PACIFIC JOURNAL OF MATHEMATICS
Vol. 142, No. 1, 1990

ASYMPTOTICS FOR CERTAIN WIENER INTEGRALS ASSOCIATED WITH HIGHER ORDER DIFFERENTIAL OPERATORS

RICHARD J. GRIEGO AND ANDRZEJ KORZENIOWSKI

The aim of this paper is to derive a large deviation principle for a certain class of higher order operators by combining the ideas of Donsker and Varadhan with the random evolution point of view of Griego and Hersh.

1. Introduction. It has been known for some time how to recover the principal eigenvalue for operators that generate Markov semigroups by means of the Large Deviation Principle of Donsker and Varadhan ([3], [4], [5]). The principal eigenvalues for such operators will be obtained as limits of certain functionals of Brownian motion.

We shall consider operators of the form $L = \frac{1}{2}\Delta_x + c(x)A_y$, where the Laplacian Δ_x is stochastic in the sense that it generates the Brownian motion semigroup; whereas A_y is analytic and does not correspond in general to a Markov process. The operator L can be interpreted as either the averaged result of randomization of the evolutions $c(x)A_y$ driven through the variable x in the coefficient $c(x)$ by Brownian motion or as a perturbation of the Laplacian by an operator-valued potential $V(x) = c(x)A_y$.

We follow the notation of [7], and we will recall some necessary facts. Let $A_y = \sum_{|\alpha|\le 2r} a_\alpha(y)D^\alpha$ be a formally self-adjoint elliptic operator of order $2r$ on a bounded open set $G \subset R^m$, with domain $D(A_y)$ being a subset of the Sobolev space $H_{2r}(G)$, such that $(A_y g, g) \le 0$ for $g \in D(A_y)$ with the inner product of $L^2(G)$.

In what follows we shall consider the following initial-boundary value problem

$$\text{(I.1)}\qquad \begin{aligned} u_t(y,y',t) &= A_y u, \qquad t>0, y, y' \in G,\\ u(y,y',0) &= \delta(y-y') \end{aligned}$$

subject to given homogeneous conditions on the boundary ∂G. We shall assume that the (fundamental) solution to this problem can be written as follows:

$$u(y, y', t) = \sum_{n=1}^{\infty} e^{-\alpha_n t} \phi_n(y) \phi_n(y'), \qquad \cdots \leq -\alpha_2 \leq -\alpha_1 < 0 \tag{I.2}$$

where $\{\phi_n\}$ are complete and orthonormal eigenfunctions of A_y with corresponding eigenvalues $\{-\alpha_n\}$. We assume the multiplicity of α_1 is k.

As shown by Gårding [6], this assumption is satisfied whenever Dirichlet boundary conditions are imposed, i.e., $\partial^j u / \partial \nu^j = 0$ for $j = 0, 1, \ldots, r-1$, where $\partial / \partial \nu$ is differentiation with respect to the outward normal to G.

Let $c : R^n \to [0, \infty)$ be a locally Hölder function that approaches infinity for large x, i.e.,

$$\begin{aligned} &c(x) \to \infty, \quad \text{as } |x| \to \infty \quad \text{and} \\ &|c(x) - c(x')| < M(x)|x - x'|^{\alpha}, \qquad 0 < \alpha < 1 \end{aligned}$$

for x' in some neighborhood of x for every x in R.

Then, by Lemma 2 of [7], $L = \frac{1}{2}\Delta_x + c(x)A_y$ acting on $\psi(x, y) \in L^2(R^n \times G)$ with $\psi(\cdot, y) \in D(\Delta_x)$ and $\psi(x, \cdot) \in D(A_y)$ itself has negative eigenvalues $\{-\lambda_n\}$, $\cdots - \lambda_2 \leq -\lambda_1 < 0$, with a complete set of orthonormal eigenfunctions $\psi_n(x, y) \in L^2(R^n \times G)$. See also below for a direct derivation of these facts.

