p-special position on A. Assume that*

$$nk < (k-p+1)(q-n-1). \tag{4.3}$$

Then $f_1, \dots, f_k$ are in special position on M. In particular, $f_1, \dots, f_k$ are algebraically dependent.

Proof. Assume that $f_1, \dots, f_k$ are in general position on M. Since $n\nu_{A_j} \geq \mu^{(n)}_{f_\lambda, a_j}$ for $j = 1, \dots, q$ and $\lambda = 1, \dots, k$, and since $\nu_A = \nu_{A_1} + \dots + \nu_{A_q}$, Theorem 3.1 implies

$$\begin{aligned} (q-n-1)T_{f_\lambda}(r,s) &\,\dot{\leq}\, nN_A(r,s) + \tfrac{1}{2}n(n+1)N_\beta(r,s) \\ &\quad + c_\lambda(\log T_{f_\lambda}(r,s) + \log^+ N_\beta(r,s) + \log r/s). \end{aligned} \tag{4.4}$$

Define $T = T_{f_1} + \dots + T_{f_k}$ and $c = c_1 + \dots + c_k > 0$. Addition yields

$$\begin{aligned} (q-n-1)T(r,s) &\,\dot{\leq}\, nkN_A(r,s) + \tfrac{1}{2}n(n+1)kN_\beta(r,s) \\ &\quad + ck(\log T(r,s) + \log N_\beta(r,s) + \log r/s). \end{aligned} \tag{4.5}$$

Here

$$\frac{N_\beta(r,s)}{T(r,s)} \to 0 \quad \text{and} \quad \frac{\log r/s}{T(r,s)} \to 0 \quad \text{for } r \to \infty. \tag{4.6}$$

Hence (4.1), (4.5) and (4.6) yield

$$(q-n-1)(k-p+1) \leq nk \tag{4.7}$$

which contradicts (4.3). Therefore $f_1, \dots, f_k$ are in special position on M. By Proposition 1.1, $f_1, \dots, f_k$ are algebraically dependent. □

If $M = \mathbb{C}^m$ and if $\pi : M \to \mathbb{C}^m$ is the identity, $\beta \equiv 0$. Thus Theorem 4.2 extends Theorem B of Ji **[J1]** who considers the case $M = \mathbb{C}^m, p = 2, k = 3$ and $p = 3n+1$ only. He concludes that f_1, f_2, f_3 satisfy a certain condition (P), which is perhaps a bit stronger but rather incomprehensible.

If we assume $p = k$ in Theorem 4.1 we obtain a special case of a Theorem of Smiley **[S3]**, **[S4]** see also Stoll **[S7**, Theorem 13.8 and Theorem 13.10].

5. Value distribution theory for dominant maps. Let M and N be connected, complex manifolds. Put $m = \dim M$ and $n = \dim N$. A meromorphic map $f\colon M \to N$ is said to be *dominant* if and only if $\operatorname{rank} f = n$, which is the case if and only if $f(M)$ has an interior point. In the case of an algebraic map this means precisely that $f(M)$ is dense in N. In the non-algebraic case, $f(N)$ may not be dense if f is dominant. If $f\colon M \to N$ is a dominant meromorphic map, then $m \geq n$. The Carlson-Griffiths-King theory of value distribution **[C1]**, **[G1]** applies to dominant meromorphic maps. We will not outline the most general setting of this theory (see Stoll **[S5]**) but restrict ourself to a special case. We will make the following assumptions.

(A1) Let (M, τ) be a parabolic covering manifold of $\mathbf{C}^m$ with branching divisor β.

(A2) Let V be a finite dimensional hermitian vector space.

(A3) Let N be a compact, connected, complex submanifold of $\mathbb{P}(V)$.

(A4) Put $\dim N = n$ and let $\imath\colon N \to \mathbb{P}(V)$ be the inclusion map.

(A5) Assume that N is not contained in any hyperplane of $\mathbb{P}(V)$.

(A6) Let K be the canonical bundle of N. Let H be the hyperplane section bundle of $\mathbb{P}(V)$ and define $L = H|N$.

