

# TRIANGLE IDENTITIES AND SYMMETRIES OF A SUBSHIFT OF FINITE TYPE

J. B. WAGONER

**We prove the group  $\text{Aut}(\sigma_A)$  of symmetries of a subshift of finite type is isomorphic to the fundamental group of the space  $\text{RS}(\mathcal{E})$  of strong shift equivalences built from the algebraic RS Triangle Identities for zero-one matrices which arise from triangles in the contractible simplicial complex of Markov partitions. Moreover, we show the higher homotopy groups of  $\text{RS}(\mathcal{E})$  are zero.  $\text{RS}(\mathcal{E})$  is therefore homotopy equivalent to the classifying space of  $\text{Aut}(\sigma_A)$ .**

**1. Introduction and statement of results.** First we briefly review Williams' strong shift equivalence criterion for conjugacy of subshifts of finite type. See [3, 4, 8]. Let  $A: \mathcal{S} \times \mathcal{S} \rightarrow \{0, 1\}$  and  $B: \mathcal{T} \times \mathcal{T} \rightarrow \{0, 1\}$  be zero-one matrices on the finite state spaces  $\mathcal{S}$  and  $\mathcal{T}$ . An *elementary strong shift equivalence*

$$(R, S): A \rightarrow B$$

is a pair of zero-one matrices  $R: \mathcal{S} \times \mathcal{T} \rightarrow \{0, 1\}$  and  $S: \mathcal{T} \times \mathcal{S} \rightarrow \{0, 1\}$  satisfying

$$RS = A \quad \text{and} \quad SR = B.$$

Let  $(X_A, \sigma_A)$  and  $(X_B, \sigma_B)$  be the subshifts of finite type (SFT) constructed from  $A$  and  $B$  respectively. The strong shift equivalence  $(R, S)$  gives rise to an *elementary symbolic conjugacy*

$$c(R, S): X_A \rightarrow X_B$$

defined as follows: Let  $x = \{x_n\}$  be in  $X_A$ . Then  $y = c(R, S)(x)$  is the unique point  $y = \{y_n\}$  in  $X_B$  such that  $1 = A(x_n, x_{n+1}) = R(x_n, y_n)S(y_n, x_{n+1})$  for all  $n$ . Similarly, one has

$$c(S, R): X_B \rightarrow X_A$$

and it is easy to verify the identities

$$c(S, R)c(R, S) = \sigma_A \quad \text{and} \quad c(R, S)c(S, R) = \sigma_B$$

which show that  $c(R, S)$  and  $c(S, R)$  are conjugacies. More generally, let  $\mathcal{E}$  denote the set of zero-one matrices on finite state spaces. We shall assume that any matrix in  $\mathcal{E}$  has at least one non-zero entry

in each row and in each column. Williams defined  $A$  and  $B$  to be *strong shift equivalent in  $\mathcal{E}$*  provided there is a chain of elementary strong shift equivalences from  $A$  to  $B$  through intermediate matrices in  $\mathcal{E}$ . The composition of the corresponding elementary conjugacies gives a conjugacy from  $(X_A, \sigma_A)$  to  $(X_B, \sigma_B)$ . Williams' proof of the converse that conjugacy implies strong shift equivalence brings in the set  $P_A$  of Markov partitions for  $\sigma_A$  on  $X_A$ . In [8]  $P_A$  is given the structure of a locally finite simplicial complex in which each Markov partition is a vertex, and the key step in Williams' argument really amounts to showing that  $P_A$  is connected. In fact,  $P_A$  turns out to be contractible [8], and in this paper we make use of the fact that it is simply connected to prove (1.5) and contractibility to prove (1.13) below.

The definition of Markov partition used in [8] for  $P_A$  is the one in [5]. A similar theory goes through using the Markov partitions by rectangles as presented, say, in [4, p. 100]. Let  $U = \{U_i\}$  be in  $P_A$  and let  $M = M(U) = \{M(i, j)\}$  be the zero-one matrix where

$$M(i, j) = 1 \quad \text{iff} \quad U_i \cap \sigma_A^{-1}(U_j) \neq \emptyset.$$

For example,  $A = M(U^A)$  where  $U^A = \{U_i^A\}$  is the "standard" Markov partition with  $U_i^A$  equal to the cylinder set of those  $x = \{x_n\}$  such that  $x_0 = i$ . Let  $V = \{V_k\}$  also be in  $P_A$ . Define zero-one matrices  $R = R(U, V) = \{R(i, k)\}$  and  $S(V, U) = \{S(k, i)\}$  by the formulas

$$\begin{aligned} R(i, k) &= 1 \quad \text{if and only if} \quad U_i \cap V_k \neq \emptyset, \\ S(k, i) &= 1 \quad \text{if and only if} \quad V_k \cap \sigma_A^{-1}(U_i) \neq \emptyset. \end{aligned}$$

Write  $U < V$  to mean that  $V$  refines  $U$ . As in [8] we write  $U \xrightarrow{+} V$  provided  $U < V < U \cap \sigma_A^{-1}(U)$ , and we write  $U \xrightarrow{-} V$  provided  $U < V < \sigma_A(U) \cap U$ . Finally, we write  $U \rightarrow V$  iff  $U \xrightarrow{+} U \cap V \xleftarrow{-} V$ . An (ordered) 1-simplex of  $P_A$  is a pair  $[U, V]$  where  $U \neq V$  and  $U \rightarrow V$ . Let  $P = M(U)$ ,  $Q = M(V)$ , and  $R$  and  $S$  be as above. It was shown in [8] that

$$P = RS \quad \text{and} \quad Q = SR$$

whenever  $U \rightarrow V$ ; that is,  $(R, S): P \rightarrow Q$ . In particular, connectivity of  $P_A$  implies the transition matrices  $M(U)$  and  $M(V)$  of any two Markov partitions  $U$  and  $V$  in  $P_A$  are strong shift equivalent in  $\mathcal{E}$ . To finish off the outline of Williams' proof, let  $\alpha: (X_A, \sigma_A) \rightarrow (X_B, \sigma_B)$  be a topological conjugacy. Let  $U = \{U_i\}$  be in  $P_A$  where