We will consider functionals Φ satisfying the conditions of [3]. Thus, let $\mathscr{F}$ be the space of probability distribution functions on R^n with the Levy metric. Let $\Phi : \mathscr{F} \to [0, \infty]$ be a function such that (a) Φ is lower semicontinuous on $\mathscr{F}$; (b) $\Phi^{-1}((-\infty, K])$ is compact in $\mathscr{F}$ for each $0 < K < \infty$; (c) if $F_m \to F$ and $\operatorname{supp} F_m \subset [a_1, b_1] \times \cdots \times [a_n, b_n]$, then $\Phi(F_m) \to \Phi(F)$; (d) let $F \in \mathscr{F}$ with $\Phi(F) < \infty$ and let $g_m : R^n \to [0, 1]$ be continuous with $g_m(x) = 1$ for $x \in [-n, n] \times \cdots \times [-n, n]$. If F_m is defined by

$$f_m(x) = \frac{\int_{-\infty}^{x_1} \cdots \int_{-\infty}^{x_n} g_m(y)\, dF(y)}{\int \cdots \int_{R^n} g_m(y)\, dF(y)}; \quad \text{where } x = (x_1, \ldots, x_n),$$

then $\Phi(F_m) \to \Phi(F)$.

If F has density f, we write $\Phi(f)$ for $\Phi(F)$.

Let Ω be the space of continuous functions $\omega : [0, \infty) \to R^n$ and let $X(s, \omega) = \omega(s)$. Define for $B \subset R^n$,

$$L(t, \omega, B) = (1/t) \int_0^t I_B(X(s, \omega))\, ds,$$

which is the proportion of time in $[0,t]$ that the path $X(\cdot,\omega)$ spends in the set B. Also, let E_x denote expectation with respect to Wiener measure P_x on Ω, so that $P_x(\omega : X(0,\omega) = x) = 1$.

Finally, let $\mathscr{F}^+ = \{f \in L^2(R^n) \cap C^1(R^n) : \|f\|_1 = 1$ and $f > 0$ on R^n or on Interior(supp(f)) if supp(f) is compact$\}$ and let $\mathscr{G} = \{g \in L^2(G) \cap D(A_y) \colon \|g\|_2 = 1\}$.

Under the above conditions on the functional Φ, Donsker and Varadhan [**3**, Theorem 2.1] have proved the following asymptotic evaluation.

THEOREM (*Donsker-Varadhan*). *For each x in R^n,*

$$\begin{aligned}\text{(I.3)}\qquad &\lim_{t\to\infty} \frac{1}{t} \log E_x[e^{-t\Phi(L(t,\omega,\cdot))}] \\ &\quad = -\inf\left\{\frac{1}{8}\int_{R^n} \frac{|\nabla f(x)|^2}{f(x)}\,dx + \Phi(f) : f \in \mathscr{F}^+\right\}.\end{aligned}$$

The following corollary follows by considering

$$\Phi(f) = \int V(x) f(x)\,dx.$$

COROLLARY. *If $V \geq 0$ is continuous and $V(x) \to \infty$ as $|x| \to \infty$, then the principal eigenvalue $-\lambda_1$ of the operator $\frac{1}{2}\Delta_x - V(x)$ on $L^2(R^n)$ is given by*

$$\begin{aligned}\text{(I.4)}\qquad -\lambda_1 &= \lim_{t\to\infty} \frac{1}{t} \log E_x\left[\exp\left(-\int_0^t V(X(x))\,ds\right)\right] \\ &= -\inf\left\{\frac{1}{8}\int_{R^n} \frac{|\nabla f(x)|^2}{f(x)}\,dx + \int_{R^n} V(x) f(x)\,dx : f \in \mathscr{F}^+\right\} \\ &= -\inf\left\{\frac{1}{2}\int_{R^n} |\nabla \psi(x)|^2\,dx \right. \\ &\qquad \left. + \int_{R^n} V(x)\psi^2(x)\,dx : \psi \in L^2(R^n), \|\psi\|_2 = 1\right\}.\end{aligned}$$

These results appear in [**3**] for $n = 1$, but they readily carry over to higher dimensions by mimicking the proofs there.

The main result of this paper is the following version of the Donsker-Varadhan theorem in our setup.

MAIN THEOREM. *Let $u(y,y',t)$ be as in* (I.1). *Then,*

$$\text{(I.5)}\quad \begin{aligned}&\lim_{t\to\infty}\frac{1}{t}\log E_x\left[\int_G u(y,y,t\Phi(L(t,\omega,\cdot)))\,dy\right]\\ &\quad=-\inf_f\inf_g\left\{\frac{1}{8}\int_{R^n}\frac{|\nabla f(x)|^2}{f(x)}\,dx\right.\\ &\qquad\qquad\left.+(-A_y g,g)\Phi(f):f\in\mathscr{F}^+,g\in\mathscr{G}\right\}.\end{aligned}$$