Here (A5) is equivalent to the requirement that $\imath$ is not linearly degenerated. Take $a \in \mathbb{P}(V^*)$. Then $a = \mathbb{P}(\mathfrak{a})$ with $\mathfrak{a} \in V^*$. Then $\mathfrak{a}$ defines a section $\breve{\mathfrak{a}} \in \Gamma(\mathbb{P}(V), H)$ with $E[a] = \operatorname{supp} \mu_{\breve{\mathfrak{a}}}$. This section restricts to a holomorphic section $\hat{\mathfrak{a}} = \breve{\mathfrak{a}} \circ \imath = \breve{\mathfrak{a}}|\Gamma(N, L)$ with

$$E_L[a] = \operatorname{supp} \mu_{\hat{\mathfrak{a}}} = E[a] \cap N. \tag{5.1}$$

LEMMA 5.1. *Assume* (A1)–(A6). *Let $f\colon M \to N$ be a dominant meromorphic map. Define $g = \imath \circ f\colon M \to \mathbb{P}(V)$. Take $a \in \mathbb{P}(V^*)$. Then g, a are free.*

Proof. Assume that g, a is not free. Then $f(M - I(f)) \subseteq E_L[a]$. Because $f(M - I(f))$ contains an interior point of N, we obtain $N = E_L[a]$ which contradicts (A5). □

Therefore we define the value distribution functions of f as those of $g = \imath \circ f$. Hence $A_f = A_g, T_f = T_g, n_{f,a} = n_{g,a}, N_{f,a} = N_{g,a}$,

$m_{f,a} = m_{g,a}$ and the *First Main Theorem* holds

(5.2) $\qquad T_f(r,s) = N_{f,a}(r,s) + m_{f,a}(r) - m_{f,a}(s) \quad \text{if } 0 < s < r \in \mathbb{R}.$

Take $a_1, \ldots, a_q$ in $\mathbb{P}(V^*)$. Then $E_L[a_1], \ldots, E_L[a_q]$ are said to have *strictly normal crossings* at $x \in E_L[a_1] \cup \cdots \cup E_L[a_q] = E$ if and only if the following property holds:

(H) Take any holomorphic section s of L over an open, connected neighborhood U of x in N with $Z(s) = \varnothing$. Pick $\mathfrak{a}_j \in V_*^*$ with $\mathbb{P}(\mathfrak{a}_j) = a_j$ for $j = 1, \ldots, q$. Then there are holomorphic functions $h_j \not\equiv 0$ on U such that $\hat{\mathfrak{a}}_j | U = h_j s$ for $j = 1, \ldots, q$. Let $1 \le j_1 < \cdots < j_t \le q$ be any collection of integers with $x \in E_L[a_{j_\lambda}]$ for $\lambda = 1, \ldots, t$. Then

(5.3) $$dh_{j_1}(x) \wedge \cdots \wedge dh_{j_t}(x) \neq 0.$$

Thus if $E_L[a_1], \ldots, E_L[a_q]$ have strictly normal crossings and if $I_x = \{j \in \mathbb{N}[1,q] \mid x \in E_L[a_j]\}$ then $\# I_x \le n$.

LEMMA 5.2. *Assume* (A2)–(A6) *with* $N = \mathbb{P}(V)$. *Take* $a_1, \ldots, a_q$ *in* $\mathbb{P}(V^*)$. *Then* $E[a_j] = E_L[a_j]$ *for* $j = 1, \ldots, q$ *and* $a_1, \ldots, a_q$ *are in general position if and only if* $E[a_1], \ldots, E[a_q]$ *have strictly normal crossings.*