$i$  runs through an indexing set  $I$ . Then  $\alpha(U) = \{\alpha(U_i)\}$  is in  $P_B$ . In fact, let  $V = \{V_k\}$  be in  $P_B$  where  $k$  is in the index set  $K$ , and suppose we have  $V = \alpha(U)$ ; that is, each  $V_k = \alpha(U_i)$  for exactly one  $U_i$ . Then  $\alpha$  gives a bijection  $\alpha: I \rightarrow K$ . Throughout this paper we will follow the convention that  $\alpha$  also denotes the  $K \times I$  permutation matrix with  $\alpha(k, i) = 1$  iff  $\alpha(i) = k$ . Let  $P = M(U)$  and  $Q = M(V)$ . Since  $U_i \cap \sigma_A^{-1}(U_j) \neq \emptyset$  iff  $\alpha(U_i) \cap \sigma_A^{-1}(\alpha(U_j)) \neq \emptyset$ , we have  $Q = \alpha P \alpha^{-1}$ . Therefore

$$(\alpha^{-1}, \alpha P): P \rightarrow Q.$$

Hence  $A$  is strong shift equivalent to  $B$ .

Let  $\text{Aut}(\sigma_A)$  be the group of symmetries of  $(X_A, \sigma_A)$ . By definition, this is the group of homeomorphisms of  $X_A$  which commute with  $\sigma_A$ . It is discrete in the topology of uniform convergence, because  $\sigma_A$  is expansive. See [1, 2, 8] for some recent information about this group. The preceding discussion suggests that elements of  $\text{Aut}(\sigma_A)$  can be described as products of various elementary conjugacies  $c(R, S)$  modulo certain relations. This turns out to be the case, and a very natural set of relations which do work come from triangles in  $P_A$ .

By definition a triangle in  $P_A$  is an ordered triple  $[U, V, W]$  of Markov partitions such that  $U \rightarrow V$ ,  $V \rightarrow W$ , and  $U \rightarrow W$ . Let

$$(1.1) \quad \begin{aligned} M &= M(U), & P &= M(V), & Q &= M(W), \\ R_1 &= R(U, V), & S_1 &= S(V, U), \\ R_2 &= R(V, W), & S_2 &= S(W, V), \\ R_3 &= R(U, W), & S_3 &= S(W, U). \end{aligned}$$

In §2 we will verify the *RS Triangle Identities*:

$$(1.2) \quad R_1 R_2 = R_3, \quad R_2 S_3 = S_1, \quad S_3 R_1 = S_2.$$

Upon either multiplying the second equation on the left by  $S_2$  or by multiplying the third equation on the right by  $S_1$ , we derive the *SS Triangle Identities* found in [8]:

$$(1.3) \quad \begin{aligned} R_1 R_2 &= R_3, \\ S_2 S_1 &= Q S_3 = S_3 M. \end{aligned}$$

**DEFINITION 1.4** The space  $RS(\mathcal{E})$  of strong shift equivalence in  $\mathcal{E}$  based on the *RS Triangle Identities* is the geometric realization of the simplicial set where an  $n$ -simplex consists of the following data:

- (a) an  $(n+1)$ -tuple  $\langle A_0, \dots, A_n \rangle$  of square matrices in  $\mathcal{E}$  and
- (b) for each  $i < j$  a strong shift equivalence  $(R_{ij}, S_{ji}): A_i \rightarrow A_j$  in

$\mathcal{E}$  such that the RS Triangle Identities hold for  $i < j < k$ ; that is,

$$R_{ij}R_{jk} = R_{ik}, \quad R_{jk}S_{ki} = S_{ji}, \quad S_{ki}R_{ij} = S_{kj}.$$

The face operators are the usual forgetful ones and the degeneracies insert the strong shift equivalence  $(1, A_i)$  from  $A_i$  to itself. See [6] or [7] for background on simplicial sets and CW complexes. It is immediate from the definition that the set of components  $\pi_0(\mathbf{RS}(\mathcal{E}))$  is exactly the set of strong shift equivalences in  $\mathcal{E}$ .

**THEOREM 1.5.** *There is an isomorphism*

$$\Phi_A: \text{Aut}(\sigma_A) \rightarrow \pi_1(\mathbf{RS}(\mathcal{E}), A).$$

The explicit formula for  $\Phi_A$  is given in (3.5) and (3.8) below.

For any strong shift equivalence  $(R, S): P \rightarrow Q$ , let  $\gamma(R, S)$  denote the corresponding homotopy class of paths from  $P$  to  $Q$  in  $\mathbf{RS}(\mathcal{E})$ . Elementary arguments in algebraic topology show that elements  $\gamma$  in  $\pi_1(\mathbf{RS}(\mathcal{E}), A)$  can be represented as products

$$(1.6) \quad \gamma = \prod_{i=1}^n \gamma(R_i, S_i)^{\varepsilon_i}$$

where  $\varepsilon_i = \pm 1$ . The defining relations are

$$\gamma(R, S)\gamma(R, S)^{-1} = 1$$

and

$$\gamma(R_1, S_1)\gamma(R_2, S_2) = \gamma(R_3, S_3)$$

whenever the RS Triangle Identities hold; that is,

$$(1.7) \quad \begin{array}{ccc} & P & \\ \nearrow (R_1, S_1) & & \searrow (R_2, S_2) \\ M & \xrightarrow{(R_3, S_3)} & Q \end{array}$$

is a triangle in  $\mathbf{RS}(\mathcal{E})$ . Given this presentation of  $\pi_1(\mathbf{RS}(\mathcal{E}), A)$ , the inverse

$$\Theta_A: \pi_1(\mathbf{RS}(\mathcal{E}), A) \rightarrow \text{Aut}(\sigma_A)$$

of  $\Phi_A$  is easy to describe: namely,  $\Theta_A$  takes the product (1.6) to

$$(1.8) \quad \Theta_A(\gamma) = \prod_{i=1}^n c(R_i, S_i)^{-\varepsilon_i}$$

where composition is read from right to left. That this is well defined follows immediately from

LEMMA 1.9. *Suppose  $(R_1, S_1): M \rightarrow P$ ,  $(R_2, S_2): P \rightarrow Q$ , and  $(R_3, S_3): M \rightarrow Q$ . Assume that  $R_1 R_2 = R_3$ . Then*

$$c(R_2, S_2)c(R_1, S_1) = c(R_3, S_3)$$

*if and only if  $R_2 S_3 = S_1$  and  $S_3 R_1 = S_2$ .*

The first equation of the RS Triangle Identities is very reasonable. This lemma shows that the last equations, which were harder to guess, are exactly what is needed to make composition of elementary conjugacies behave well around a triangle.