COROLLARY. *The principal eigenvalue $-\lambda_1$ of $L = 1/2\Delta_x + c(x)A_y$ on $L^2(R^n\times G)$ is given by*

$$\text{(I.6)}\quad \begin{aligned}-\lambda_1&=\lim_{t\to\infty}\frac{1}{t}\log E_x\left[\int_G u(y,y,\alpha(t))\,dy\right]\\ &=-\inf_f\inf_g\left\{\frac{1}{8}\int_{R^n}\frac{|\nabla f(x)|^2}{f(x)}\,dx\right.\\ &\qquad\left.+(-A_y g,g)\int_{R^n}c(x)f(x)\,dx:f\in\mathscr{F}^+,g\in\mathscr{G}\right\}\\ &=-\inf_\psi\inf_g\left\{\frac{1}{2}\int_{R^n}|\nabla\psi(x)|^2\,dx\right.\\ &\qquad+(-A_y g,g)\int_{R^n}c(x)\psi^2(x)\,dx:\\ &\qquad\qquad\left.g\in\mathscr{G},\psi\in L^2(R^n),\|\psi\|_2=1\right\}\end{aligned}$$

where $\alpha(t)=\int_0^t c(x(s))\,ds$.

Observe that when $A_y\equiv -1$, then $u(y,y',t)=e^{-t}\delta(y-y')$ and we recover (I.4).

Proofs are given in the next section.

EXAMPLES. As a simple illustration, consider the fourth order operator

$$L=\frac{1}{2}\frac{\partial^2}{\partial x^2}-c(x)\frac{\partial^4}{\partial y^4},\quad \text{for } (x,y)\in R^1\times(0,1),$$

which arises in elasticity theory. We impose boundary conditions $g(0)=g(1)=g''(0)=g''(1)=0$ on A_y. Then $(-A_y g,g)=(g'',g'')$ and we obtain

$$\lambda_1=\inf\left\{\frac{1}{8}\int_{-\infty}^{\infty}\frac{(f'(x))^2}{f(x)}\,dx+\pi^4\int_{-\infty}^{\infty}c(x)f(x)\,dx:f\in\mathscr{F}^+\right\}.$$

Since $\alpha_1 = \inf\{(g'', g'') : g \in \mathscr{G}\} = \pi^4$ with corresponding eigenfunctions $\phi(y) = \sin \pi y$. See [**1**, p. 146] for other examples of possible boundary conditions.

Another example is $A_y = -\Delta^2$ with $u = 0$ and $\partial u/\partial \nu = 0$ on ∂G in R^2 that corresponds to a vibrating plate; see [**2**, p. 460].

II. Proofs.

LEMMA. *Let $0 < p < \infty$ and $0 < \varepsilon < a < \infty$ be given. Then for real numbers such that $0 < \alpha_1 < \alpha_2 \le \cdots$ and $\alpha_n \sim Cn^p$ as $n \to \infty$, there is a constant $M = M(\alpha_1, p, C, \varepsilon)$ such that $\sum_{n=2}^{\infty} e^{-\alpha_n a} \le Me^{-\alpha_1 a}$.*

Proof. Let $b = C/2\alpha_1$. Then by the assumption there is an N such that $\alpha_n \ge \alpha_1 bn^p$ and $bN^p \ge 1$ for $n \ge N$. Hence, $\sum_{n=2}^{\infty} e^{-\alpha_n a} \le Ne^{-\alpha_1 a} + \sum_{n=N+1}^{\infty} e^{-\alpha_n a}$ and the second term on the right is dominated by

$$\begin{aligned}\int_N^{\infty} e^{-\alpha_1 abx^p}\, dx &= \frac{e^{-\alpha_1 a}}{pb^{1/p}} \int_{bN^p}^{\infty} e^{-\alpha_1 a(z-1)} z^{(1-p)/p}\, dz \\ &\le e^{-\alpha_1 a}\left[\frac{1}{pb^{1/p}} \int_0^{\infty} e^{-\alpha_1 \varepsilon z}(1+z)^{(1-p)/p}\, dz\right].\end{aligned}$$

Letting M be the finite constant in the brackets we otain the result of the lemma.