Proof. Take any $x = \mathbb{P}(\mathfrak{x}) \in \mathbb{P}(V)$. Then $E(x) = \{\lambda \mathfrak{x} \mid \lambda \in \mathbb{C}\}$ is the complex line defined by x. Take any $\mathfrak{b} \in V_*^*$. Then $\mathfrak{b}\colon V \to \mathbb{C}$ is a linear map and $E[b] = \mathbb{P}(\ker \mathfrak{b})$. Also $A(\mathfrak{b}) = \{\mathfrak{x} \in V \mid \mathfrak{b}(\mathfrak{x}) = 1\}$ is an n-dimensional affine plane in V and $\mathbb{P}_{\mathfrak{b}} = \mathbb{P}\colon A(\mathfrak{b}) \to \mathbb{P}(V) - E[b]$ is biholomorphic. Take $z \in \mathbb{P}(V)$ and let T_z be the holomorphic tangent space of $\mathbb{P}(V)$ at z. If $z \in \mathbb{P}(V) - E[b]$, then T_z can be identified via $\mathbb{P}_{\mathfrak{b}}$ with $\ker \mathfrak{b}$ affixed to $\mathfrak{z} = \mathbb{P}_{\mathfrak{b}}^{-1}(z) \in A(\mathfrak{b})$ as tangent space of $A(\mathfrak{b})$. Thus $\mathbb{P}(\mathfrak{z}) = z$ with $\mathfrak{b}(\mathfrak{z}) = 1$.

Now $\mathfrak{b}$ defines a section $\check{\mathfrak{b}} = \hat{\mathfrak{b}}$ of $H = L$ on $\mathbb{P}(V)$ by $\hat{\mathfrak{b}}(x) = \mathfrak{b} | E(x)$ since H is the dual bundle to the tautological bundle $\{(x, \mathfrak{x}) \in \mathbb{P}(V) \times V \mid \mathfrak{x} \in E(X)\}$. Then $Z(\hat{\mathfrak{b}}) = E[b]$.

For each $j \in \mathbb{N}[1,q]$ take $\mathfrak{a}_j \in V_*^*$ such that $\mathbb{P}(\mathfrak{a}_j) = a_j$. Take $x = \mathbb{P}(\mathfrak{x})$ in $\mathbb{P}(V)$ and $b = \mathbb{P}(\mathfrak{b}) \in \mathbb{P}(V^*)$ with $\mathfrak{b}(\mathfrak{x}) = 1$. Then there is a holomorphic function h_j on $\mathbb{P}(V) - E[b]$ such that $\hat{\mathfrak{a}}_j = h_j \hat{\mathfrak{b}}$. Take any $z \in \mathbb{P}(V) - E[b]$. Then $\hat{\mathfrak{a}}_j(z) = h_j(z)\hat{\mathfrak{b}}(z)$. If $\mathfrak{z} \in A(b)$ with $z = \mathbb{P}(\mathfrak{z})$, then $\hat{\mathfrak{a}}_j(z) = \mathfrak{a}_j(\mathfrak{z})$ and $\hat{\mathfrak{b}}(z)(\mathfrak{z}) = 1$. Hence $h_j(z) = \mathfrak{a}_j(\mathfrak{z})/\mathfrak{b}(\mathfrak{z})$. If $\mathfrak{v} \in \ker \mathfrak{b} = T_z$, then

(5.4) $$dh_j(z, \mathfrak{v}) = \mathfrak{a}_j(\mathfrak{v}).$$

Assume that $a_1, \ldots, a_q$ are in general position. Take $x \in E[a_1] \cup \cdots \cup E[a_q]$. Take any collection of integers $1 \leq j_1 < \cdots < j_t \leq q$ with $x \in E[a_{j_\lambda}]$ for $\lambda = 1, \ldots, q$. Determine $b, \mathfrak{b}, \mathfrak{a}_j, \mathfrak{x}$ as above. Then $\mathfrak{a}_{j_\lambda}(\mathfrak{x}) = 0$ for $\lambda = 1, \ldots, t$. If $t \geq n+1$ then $\mathfrak{a}_{j_1}, \ldots, \mathfrak{a}_{j_{n+1}}$ are linearly independent. Hence $\mathfrak{x} = 0$ which contradicts $\mathfrak{b}(\mathfrak{x}) = 1$. Therefore $t \leq n$. By general position, $\mathfrak{a}_{j_1}, \ldots, \mathfrak{a}_{j_t}$ are linearly independent. If $\mathfrak{a}_{j_1} | \ker \mathfrak{b}, \ldots, \mathfrak{a}_{j_t} | \ker \mathfrak{b}$ are linearly dependent, there are constants $c_1, \ldots, c_t$ not all zero such that $\mathfrak{a} = c_1 \mathfrak{a}_{j_1} + \cdots + c_t \mathfrak{a}_{j_t} \in V^*$ with $\ker \mathfrak{b} \subseteq \ker \mathfrak{a}$. Also $0 \neq \mathfrak{x} \in \ker \mathfrak{a} - \ker \mathfrak{b}$. Hence $\dim \ker \mathfrak{a} = n+1$ and $\mathfrak{a} \equiv 0$. Since $\mathfrak{a}_{j_1}, \ldots, \mathfrak{a}_{j_t}$ are linearly independent, $\mathfrak{a} \equiv 0$ is impossible. Thus $dh_{j_1}(z) = \mathfrak{a}_{j_1} | \ker \mathfrak{b}, \ldots, dh_{j_t}(z) = \mathfrak{a}_{j_t} | \ker \mathfrak{b}$ are linearly independent. Hence $E[a_1], \ldots, E[a_q]$ have strictly normal crossings.