There is also the *space*  $\text{SS}(\mathcal{E})$  of strong shift equivalence in  $\mathcal{E}$  based on the SS Triangle Identities with the same face and degeneracy operators as used for  $\text{RS}(\mathcal{E})$ . Since the RS Identities imply the SS identities, there is a natural continuous map

$$\text{RS}(\mathcal{E}) \rightarrow \text{SS}(\mathcal{E}).$$

The set of components  $\pi_0(\text{SS}(\mathcal{E}))$  is also the set of strong shift equivalence in  $\mathcal{E}$  and the induced map  $\pi_0(\text{RS}(\mathcal{E})) \rightarrow \pi_0(\text{SS}(\mathcal{E}))$  is a bijection. The commutative diagram (4.30) of [8] expands to

$$(1.10) \quad \begin{array}{ccccc} & & \pi_1(\text{RS}(\mathcal{E}), A) & & \\ & \nearrow \Phi_A & \downarrow & \nwarrow & \\ \text{Aut}(\sigma_A) & \xrightarrow{\quad} & \pi_1(\text{SS}(\mathcal{E}), A) & & \\ & \searrow \Psi_A & \downarrow \Psi_A & \swarrow & \\ & & \pi_1(S(\overline{\mathcal{E}}), A) & & \end{array}$$

where  $S(\overline{\mathcal{E}})$  is the space of shift equivalences of non-negative integral matrices. It was shown in [8] there is an isomorphism between  $\pi_1(S(\overline{\mathcal{E}}), A)$  and the group  $\text{Aut}(s_A) = \text{Aut}(G(A), G(A)_+, s_A)$  of order preserving automorphisms of the dimension group  $G(A)$  which commute with the automorphism  $s_A$  induced on  $G(A)$  by  $A$ . The Finite

Order Generation Conjecture (FOG) can therefore be reformulated to state

The kernel of

$$(1.11) \quad \delta_A: \pi_1(\text{RS}(\mathcal{E}), A) \rightarrow \pi_1(S(\overline{\mathcal{E}}), A)$$

is generated by elements of finite order.

In [9] we use the spaces  $\text{RS}(\mathcal{E})$  and  $S(\overline{\mathcal{E}})$  together with methods of this paper to prove the following eventual version of this conjecture:

**Eventual fog 1.12.** Let  $\alpha$  be in  $\ker(\delta_A)$ . There is an integer  $k_0 \geq 1$  such that if  $k \geq k_0$ , then  $\alpha$  is a product of homeomorphisms of  $X_A$  of finite order which commute with  $\sigma_A^k$ .

In the proof of (1.12) given in [9], the integer  $k_0$  seems to depend on  $\alpha$  and the product expressions are possibly different for different  $k$ . Of course, FOG conjectures that we can take  $k_0 = 1$ , in which case the product expression for  $k = 1$  works for all  $k$ . M. Boyle has shown  $\alpha$  in (1.12) is a product of just two finite order elements commuting with  $\sigma_A^k$  for large  $k$ , although in practice one must take  $k$  larger than  $k_0$  to do this.

In §4 we prove

**THEOREM 1.13.**  $\pi_n(\text{RS}(\mathcal{E}), A) = 0$  for  $n \geq 2$ .

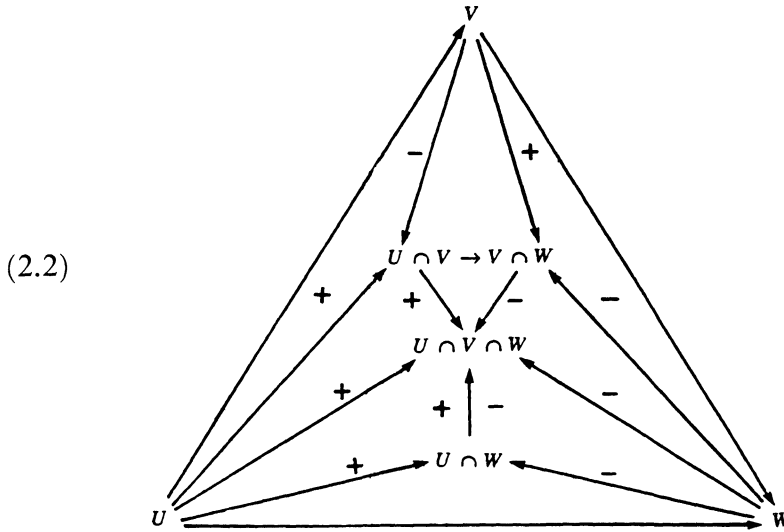
For simplicity, we assume throughout this paper that all the matrices in  $\mathcal{E}$  are finite. However, (1.5) and (1.13) generalize without change when  $\mathcal{E}$  consists of infinite matrices in one of the three classes considered in (4.1) of [8]. There is also a stochastic version for symmetries of  $(X_A, \sigma_A)$  which preserve a  $\sigma_A$ -invariant Markov measure.