We also need some information about the eigenvalues $\{-\alpha_n\}$ of $-A_y$ in the form of a Weyl-type theorem. First, define $w(y) = m(\xi : 0 < a^0(y,\xi) < 1)$ where m is Lebesgue measure and $W(G) = \int_G w(y)\, dy$, where $a^0(y,\xi) = \sum_{|\alpha|=2r} a_\alpha(y)\xi^\alpha$. Gårding has shown the following asymptotic expression for the eigenvalues:

$$N(\alpha) = \sum_{\alpha_n \le \alpha} 1 \sim (2\pi)^{-m} W(G)\alpha^{m/2r} \quad \text{as } \alpha \to \infty \text{ (see [\textbf{6}, p. 239])}.$$

Since $N(\alpha) = n$ for $\alpha = \alpha_n$ we have that

$$\text{(II.1)} \qquad \alpha_n \sim Cn^p \quad \text{as } n \to \infty$$

with $p = 2r/m$ and $C = (2\pi)^{2r} W(G)^{-p}$.

Comparing this with our examples, we have in the case $A_y = -\partial^4/\partial y^4$ that $\alpha_n = \pi^4 n^4$ and for $A_y = -\Delta^2$ we obtain

$$\alpha_n \sim \left(\frac{4\pi}{\text{area of the plate}}\right)^2 n^2.$$

Both of these results agree with (II.1).

Returning to the proof of the Main Theorem, in order to show (I.5) put $\alpha = \Phi(L(t,\omega,\cdot))$ and assume for now that $a \geq \varepsilon > 0$. Then by (I.2) we have

$$u(y,y,at) = \sum_{n=1}^{\infty} e^{-\alpha_n at}\phi_n^2(y).$$

Integrating with respect to y in this expression we obtain by orthonormality of the ϕ_n's that

$$\int_G u(y,y,at)\,dy = \sum_{n=1}^{\infty} e^{-\alpha_n at}$$

Applying the Lemma and (II.1) we get

$$ke^{-\alpha_1 at} \leq \sum_{n=1}^{\infty} e^{-\alpha_n at} \leq e^{-\alpha_1 at}(k+M) \tag{II.2}$$

where k is the multiplicity of α_1 and the constant M does not depend on the sample path ω. Taking E_x and then $(1/t)\log$ of both sides of (II.2) and letting $t \to \infty$ we obtain

$$\begin{aligned}\lim_{t\to\infty} \frac{1}{t}\log E_x\left[\int_G u(y,y,t\Phi(L(t,\omega,\cdot)))\right] \\ = \lim_{t\to\infty} \frac{1}{t}\log E_x[e^{-\alpha_1 t\Phi(L(t,\omega,\cdot))}] \\ = -\inf_{f\in\mathcal{F}^+}\left\{\frac{1}{8}\int_G \frac{|\nabla f(x)|^2}{f(x)}\,dx + \alpha_1\Phi(f)\right\}\end{aligned} \tag{II.3}$$

where the second equality comes from the Donsker-Varadhan result (I.3).

This proves (I.5) for $\Phi \geq \varepsilon > 0$ because $\alpha_1 = \inf_{g\in\mathcal{G}}(-A_y g, g)$. To see that the ε condition is immaterial, apply (II.3) for $\tilde{\Phi} = \Phi + \varepsilon$ then $-\alpha_1\varepsilon$ cancels on all sides of (II.3) proving (I.5) for all functionals satisfying the conditions of Donsker-Varadhan as required by the Main Theorems. This proves the Main Theorem.

Turning to the proof of the Corollary and (I.6), the way that one obtains the eigenvalues and eigenfunctions for $L = \frac{1}{2}\Delta_x + c(x)A_y$ in [7] is a two-stage process that leads to a double index for both eigenvalues and eigenfunctions as follows. Fix n and consider the operator $\frac{1}{2}\Delta_x - \alpha_n c(x)$ operating on $L^2(R^n)$. By the conditions on $c(x)$ and a result of Ray [**8**, Theorem 3], this operator has negative eigenvalues $\{-\beta_{m,n},\ m = 1,2,\dots\}$ and complete orthonormal eigenfunctions

$\{u_{m,n}(x),\ m = 1, 2, \dots\}$ in $L^2(R^n)$. This implies

$$\begin{aligned}
L(u_{m,n}(x)\phi_n(y)) &= \frac{1}{2}\Delta_x u_{m,n}(x)\phi_n(y) + c(x)A_y u_{m,n}(x)\phi_n(y) \\
&= \left(\frac{1}{2}\Delta_x - \alpha_n c(x)\right) u_{m,n}(x)\phi_n(y) \\
&= -\beta_{m,n} u_{m,n}(x)\phi_n(y).
\end{aligned}$$

Hence, the $-\beta_{m,n}$ for $m, n = 1, 2, \dots$, are eigenvalues of $L = \frac{1}{2}\Delta_x + c(x)A_y$ with corresponding eigenfunctions $u_{m,n}(x)\phi_n(y)$.