Assume that $E[a_1], \ldots, E[a_q]$ have strictly normal crossings. Take any collection of integers $1 \leq j_1 < \cdots < j_t \leq q$ with $t \leq n+1$, with $t \leq n+1$, then we have to show that $\mathfrak{a}_{j_1}, \ldots, \mathfrak{a}_{j_t}$ are linearly independent. Assume that $\mathfrak{a}_{j_1}, \ldots, \mathfrak{a}_{j_t}$ are linearly dependent. Then $\mathfrak{x} \in V^*$ exists such that $\mathfrak{a}_{j_\lambda}(\mathfrak{x}) = 0$ for $\lambda = 1, \ldots, t$. Thus $x = \mathbb{P}(\mathfrak{x}) \in E[a_{j_1}] \cap \cdots \cap E[a_{j_t}]$. Strictly normal crossings implies $t \leq n$. Since $\mathfrak{a}_{j_1}, \ldots, \mathfrak{a}_{j_t}$ are linearly dependent, also $\mathfrak{a}_{j_1} | \ker \mathfrak{b}, \ldots, \mathfrak{a}_{j_t} | \ker \mathfrak{b}$ are linearly dependent. By (5.4) we obtain $dh_{j_1}(x) \wedge \cdots \wedge dh_{j_t}(x) = 0$ which contradicts (5.3). Thus $\mathfrak{a}_{j_1}, \ldots, \mathfrak{a}_{j_t}$ are linearly independent. □

Now we will make the following additional assumption:

(A7) Take $a_1, \ldots, a_q$ in $\mathbb{P}(V^*)$ such that $E_L[a_1], \ldots, E_L[a_q]$ have strictly normal crossings.

The hermitian metric on V defines a hermitian metric l along the fibers of H whose Chern form $c(H, l)$ is the Fubini Study form Ω on $\mathbb{P}(V)$. Naturally, l restricts to a hermitian metric l along the fibers of L such that $c(L, l) = \iota^*(c(H, l))$. Thus if $f \colon M \to N$ is a meromorphic map and $g = \iota \circ f$ we obtain

$$(5.5) \qquad g^*(\Omega) = g^* \iota^*(c(H, l)) = f^*(c(L, l)),$$
$$T_f(r, s) = T_g(r, s) = \int_s^r t^{2-2m} \int_{M[t]} f^*(c(L, l)) \wedge v^{m-1} \frac{dt}{t}$$

such that our definition of the characteristic agrees with [S5].

Let Φ be the set of all real numbers $v \in \mathbb{R}_+$ such that there is a hermitian metric κ along the fibers of K such that

$$(5.6) \qquad c(K, \kappa) + v c(L, l) \geq 0 \quad \text{on } N.$$

Since $c(L, l) = \Omega > 0$ we see that $\Phi \neq \varnothing$. Define

$$(5.7) \qquad [K^* : L] = \inf \Phi.$$

THEOREM 5.3. *Second Main Theorem for dominant maps. Assume* (A1)–(A7). *Assume that* $q > [K^* : L]$. *Let* $f\colon M \to N$ *be a dominant holomorphic map. Take* $s > 0$ *and* $\varepsilon > 0$ *with* $\varepsilon < q - [K^* : L]$. *Define* $A_j = \operatorname{supp} f^{-1}(E_L[a_j])$ *for* $j = 1, \ldots, q$ *and* $A = A_1 \cup \cdots \cup A_q$. *Then there is a constant* $c > 0$ *such that*

$$(5.8) \qquad (q - [K^* : L] - \varepsilon)T_f(r,s) \\ \mathrel{\dot{\le}} N_A(r,s) + N_\beta(r,s) + c \log T_f(r,s) + \varepsilon \log r.$$