**2. Proof of the RS triangle identities.** Consider the triangle

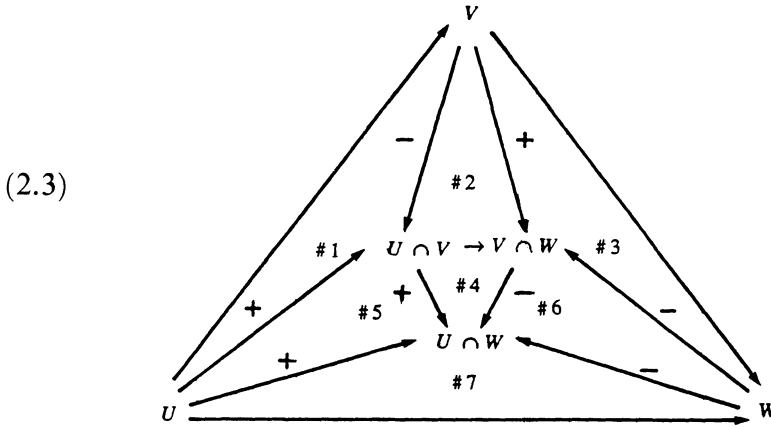
$$(2.1) \quad \begin{array}{ccc} & v & \\ \nearrow & & \searrow \\ u & \xrightarrow{\quad} & w \end{array}$$

in  $P_A$  with the corresponding matrices (1.1). The equation  $R_1 R_2 = R_3$  was verified in (3.3) of [8]. So it remains to prove the last two

equations in (1.2). As in §3 of [8] this subdivides into 9 triangles



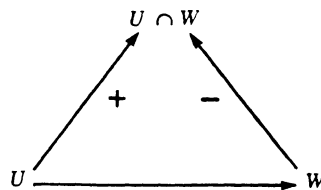
As in (3.9) of [8] we have  $U \cap W = U \cap V \cap W$  and the diagram



The first step is to verify the RS Triangle Identities for each triangle in (2.3). In all that follows, “LHS” is the left hand side of an equation and “RHS” is the right hand side. Let  $\sigma = \sigma_A$ .

*Triangles #1, #3, #4, #7:*

These are all of the form



(I)  $S_1 = R_2 S_3$ .

LHS:  $S_1((i, p), j) = 1$  iff  $U_i \cap W_p \cap \sigma^{-1}(U_j) \neq \emptyset$ .

RHS: The term  $R_2((i, p), q) S_3(q, j) = 1$  iff  $U_i \cap W_p \cap W_q \neq \emptyset$  and  $W_q \cap \sigma^{-1}(U_j) \neq \emptyset$ .

In particular  $q = p$ ; so there is at most one non-zero term and the RHS is a zero-one matrix.

Write

$$U_i = \bigcup \sigma(W_a) \cap W_b, \quad U_j = \bigcup \sigma(W_c) \cap W_d.$$

Assume RHS  $\neq 0$ . Then  $U_i \cap W_p \neq \emptyset$  implies some  $b = p$  and  $W_q \cap \sigma^{-1}(U_j) \neq \emptyset$  implies some  $c = q = p$ . Therefore,

$$U_i \cap \sigma^{-1}(U_j) \supset \sigma(W_a) \cap W_p \cap \sigma^{-1}(W_d) \neq \emptyset$$

for some  $a$  and some  $d$ . This shows  $U_i \cap W_p \cap \sigma^{-1}(U_j) \neq \emptyset$  implying LHS  $\neq 0$ . Conversely, suppose that LHS  $\neq 0$ . Then  $U_i \cap W_p \cap \sigma^{-1}(U_j) \neq \emptyset$ . Let  $q = p$ . Then  $U_i \cap W_p \cap W_q \neq \emptyset$  and  $W_q \cap \sigma^{-1}(U_j) \neq \emptyset$  implying RHS  $\neq 0$ .

(II)  $S_2 = S_3 R_1$ .

LHS:  $S_2(p, (j, q)) = 1$  iff  $W_p \cap \sigma^{-1}(U_j) \cap \sigma^{-1}(W_q) \neq \emptyset$ .

RHS: Consider a single term  $S_3(p, i) R_1(i, (j, q))$ . This is equal to 1 iff  $W_p \cap \sigma^{-1}(U_i) \neq \emptyset$  and  $U_i \cap U_j \cap W_q \neq \emptyset$ ; which implies  $i = j$  and  $U_i \supset U_j \cap W_q$ .

Thus there is at most one  $i$  giving a non-zero term on the RHS, and so  $S_3 R_1$  is a zero-one matrix.

Now write

$$W_p = \bigcup U_a \cap \sigma^{-1}(U_b), \quad W_q = \bigcup U_e \cap \sigma^{-1}(U_f).$$

Assume RHS  $\neq 0$ . Then  $i = j$  and

$$W_p \cap \sigma^{-1}(U_i) = \bigcup U_a \cap \sigma^{-1}(U_b) \cap \sigma^{-1}(U_j)$$

implying some  $b = j$ . Similarly,

$$U_j \cap W_q = \bigcup U_j \cap U_e \cap \sigma^{-1}(U_f)$$

implying some  $e = j$ . Therefore,

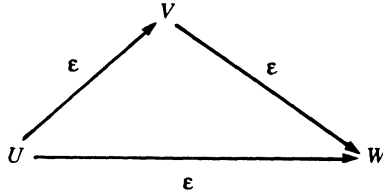
$$W_p \cap \sigma^{-1}(U_j) \cap \sigma^{-1}(W_q) \supset U_a \cap \sigma^{-1}(U_j) \cap \sigma^{-2}(U_f) \neq \emptyset$$

for some  $a$  and some  $f$ . Hence LHS  $\neq 0$ . Conversely, suppose LHS  $\neq 0$ . Then  $W_p \cap \sigma^{-1}(U_j) \neq \emptyset$  and  $U_j \cap U_j \cap W_q \neq \emptyset$  implying RHS  $\neq 0$ .



*Triangles #5, #6:*

These are of the form



where all  $\varepsilon = \pm 1$  are equal.

First, we give the argument for  $\varepsilon = +1$ .

(I)  $S_1 = R_2 S_3$ .

LHS:  $S_1(k, i) \neq 0$  iff  $V_k \cap \sigma^{-1}(U_i) \neq \emptyset$ .

RHS:  $R_2(k, p)S_3(p, i) \neq 0$  iff  $V_k \cap W_p \neq \emptyset$  and  $W_p \cap \sigma^{-1}(U_i) \neq \emptyset$ .

