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NOTE ON THE INEQUALITY OF THE ARITHMETIC AND GEOMETRIC MEANS

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We show how to insert a continuum of additional terms (defined by an integral and depending on an arbitrary positive parameter) between the two sides of the generalized arithmetic-geometric mean inequality with weights. Applications give an inequality involving positive definite matrices and also a refinement of the inequality connecting the inscribed and circumscribed radii of a triangle.

We suppose throughout that

(1) $n \in \mathbb{N}$ and $a_j > 0, \quad q_j > 0\ (j = 1, \dots, n), \quad q_1 + \cdots + q_n = 1.$

Then we have the well-known inequality of the means (e.g. [**2**, #9])

$$\prod_{j=1}^{n} a_j^{q_j} \le \sum_{j=1}^{n} q_j a_j, \tag{2}$$

with equality if and only if $a_j = a_1 \quad (j = 1, \dots, n)$.

THEOREM 1. *If* (1) *holds and if* $p > 0$, *then*

$$\prod_{j=1}^{n} a_j^{q_j} \le \left\{ p \int_0^\infty \left[\prod_{j=1}^{n} (x + a_j)^{q_j} \right]^{-p-1} dx \right\}^{-1/p} \le \sum_{j=1}^{n} q_j a_j. \tag{3}$$

Proof. For $x \ge 0$, we replace a_j by $x + a_j$ in (2); then

$$0 < \prod_{j=1}^{n} (x + a_j)^{q_j} \le \sum_{j=1}^{n} q_j (x + a_j) = x + \sum_{j=1}^{n} q_j a_j.$$

Hence (for $p > 0$)

$$\int_0^\infty \left[\prod_{j=1}^{n} (x + a_j)^{q_j} \right]^{-p-1} dx \tag{4}$$

$$\ge \int_0^\infty \left[x + \sum_{j=1}^{n} q_j a_j \right]^{-p-1} dx = \frac{1}{p} \left(\sum_{j=1}^{n} q_j a_j \right)^{-p}.$$

In addition, by Hölder's integral inequality for n functions [**2**, #188],

$$(5)\quad \int_0^\infty \left[\prod_{j=1}^n (x+a_j)^{q_j}\right]^{-p-1} dx = \int_0^\infty \prod_{j=1}^n [(x+a_j)^{-p-1}]^{q_j}\,dx$$
$$\le \prod_{j=1}^n \left[\int_0^\infty (x+a_j)^{-p-1}dx\right]^{q_j}$$
$$= \prod_{j=1}^n \left[\int_0^\infty (x+a_j)^{-p-1}dx\right]^{q_j} = \prod_{j=1}^n \frac{1}{p}(a_j)^{-pq_j}.$$

Multiplying (4) and (5) through by p and raising both sides to the power $-1/p$, we obtain respectively the right and left sides of (3). □

REMARKS. (a) As examination of the proof of Theorem 1 shows, there is strict inequality in each part of (3) unless $a_j = a_1$ ($j = 1, \dots, n$).

(b) If $q_1 = \cdots = q_n = 1/n$ in (3) then, for any $p > 0$,

$$(6)\quad (a_1a_2\cdots a_n)^{1/n} \le \left\{p\int_0^\infty [(x+a_1)\cdots(x+a_n)]^{-(p+1)/n}\,dx\right\}^{-1/p}$$
$$\le \frac{1}{n}(a_1+\cdots+a_n).$$

Suppose $a_{ij} > 0$ ($i = 1, \dots, m$; $j = 1, \dots, n$); then ([**2**, #11])

$$(7)\qquad \sum_{i=1}^m \prod_{j=1}^n a_{ij}^{q_j} \le \prod_{j=1}^n \left(\sum_{i=1}^m a_{ij}\right)^{q_j}.$$

If we then use $a_j := \sum_{i=1}^m a_{ij}$ in (3) and combine (6) with the left side of (3), we obtain the apparently more general result (reducing to (3) for $m = 1$):