Proof. We want to apply Theorem 18.13E in **[S5]**. Obviously assumptions (D1)–(D8) are satisfied. Assumption (D9) requires: "Let F be an effective Jacobian section of f dominated by τ. Let Y be the dominator". We will not discuss the definition of these terms. An effective Jacobian section is a holomorphic section $F \not\equiv 0$ in a certain line bundle on M. Under our assumption **[S5**, Proposition 18.6**]** provides us with an effective Jacobian section F dominated by τ with $Y \equiv m$. Hence the assumptions (D1)–(D9) are satisfied. Let μ_F be the zero divisor of F in Theorem 18.13E; the exceptional set E in $\mathbb{R}^+$ is picked such that $\int_E x^\varepsilon\, dx < \infty$. Because $x^\varepsilon \ge 1$ if $x \ge 1$, the set E has finite measure. Thus Theorem 18.13E with 18.17 implies

$$(5.9) \qquad N_{\mu_F}(r,s) + (q - [K^* : L] - \varepsilon)T_f(r,s) \\ \mathrel{\dot{\le}} \sum_{j=1}^{q} N_{f,a_j}(r,s) + \operatorname{Ric}_\tau(r,s) \\ + c_1 \log T_f(r,s) + c_2 \log m + c_3 \log r$$

where $c_1 > 0$ and $c_2 > 0$ are some constants and $c_3 = 2\varepsilon\varsigma n$. We have $\operatorname{Ric}_\tau(r,s) = N_\beta(r,s)$ in our situation. Replacing ε by another smaller ε, we can replace $c_2 \log m + c_3 \log r$ by $\varepsilon \log r$. Lemma 4.1 by Smiley **[S3]** (see also Drouilhet **[D1]**) ascertains

$$(5.10) \qquad \sum_{j=1}^{q} N_{f,a_j}(r,s) - N_{\mu_F}(r,s) \le N_A(r,s).$$

Thus (5.9) and (5.10) imply (5.8). □

Now we proceed to replace the holomorphic map f in Theorem 5.3 by a meromorphic map. Assume that (A1)–(A7) holds and that $f\colon M \to N$ is a dominant meromorphic map. Recall that we are given a proper, surjective holomorphic map $\pi\colon M \to \mathbb{C}^m$ such that $\tau = \|\pi\|^2$ and that β is the branching divisor of π. Let $\Gamma(f)$ be the closed graph of f in $M \times N$ and let $\zeta\colon \Gamma(f) \to M$ and $\tilde{f}\colon \Gamma(f) \to N$ be the projections. The map ζ is proper and $\zeta\colon \Gamma(f) - \zeta^{-1}(I(f)) \to M - I(f)$ is biholomorphic. Let $\lambda\colon \hat{M} \to \Gamma(f)$ be a resolution of singularities of $\Gamma(f)$. Then $\hat{M}$ is a connected, complex manifold of dimension m. The map $\lambda\colon \hat{M} \to \Gamma(f)$ is proper, surjective and holomorphic. The set $\hat{I}(f) = \lambda^{-1}(\zeta^{-1}(I(f)))$ is analytic with $\hat{I}(f) \neq \hat{M}$. The map

$$(5.11) \qquad \lambda\colon \hat{M} - \hat{I}(f) \to \Gamma(f) - \zeta^{-1}(I(f))$$

is biholomorphic. The map $\rho = \zeta \circ \lambda\colon \hat{M} \to M$ is proper, surjective and holomorphic. The map $\rho\colon \hat{M} - \hat{I}(f) \to M - I(f)$ is biholomorphic. The map $\hat{\pi} = \pi \circ \rho\colon M \to \mathbb{C}^m$ is proper, surjective and holomorphic. Then $\hat{\tau} = \tau \circ \rho = \tau_0 \circ \hat{\pi}$ is a parabolic exhaustion of $\hat{M}$. Therefore $(\hat{M}, \tau)$ is a parabolic covering manifold of $\mathbb{C}^m$. Let $\hat{\beta}$ be the branching divisor of $\hat{\pi}$. Because $\rho\colon \hat{M} - \hat{I}(f) \to M - I(f)$ is biholomorphic we have $\hat{\beta}(x) = \beta(\rho(x))$ for all $x \in \hat{M} - \hat{I}(f)$.