We claim that RHS is a zero-one matrix. Given  $k$ , choose some  $j$  such that  $R_1(j, k) \neq 0$ . Then  $R_1(j, k)R_2(k, p)S_3(p, i) = R_3(j, p)S_3(p, i)$ . Since  $R_3 S_3 = Q$  is a zero-one matrix, there can be at most one  $p$  for which this term is non-zero.

Assume LHS  $\neq 0$ . We have  $W < U \cap \sigma^{-1}(U) < V \cap \sigma^{-1}(U)$ . So choose the unique index  $p$  with  $W_p \supset V_k \cap \sigma^{-1}(U_i)$ . Then  $V_k \cap W_p \neq \emptyset$  and  $W_p \cap \sigma^{-1}(U_i) \neq \emptyset$  implying RHS  $\neq 0$ . Conversely, let RHS  $\neq 0$ . Since  $V < W$ ,  $V_k \cap W_p \neq \emptyset$  implies  $V_k \supset W_p$ . Thus  $W_p \cap \sigma^{-1}(U_i) \neq \emptyset$  implies  $V_k \cap \sigma^{-1}(U_i) \neq \emptyset$ . This shows LHS  $\neq 0$ .

(II)  $S_2 = S_3 R_1$ .

LHS:  $S_2(p, k) \neq 0$  iff  $W_p \cap \sigma^{-1}(V_k) \neq \emptyset$ .

RHS:  $S_3(p, i)R_1(i, k) \neq 0$  iff  $W_p \cap \sigma^{-1}(U_i) \neq \emptyset$  and  $U_i \cap V_k \neq \emptyset$ .

Since  $U < V$ , we have  $U_i \supset V_k$  and thus  $i$  is determined by  $k$ . This shows RHS is a zero-one matrix.

Assume LHS  $\neq 0$ . Let  $i$  be such that  $U_i \supset V_k$ . Then  $W_p \cap \sigma^{-1}(U_i) \neq \emptyset$  and  $U_i \cap V_k \neq \emptyset$  showing that RHS  $\neq 0$ . Conversely, suppose RHS  $\neq 0$ . Then  $U_i \supset V_k$  as remarked above. Write

$$W_p = \bigcup U_a \cap \sigma^{-1}(U_b), \quad V_k = \bigcup U_i \cap \sigma^{-1}(U_d).$$

The condition  $W_p \cap \sigma^{-1}(U_i) \neq \emptyset$  implies some  $b = i$ . Hence,

$$W_p \cap \sigma^{-1}(V_k) \supset U_a \cap \sigma^{-1}(U_i) \cap \sigma^{-2}(U_d) \neq \emptyset$$

for some  $a$  and some  $d$ . Thus LHS  $\neq 0$ .

Next, we give the argument for  $\varepsilon = -1$ . Remember that now  $R_1 = R(W, V)$ ,  $R_2 = R(V, U)$ ,  $R_3 = R(W, U)$ , etc.

(I)  $S_1 = R_2 S_3$ ,

LHS:  $S_1(k, p) \neq 0$  iff  $V_k \cap \sigma^{-1}(W_p) \neq \emptyset$ .

RHS:  $R_2(k, i)S_3(i, p) \neq 0$  iff  $V_k \cap U_i \neq \emptyset$  and  $U_i \cap \sigma^{-1}(W_p) \neq \emptyset$ .

We claim that RHS is a zero-one matrix. Given  $k$  and  $p$ , choose some  $q$  such that  $R_1(q, k) \neq 0$ . Then

$$R_1(q, k)R_2(k, i)S_3(i, p) = R_3(q, i)S_3(i, p).$$

Since  $R_3 S_3 = Q$  is a zero-one matrix, there can be at most one  $i$  for which this term is non-zero.

Assume LHS  $\neq 0$ . We have  $V < U \cap \sigma^{-1}(U) < V \cap \sigma^{-1}(W)$ . So choose the unique index  $i$  with  $U_i \supset V_k \cap \sigma^{-1}(W_p) \neq \emptyset$ . Then  $V_k \cap U_i \neq \emptyset$  and  $U_i \cap \sigma^{-1}(W_p) \neq \emptyset$  implying RHS  $\neq 0$ . Conversely, the RHS  $\neq 0$ . Write

$$W_p = \bigcup \sigma(U_a) \cap U_b, \quad V_k = \bigcup \sigma(U_c) \cap U_d.$$

Since  $V_k \cap U_i \neq \emptyset$ , some  $d = i$ . Also  $U_i \cap \sigma^{-1}(W_p) \neq \emptyset$  implies some  $a = i$ . Therefore

$$V_k \cap \sigma^{-1}(W_p) \supset \sigma(U_c) \cap U_i \cap \sigma^{-1}(U_b) \neq \emptyset$$

for some  $c$  and some  $b$ . Thus LHS  $\neq 0$ .

(II)  $S_2 = S_3 R_1$ .

LHS:  $S_2(i, k) \neq 0$  iff  $U_i \cap \sigma^{-1}(V_k) \neq \emptyset$ .

RHS:  $S_3(i, p)R_1(p, k) \neq 0$  iff  $U_i \cap \sigma^{-1}(W_p) \neq \emptyset$  and  $W_p \cap V_k \neq \emptyset$ .

We claim that RHS is a zero-one matrix. Given  $i$  and  $k$ , choose some  $j$  such that  $R_2(k, j) \neq 0$ . Then

$$S_3(i, p)R_1(p, k)R_2(k, j) = S_3(i, p)R_3(p, j).$$

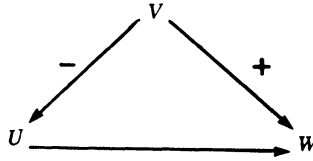
Since  $S_3 R_3 = M$  is a zero-one matrix, there can be at most one  $p$  for which this term is non-zero.

Assume LHS  $\neq 0$ . We have  $W < \sigma(U) \cap U < \sigma(U) \cap V$ . Let  $p$  be the unique index where  $W_p \supset \sigma(U_i) \cap V_k \neq \emptyset$ . Then  $U_i \cap \sigma^{-1}(W_p) \neq \emptyset$

and  $W_p \cap V_k \neq \emptyset$ . This says  $\text{RHS} \neq 0$ . Conversely, assume  $\text{RHS} \neq 0$ . Since  $V < W$ , we must have  $W_p \subset V_k$ . Hence  $U_i \cap \sigma^{-1}(V_k) \neq \emptyset$  and  $\text{LHS} \neq 0$ .