COROLLARY 1.1. *Let* $a_{ij} > 0$ ($i = 1, \dots, m$; $j = 1, \dots, n$), $p > 0$. *Then*

$$(8)\quad \sum_{i=1}^m \prod_{j=1}^n a_{ij}^{q_j} \le \left\{p\int_0^\infty \left[\prod_{j=1}^n \left(x+\sum_{i=1}^m a_{ij}\right)^{q_j}\right]^{-p-1} dx\right\}^{-1/p}$$
$$\le \sum_{i=1}^m \sum_{j=1}^n q_j a_{ij}.$$

If A is a real positive definite $n \times n$ matrix (namely the quadratic form vAv is positive for all non-trivial n-vectors v) then it is well known that the eigenvalues a_j $(j = 1, \ldots, n)$ are all positive. Indeed, since the a_j are the solutions of the polynomial equation $|\lambda I - A| := \det(\lambda I - A) = 0$, we clearly also have

$$|xI + A| = \prod_{j=1}^{n} (x + a_j) \quad \text{for any } x \in \mathbb{R}.$$

From this equation $(x = 0)$ and the definition of the trace, we therefore have

$$|A| = \prod_{j=1}^{n} a_j > 0, \qquad \operatorname{tr} A = \sum_{j=1}^{n} a_j.$$

Using these a_j in (6), we then obtain

COROLLARY 1.2. *Let A be a real positive definite $n \times n$ matrix and $p > 0$. Then*

$$(9) \qquad |A|^{1/n} \le \left\{ p \int_0^\infty |xI + A|^{-(p+1)/n}\, dx \right\}^{-1/p} \le \frac{1}{n} \operatorname{tr} A.$$

There is an analogue of this result similar to Corollary 1.1; replace A by $\sum_{i=1}^m A_i$ in Corollary 1.2 and use Minkowski's inequality for positive definite $n \times n$ matrices (e.g. [**1**, p. 70, Theorem 15]), that

$$\sum_{i=1}^{m} |A_i|^{1/n} \le \left| \sum_{i=1}^{m} A_i \right|^{1/n}$$

(this is really (7) in disguise, with $q_1 = \cdots = q_n = 1/n$). We obtain immediately

COROLLARY 1.3. *Let A_i $(i = 1, \ldots, m)$ be real positive definite $n \times n$ matrices, and $p > 0$. Then*

$$(10) \qquad \sum_{i=1}^{m} |A_i|^{1/n} \le \left\{ p \int_0^\infty \left| xI + \sum_{i=1}^{m} A_i \right|^{-(p+1)/n} dx \right\}^{-1/p} \le \frac{1}{n} \operatorname{tr} \left(\sum_{i=1}^{m} A_i \right).$$

As a further application of Theorem 1, we show how to insert additional terms between the two sides of Euler's inequality $2r \le R$ (e.g.

see [**3**, p. 79] or [**4**, v.2: §17.3, p. 161]) connecting the circumscribed radius R and the inscribed radius r of a triangle. If the triangle has angles A, B, C with sides a, b, c opposite these angles, and area Δ, then, using the sine rule,

$$R = \frac{a+b+c}{2(\sin A + \sin B + \sin C)} \geq \frac{a+b+c}{3\sqrt{3}}. \tag{11}$$

Also

$$2r(a+b+c) = 4\Delta \leq \sqrt{3}(abc)^{2/3}$$

(see [**4**, v.2: §17.3, pp. 161, 372]) and so, by the arithmetic-geometric mean inequality,

$$2r \leq \frac{\sqrt{3}(abc)^{2/3}}{a+b+c} \leq \frac{\sqrt{3}(abc)^{2/3}}{3(abc)^{1/3}} = \frac{(abc)^{1/3}}{\sqrt{3}}. \tag{12}$$

Now, by (11), (12) and (6), we have:

COROLLARY 1.4. *If a, b, c are the sides of a triangle, with inscribed radius r and circumscribed radius R, then, for any $p > 0$,*

$$2r\sqrt{3} \leq (abc)^{1/3} \leq J(a, b, c; p) \leq \frac{1}{3}(a+b+c) \leq R\sqrt{3}, \tag{13}$$

where

$$J(a, b, c; p) := \left\{p \int_0^\infty [(x+a)(x+b)(x+c)]^{-(p+1)/3}\, dx\right\}^{-1/p}.$$

There is strict inequality throughout (13) *unless $a = b = c$.*

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