LEMMA 5.4. *Assume that there are given divisors ν on M and $\hat{\nu}$ on $\hat{M}$ such that $\hat{\nu}(x) = \nu(\rho(x))$ for all $x \in \hat{M} - \hat{I}(f)$. Take $0 < s < r$. Then*

$$(5.12) \qquad N_{\hat{\nu}}(r,s) = N_{\nu}(r,s).$$

Proof. Define $S = \operatorname{supp}\nu$ and $\hat{S} = \operatorname{supp}\hat{\nu}$. Define $S_0 = S \cap I(f)$ and $\hat{S}_0 = \hat{S} \cap \hat{I}(f)$. Put $S_1 = S - S_0$ and $\hat{S}_1 = \hat{S} - \hat{S}_0$. Then $\rho\colon \hat{S}_1 \to S_1$ is biholomorphic. Let $\hat{C}$ be a branch of $\hat{S}_0$. Then $C = \rho(\hat{C})$ is an irreducible analytic subset of $I(f)$. Hence $\dim C \leq m-2$. Let $\hat{j}\colon \hat{C} \to \hat{M}$ and $j\colon C \to M$ be the inclusion maps. The map ρ restricts to $\rho_0\colon \hat{C} \to C$ such that $j \circ \rho_0 = \rho \circ \hat{j}$. Because $\dim C \leq m-2$, we have $j^*(\upsilon^{m-1}) = 0$. Thus

$$\begin{aligned}(5.13) \quad \hat{j}^*(\hat{\upsilon}^{m-1}) &= \hat{j}^*(dd^c\hat{\tau}^{m-1}) = \hat{j}^*((dd^c\tau \circ \rho)^{m-1}) \\ &= \hat{j}^*(\rho^*(dd^c\tau)^{m-1}) = (\rho \circ \hat{j})^*((dd^c\tau)^{m-1}) \\ &= (j \circ \rho_0)^*(\upsilon^{m-1}) = \rho_0^*(j^*(\upsilon^{m-1})) = \rho_0^*(0) = 0.\end{aligned}$$

Take $0 < t \in \mathbb{R}$. We obtain

$$\begin{aligned}(5.14) \quad \int_{\hat{S}[t]} \hat{\nu}\hat{\upsilon}^{m-1} &= \int_{\hat{S}_1[t]} \hat{\nu}\hat{\upsilon}^{m-1} + \int_{\hat{S}_0[t]} \hat{\nu}\hat{j}^*(\hat{\upsilon}^{m-1}) \\ &= \int_{\hat{S}_1[t]} (\nu \circ \rho_0)\rho_0^*(j^*(\upsilon^{m-1})) = \int_{S_1[t]} \nu\upsilon^{m-1} = \int_{S[t]} \nu\upsilon^{m-1}.\end{aligned}$$

Thus $n_{\hat{\nu}} = n_{\nu}$ which implies (5.12). $\square$

In particular $N_{\hat{\beta}} = N_\beta$. The map $\hat{f} = \tilde{f} \circ \lambda$ is holomorphic. If $x \in \hat{M} - \hat{I}(f)$, then $\hat{f}(x) = \tilde{f}(\lambda(x)) = f(\zeta(\lambda(x))) = f(\rho(x))$. Thus $\hat{f}$ has rank n. If $a \in \mathbb{P}(V^*)$, then $\mu_{f,a}(\rho(x)) = \mu_{\hat{f},a}(x)$. Define $A_j = \operatorname{supp} \mu_{f,a_j}$ and $\hat{A}_j = \operatorname{supp} \mu_{\hat{f},a_j} = \hat{f}^{-1}(E_L[a_j])$. Then $\hat{A}_j - \hat{I}(f) = \rho^{-1}(A_j - I(f))$. Define $A = A_1 \cup \cdots \cup A_q$ and $\hat{A} = \hat{A}_1 \cup \cdots \cup \hat{A}_q$. Then we have