*Triangle #2:*

This triangle is of the form



(I)  $S_1 = R_2 S_3$ .

LHS:  $S_1(k, i) \neq 0$  iff  $V_k \cap \sigma^{-1}(U_i) \neq \emptyset$ .

RHS:  $R_2(k, p)S_3(p, i) \neq 0$  iff  $V_k \cap W_p \neq \emptyset$  and  $W_p \cap \sigma^{-1}(U_i) \neq \emptyset$ .

The RHS is a zero-one matrix. To see this, choose  $j$  so that  $R_1(j, k) = 1$ . Then  $R_1(j, k)R_2(k, p)S_3(p, i) = R_3(j, p)S_3(p, i)$ . Since  $R_3 S_3 = M$  is a zero-one matrix, there is at most one  $p$  giving a non-zero term.

Assume  $\text{LHS} \neq 0$ . We have  $W < V \cap \sigma^{-1}(V) < V \cap \sigma^{-1}(U)$ . So choose  $p$  with  $W_p \supset V_k \cap \sigma^{-1}(U_i) \neq \emptyset$ . Then  $V_k \cap W_p \neq \emptyset$  and  $W_p \cap \sigma^{-1}(U_i) \neq \emptyset$  and  $\text{RHS} \neq 0$ . Conversely, suppose  $\text{RHS} \neq 0$ . Then  $V_k \supset W_p$  and  $V_k \cap \sigma^{-1}(U_i) \neq \emptyset$ . This gives  $\text{LHS} \neq 0$ .

(II)  $S_2 = S_3 R_1$ .

LHS:  $S_2(p, k) \neq 0$  iff  $W_p \cap \sigma^{-1}(V_k) \neq \emptyset$ .

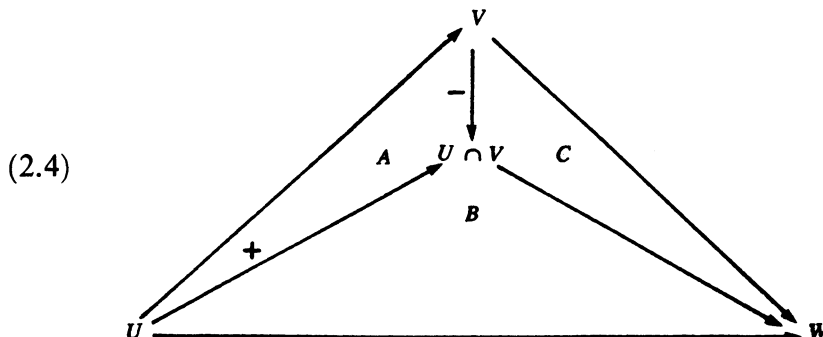
RHS:  $S_3(p, i)R_1(i, k) \neq 0$  iff  $W_p \cap \sigma^{-1}(U_i) \neq \emptyset$  and  $U_i \cap V_k \neq \emptyset$ .

The RHS is a zero-one matrix. To see this, let  $q$  be any index where  $R_2(k, q) \neq 0$ . Then  $S_3(p, i)R_1(i, k)R_2(k, q) = S_3(p, i)R_3(i, q)$ . Since  $S_3 R_3 = Q$  is zero-one, there is at most one  $i$  for which the term is non-zero.

Assume  $\text{LHS} \neq 0$ . We have  $U < \sigma(V) \cap V < \sigma(W) \cap V$ . Let  $i$  be the index for which  $U_i \supset \sigma(W_p) \cap V_k \neq \emptyset$ . Then  $W_p \cap \sigma^{-1}(U_i) \neq \emptyset$  and  $U_i \cap V_k \neq \emptyset$ . This shows  $\text{RHS} \neq 0$ . Finally, let  $\text{RHS} \neq 0$ . Then  $U_i \subset V_k$ , because  $U > V$ . Therefore  $W_p \cap \sigma^{-1}(V_k) \neq \emptyset$  and  $\text{LHS} \neq 0$ .

This completes the argument for the triangles in (2.3). We now show how these combine to prove the RS Identities for (2.1). From

(2.2) we have the diagram of three triangles



*Step 1.* We will show the RS Identities for Triangles A, B, and C in (2.4) imply the RS Identities for (2.1).

$$(I) \ S(V, U) = R(V, W)S(W, U)?$$

Since  $U < U \cap V$ , the matrix  $R(U, U \cap V)$  has exactly one non-zero term in each column. Therefore, the above equation holds provided it holds when multiplied on the right by  $R(U, U \cap V)$ . We have

$$S(V, U)R(U, U \cap V) = S(V, U \cap V)$$

and

$$\begin{aligned} R(V, W)S(W, U)R(U, U \cap V) &= R(V, W)S(W, U \cap V) \\ &= S(V, U \cap V) \end{aligned}$$

$$(II) \ S(W, V) = S(W, U)R(U, V)?$$

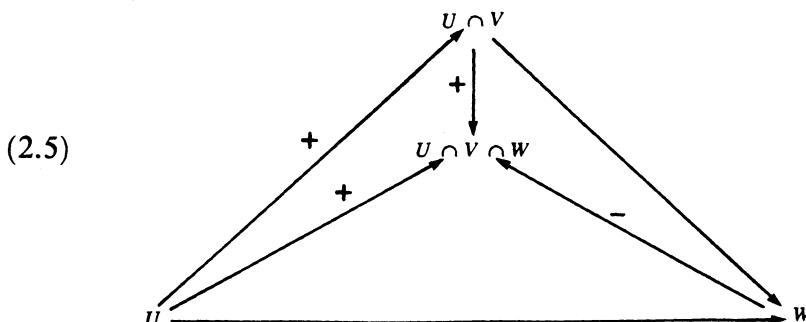
$$\begin{aligned} S(W, V) &= S(W, U \cap V)R(U \cap V, V) \\ &= S(W, U)R(U, U \cap V)R(U \cap V, V) \\ &= S(W, U)R(U, V). \end{aligned}$$

It remains to check the RS Identities for each of the triangles A, B, C.