Now, by the Feynman-Kac formula, if the largest eigenvalue of $\frac{1}{2}\Delta_x - \alpha c(x)$ is $\lambda(\alpha)$, then

$$\lambda(\alpha) = \lim_{t\to\infty} \frac{1}{t} \log E_x \left[\exp\left(-\alpha \int_0^t c(X(s))\, ds\right)\right]$$

implying that $\lambda(\alpha)$ decreases as α increases, which in turn implies $-\beta_{1,1} = \sup_{m,n}(-\beta_{m,n})$, i.e., the supremum is itself an eigenvalue. We label the eigenvalues in a linear array $\cdots \le -\lambda_2 \le -\lambda_1 < 0$, so that $-\lambda_1 = -\beta_{1,1}$.

By (I.4) we have that

$$\begin{aligned}
\text{(II.4)} \quad -\lambda_1 &= \lim_{t\to\infty} \frac{1}{t} \log E_x \left[\exp\left(-\alpha_1 \int_0^t c(X(s))\, ds\right)\right] \\
&= -\inf\left\{\frac{1}{8}\int_{R^n} \frac{|\nabla f(x)|^2}{f(x)}\, dx + \alpha_1 \int_{R^n} c(x) f(x)\, dx : f \in \mathscr{F}^+\right\} \\
&= -\inf\left\{\frac{1}{2}\int_{R^n} |\nabla \psi(x)|^2\, dx \right. \\
&\qquad \left. + \alpha_1 \int_{R^n} c(x)\psi^2(x)\, dx : \psi \in L^2(R^n), \|\psi\|_2 = 1\right\}.
\end{aligned}$$

For $\Phi(f) = \int c(x) f(x)\, dx$ we have that

$$t\Phi(L(t, \omega, \cdot)) = \int_0^t c(X(s))\, ds = \alpha(t);$$

thus by (II.3) and the fact that $\alpha_1 = \inf_{g\in\mathscr{G}}(-A_y g, g)$ we obtain (I.6) upon substitution into (II.4), there by proving our corollary.

REMARK. Under an additional weak condition on the operator A_y, it is possible to show for each $y_0 \in G$ that

$$\begin{aligned}
&\lim_{t\to\infty} \frac{1}{t} \log \left| E_x \left[\int_G u(y_0, y, t\Phi(L(t, \omega, \cdot)))\, dy\right]\right| \\
&\quad = \lim_{t\to\infty} \frac{1}{t} \log E_x \left[\int_G u(y, y, t\Phi(L(t, \omega, \cdot)))\, dy\right].
\end{aligned}$$

Note that the absolute value on the left-hand side is necessary since $u(y_0, y, t)$ is not nonnegative in general.

Our results can be extended to operators of the form $L = D + V(x)$ where D is the generator of a wide class of Markov processes satisfying the assumptions of the Donsker-Varadhan theory and for fairly general operator-valued potentials $V(x)$ treated in the theory of random evolutions.

References

[1] L. Collatz, *Eigenwerte Probleme*, Chelsea, New York, 1948.

[2] R. Courant and D. Hilbert, *Methods of Mathematical Physics*, Vol. 1, Interscience-Wiley, New York, 1953.

[3] M. D. Donsker and S. R. S. Varadhan, *Asymptotic evaluation of certain Wiener integrals for large time*, Proceedings of International Conference of Function Space Integration, Oxford, (1974), 15–33.

[4] ——, *Asymptotic evaluation of certain Markov process expectations for large time* I, Comm. Pure Appl. Math., **28** (1975), 1–47.

[5] ——, *Asymptotics for the Wiener sausage*, Comm. Pure Appl. Math., **28** (1975), 525–565.

[6] L. Gårding, *On the asymptotic distribution of the eigenvalues and eigenfunctions of elliptic differential operators*, Math. Scand., **1** (1953), 237–255.

[7] R. J. Griego and R. Hersh, *Weyl's theorem for certain operator-valued potentials*, Indiana Univ. Math. J., **27** (1978), 195–209.

[8] D. Ray, *On spectra of second-order differential operators*, Trans. Amer. Math. Soc., **77** (1954), 299–321.

Received May 2, 1988.

UNIVERSITY OF NEW MEXICO
ALBUQUERQUE, NM 87131

AND

UNIVERSITY OF TEXAS
ARLINGTON, TX 76019