$$N_{\hat{A}}(r,s) = N_A(r,s) \quad \text{for all } 0 < s < r. \tag{5.15}$$

Because $\rho\colon \hat{M} - \hat{I}(f) \to M - I(f)$ is biholomorphic and $\hat{f} = f \circ \rho$ on $\hat{M} - \hat{I}(f)$, we have

$$\begin{aligned} \int_{\hat{M}[t]} \hat{f}^*(\imath^*(\Omega)) \wedge \hat{\upsilon}^{m-1} &= \int_{\hat{M}[t]} \rho^*(f^*(\imath^*(\Omega)) \wedge \upsilon^{m-1}) \\ &= \int_{M[t]} f^*(\imath^*(\Omega) \wedge \upsilon^{m-1} \end{aligned} \tag{5.16}$$

for all $t > 0$. Thus $T_{\hat{f}} = T_f$. The assumptions of Theorem 5.3 are satisfied for $\hat{f}, \hat{M}, \hat{\tau}, \hat{A}_j, \hat{A}$. Hence (5.8) holds accordingly. With these identities we obtain

THEOREM 5.4. *Second Main Theorem for dominant meromorphic maps. Assume* (A1)–(A7). *Assume that* $q > [K^* : L]$. *Let* $f\colon M \to N$ *be a dominant meromorphic map. For* $j = 1, \ldots, q$ *define* $A_j = \operatorname{supp} \mu_{f,a_j}$. *Put* $A = A_1 \cup \cdots \cup A_q$. *Take positive real numbers* s *and* ε *with* $\varepsilon < q - [K^* : L]$. *Then there is a constant* $c > 0$ *such that*

$$\begin{aligned} &(q - [K^* : L] - \varepsilon) T_f(r,s) \\ &\quad \leq N_A(r,s) + N_\beta(r,s) + c \log T_f(r,s) + \varepsilon \log r. \end{aligned} \tag{5.17}$$

6. Propagation Theorems for dominant holomorphic maps.

THEOREM 6.1. *Second Propagation Theorem. Assume* (A1)–(A7). *Assume that* $q > [K^* : L]$. *Let* p *and* k *be integers with* $2 \leq p \leq k \leq \dim V$. *For* $\lambda = 1, \ldots, k$ *let* $f_\lambda\colon M \to N$ *be dominant, meromorphic maps. Assume that at least one of these maps* f_λ *grows quicker than the branching divisor. Assume that for each* $j = 1, \ldots, q$ *the analytic set* $A_j = \operatorname{supp} \mu_{f_\lambda, a_j}$ *does not depend on* $\lambda = 1, \ldots, k$. *Define* $A = A_1 \cup \cdots \cup A_k$. *Define* $g_\lambda = \imath \circ f_\lambda$ *for all* $\lambda = 1, \ldots, k$. *Assume that* $g_1, \ldots, g_k$ *are in p-special position on* A. *Assume that*

$$k < (k - p + 1)(q - [K^* : L]). \tag{6.1}$$

Then $g_1, \dots, g_k$ are in special position on M and $f_1, \dots, f_k$ are algebraically dependent.

Proof. Since $T_{f_\lambda} = T_{g_\lambda}$ for $\lambda = 1, \dots, k$, $T = T_{f_1} + \cdots + T_{f_k} = T_{g_1} + \cdots + T_{g_k}$. Assume that $g_1, \dots, g_k$ are in general position. Then (4.1) implies

$$(6.2) \qquad (k - p + 1) N_A(r,s) \leq T(r,s) + m_{g_1 \wedge \dots \wedge g_k}(s).$$