*Step 2.* Triangle A is just Triangle #1 of (2.3).

*Step 3.* Triangle B.

From (2.2) and (2.3) we obtain the diagram



Recall that  $U \cap W = U \cap V \cap W$  and that the RS Identities for each of the subtriangles was verified above for (2.3).

$$(I) \ S(U \cap V, U) = R(U \cap V, W)S(W, U)?$$

$$\begin{aligned} S(U \cap V, U) &= R(U \cap V, U \cap V \cap W)S(U \cap V \cap W, U) \\ &= R(U \cap V, U \cap V \cap W)R(U \cap V \cap W, W)S(W, U) \\ &= R(U \cap V, W)S(W, U). \end{aligned}$$

$$(II) \ S(W, U \cap V) = S(W, U)R(U, U \cap V)?$$

Similarly to (I) of Step 1 this equation holds iff it holds when multiplied on the right by  $R(U \cap V, U \cap V \cap W)$ . We have

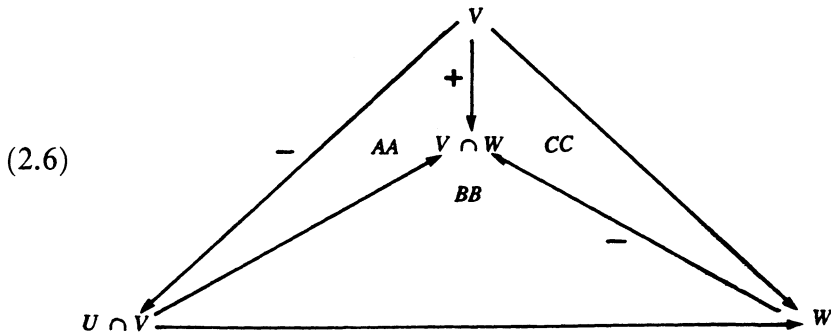
$$S(W, U \cap V)R(U \cap V, U \cap V \cap W) = S(W, U \cap V \cap W)$$

and

$$\begin{aligned} S(W, U)R(U, U \cap V)R(U \cap V, U \cap V \cap W) \\ = S(W, U)R(U, U \cap V \cap W) = S(W, U \cap V \cap W). \end{aligned}$$

*Step 4. Triangle C.*

From (2.2) and (2.3) we get



Assuming the RS Identities hold for the Triangles AA, BB, and CC, we now verify the RS Identities for Triangle C. In Step 5 below the RS Identities for AA, BB, CC are checked.

$$(I) \ S(V, U \cap V) = R(V, W)S(W, U \cap V)?$$

$$\begin{aligned} S(V, U \cap V) &= R(V, V \cap W)S(V \cap W, U \cap V) \\ &= R(V, V \cap W)R(V \cap W, W)S(W, U \cap V) \\ &= R(V, W)S(W, U \cap V) \end{aligned}$$

$$(II) \ S(W, V) = S(W, U \cap V)R(U \cap V, V)?$$

Similarly to above remarks, this equation holds iff it holds when multiplied on the right by  $R(V, V \cap W)$ . We have

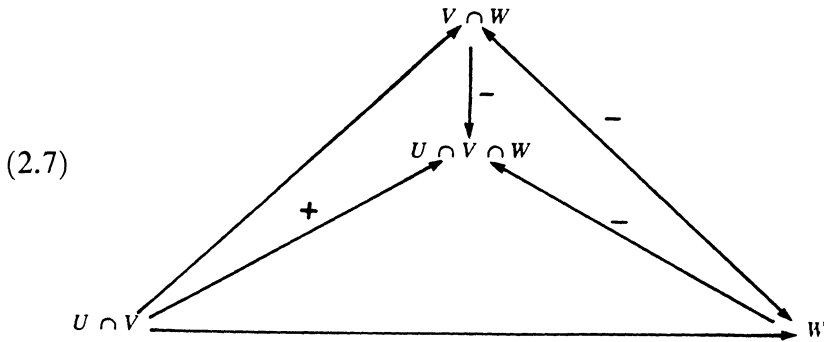
$$S(W, V)R(V, V \cap W) = S(W, V \cap W)$$

and

$$\begin{aligned} S(W, U \cap V)R(U \cap V, V)R(V, V \cap W) \\ = S(W, U \cap V)R(U \cap V, V \cap W) = S(W, V \cap W). \end{aligned}$$

*Step 5. Triangles AA, BB, CC*

Triangles AA and CC are just like Triangles #2 and #3 in (2.3). From diagram (2.2) we see that Triangle BB subdivides into



Each of the subtriangles in (2.7) satisfies the RS Identities by Step 1. So proceed as before:

$$(I) \ S(V \cap W, U \cap W) = R(V \cap W, W)S(W, U \cap V)?$$

The matrix  $R(U \cap V \cap W, V \cap W)$  has the property that each row has exactly one non-zero entry. Therefore the above equation holds iff it holds when multiplied on the left by  $R(U \cap V \cap W, V \cap W)$ . We have

$$\begin{aligned} R(U \cap V \cap W, V \cap W)S(V \cap W, U \cap V) \\ = S(U \cap V \cap W, U \cap V) \end{aligned}$$

and

$$\begin{aligned} R(U \cap V \cap W, V \cap W)R(V \cap W, W)S(W, U \cap V) \\ = R(U \cap V \cap W, W)S(W, U \cap V) = S(U \cap V \cap W, U \cap V). \end{aligned}$$

$$(II) \ S(W, V \cap W) = S(W, U \cap V)R(U \cap V, V \cap W)?$$

$$\begin{aligned} S(W, V \cap W) &= S(W, U \cap V \cap W)R(U \cap V \cap W, V \cap W) \\ &= S(W, U \cap V)R(U \cap V, U \cap V \cap W)R(U \cap V \cap W, V \cap W) \\ &= S(W, U \cap V)R(U \cap V, V \cap W). \end{aligned}$$

This finally completes the proof of the RS Triangle Identities.