Take $\varepsilon \in \mathbb{R}$ with $0 < \varepsilon < q - [K^* : L]$ and $0 < s \in \mathbb{R}$. Then (5.17) holds for each f_λ where A and β do not depend on λ. Also c can be taken independently of λ and $T_{f_\lambda} \leq T$ for $\lambda = 1, \dots, k$. Hence addition implies

$$(6.3) \qquad \begin{aligned} &(q - [K^* : L] - \varepsilon) T(r,s) \\ &\quad \leq k N_A(r,s) + k(N_\beta(r,s) + c \log T(r,s) + \varepsilon \log r). \end{aligned}$$

The constant in (6.2) can be absorbed into $\varepsilon \log r$. Hence (6.2) yields

$$(6.4) \qquad \begin{aligned} &(q - [K^* : L] - \varepsilon) \\ &\quad \leq \frac{k}{k - p + 1} + \frac{k}{T(r,s)} (N_\beta(r,s) + c \log T(r,s) + \varepsilon \log r). \end{aligned}$$

Here $T(r,s)/\log r \to A_{g_1}(0) + \cdots + A_{g_k}(0) \leq \infty$ for $r \to \infty$ where the limit is positive. Hence a constant $B \geq 0$ exists such that

$$(6.5) \qquad (q - [K^* : L] - \varepsilon) \leq \frac{k}{k - p + 1} + \varepsilon B.$$

Thus $\varepsilon \to 0$ yields $(k - p + 1)(q - [K^* : L]) \leq k$ which contradicts (6.1). □

If $(M, \tau) = (\mathbb{C}^m, \tau_0)$ and if $k = 3, p = 2$ and $[K^* : L] \leq q - 2$, we obtain Theorem C of Ji **[J1]** except that his "Property (P)" is replaced by special position.

Assume that $K \otimes L^{q-2}$ is positive. Then $[K^* : L] < q - 2$ and $k = 2 = p$ satisfies (6.1). Hence f_1, f_2 are in special position on M, which means $f_1 = f_2$. We retrieve a Uniqueness Theorem of Drouilhet **[D1]**.

If $N = \mathbb{P}(V)$, then $K = H^{-n-1}$ and $L = H$. Thus $[K^* : L] = n + 1$. Lemma 5.2 and Theorem 6.1 imply

THEOREM 6.2. *Third Propagation Theorem. Let (M, τ) be a parabolic covering manifold of $\mathbb{C}^m$ with branching divisor β. Let V be a hermitian vector space of dimension $n + 1 > 1$. Let p and k be integers* *with* $2 \le p \le k \le n+1$. *For* $\lambda = 1,\dots,k$ *let* $f_\lambda\colon M \to \mathbb{P}(V)$ *be dominant meromorphic maps. Assume that at least one of the maps* f_λ *grows quicker than the branching divisor* β. *Let* $a_1,\dots,a_q$ *be in general position in* $\mathbb{P}(V^*)$ *with* $q \ge n+1$. *Assume that for each* $j = 1,\dots,1$ *the analytic set* $A_j = \operatorname{supp}\mu_{f_\lambda,a_j}$ *does not depend on* $\lambda = 1,\dots,k$. *Put* $A = A_1 \cup \cdots \cup A_q$. *Assume that* $f_1,\dots,f_k$ *are in p-special position on* A. *Assume that*

$$k < (k-p+1)(q-n-1). \tag{6.6}$$

Then $f_1,\dots,f_k$ *are in special position. In particular they are algebraically dependent.*

Thus for dominant maps, Theorem 4.2 *is improved. No* f_λ *needs to have transcendental growth. Different* A_j *may have common branches and* kn *in* (4.3) *is replaced by* k *in* (6.4).

If $(M,\tau) = (\mathbb{C}^m,\tau_0)$ and if $k = 3, p = 2$ and $q = n+3$, then (6.6) is satisfied and we obtain Theorem A of Ji **[J1]** with "Property (P)" replaced by "special position". If $k = 2 = p$ and $q > n+3$, then (6.6) is satisfied. Hence f_1, f_2 are in special position on M. Thus $f_1 = f_2$. Therefore we retrieve a Uniqueness Theorem of Drouilhet **[D1]**.

If each map f_λ does not grow quicker than the branching divisor, but if at least one map f_λ separates the fibers of π, we still obtain propagation theorems by a Theorem of Noguchi [N2]. Also see Stoll **[S9]**.

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