**3. The isomorphisms  $\Phi_A$  and  $\Theta_A$ .** Our first goal is to give a formula in (3.5) and (3.8) for the homomorphism

$$\Phi_A: \text{Aut}(\sigma_A) \rightarrow \pi_1(\text{RS}(\mathcal{E}), A).$$

If the square matrices  $A$  and  $B$  lie in the same component of  $\text{RS}(\mathcal{E})$ , let

$$\pi_1(\text{RS}(\mathcal{E}); A, B)$$

denote the set of homotopy classes of paths starting at  $A$  and ending at  $B$ . Concatenation of paths gives a pairing

$$\pi_1(\text{RS}(\mathcal{E}); A, B) \times \pi_1(\text{RS}(\mathcal{E}); B, C) \rightarrow \pi_1(\text{RS}(\mathcal{E}); A, C)$$

denoted by “ $*$ ”. When  $A = B$  we just have the fundamental group  $\pi_1(\text{RS}(\mathcal{E}), A)$ .

Let  $\text{Isom}(\sigma_A, \sigma_B)$  denote the set of conjugacies  $\alpha: (X_A, \sigma_A) \rightarrow (X_B, \sigma_B)$ . Suppose  $U \rightarrow V$  in  $P_A$  with  $P = M(U)$  and  $Q = M(V)$ , and let  $R$  and  $S$  be defined as in §1 giving a strong shift equivalence  $(R, S): P \rightarrow Q$ . Let  $\alpha$  be in  $\text{Isom}(\sigma_A, \sigma_B)$ . Let  $U' = \alpha(U)$  and  $V' = \alpha(V)$  in  $P_B$ . Let  $P' = M(U')$  and  $Q' = M(V')$ . We have  $U' \rightarrow V'$  in  $P_B$ . Let  $R'$  and  $S'$  be the corresponding matrices giving a strong shift equivalence from  $P'$  to  $Q'$ . These matrices satisfy the identities

$$\begin{aligned} P' &= \alpha P \alpha^{-1}, & Q' &= \alpha Q \alpha^{-1}, \\ R' &= \alpha R \alpha^{-1}, & S' &= \alpha S \alpha^{-1}, \end{aligned}$$

which translate into the following diagrams of triangles in  $\text{RS}(\mathcal{E})$ :

$$(3.1) \quad \begin{array}{ccc} P & \xrightarrow{(R, S)} & Q \\ (\alpha^{-1}, \alpha P) \downarrow & \nearrow (\alpha R, S \alpha^{-1}) & \downarrow (\alpha^{-1}, \alpha Q) \\ P' & \xrightarrow{(R', S')} & Q' \end{array}$$

$$(3.2) \quad \begin{array}{ccc} P & \xrightarrow{(R, S)} & Q \\ (\alpha^{-1}, \alpha P) \downarrow & \searrow (R \alpha^{-1}, \alpha S) & \downarrow (\alpha^{-1}, \alpha Q) \\ P' & \xrightarrow{(R', S')} & Q' \end{array}$$

Incidentally, these two diagrams form the boundary of the tetrahedron  $\langle P, P', Q, Q' \rangle$  in  $RS(\mathcal{E})$ .

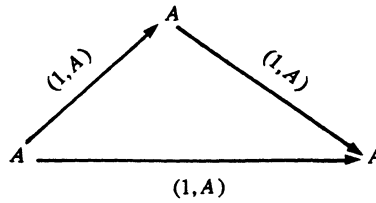
LEMMA 3.3. (a)  $\gamma(1, A) = 1$  in  $\pi_1(RS(\mathcal{E}), A)$ .

(b) If  $\gamma \in \pi_1(RS(\mathcal{E}); A, B)$ , then  $\gamma(A, 1) * \gamma = \gamma * \gamma(B, 1)$ .

(c)  $\gamma(R, S) * \gamma(S, R) = \gamma(A, 1)$ .

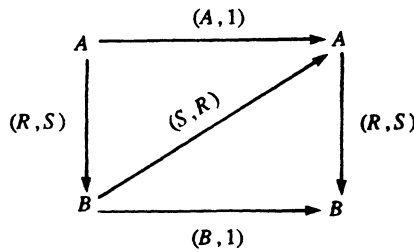
(d)  $\gamma(\alpha^{-1}, \alpha P)^{-1} = \gamma(\alpha, P\alpha^{-1})$ .

*Proof of (a).* The triangle



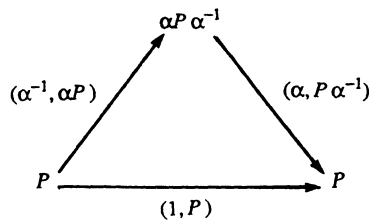
shows that  $\gamma(1, A) * \gamma(1, A) = \gamma(1, A)$ . Now cancel.

*Proof of (b) and (c).* Since  $\gamma$  is a product of paths  $\gamma(R, S)$  and their inverses, it suffices to consider the case  $\gamma = \gamma(R, S)$ . The formula (b) then follows from the diagram



The formula (c) is a consequence of the top triangle in this diagram.

*Proof of (d).* This follows from the triangle





**PROPOSITION 3.4.** *Let  $U$  and  $V$  be in  $P_A$  with  $P = M(U)$  and  $Q = M(V)$ . Then there is a well defined path  $\Gamma(U, V)$  in  $\pi_1(\text{RS}(\mathcal{E}); P, Q)$  such that*

$$\begin{aligned}\Gamma(U, U) &= 1, \\ \Gamma(U, W) &= \Gamma(U, V) * \Gamma(V, W).\end{aligned}$$

Moreover, if  $\alpha \in \text{Isom}(\sigma_A, \sigma_B)$ , then

$$\begin{aligned}\Gamma(\alpha(U), \alpha(V)) &= \gamma(\alpha^{-1}, \alpha P)^{-1} * \Gamma(U, V) * \gamma(\alpha^{-1}, \alpha Q) \\ &= \gamma(\alpha, P\alpha^{-1}) * \Gamma(U, V) * \gamma(\alpha^{-1}, \alpha Q).\end{aligned}$$

*Proof.* For the special case  $U \rightarrow V$  in  $P_A$ , define  $\Gamma(U, V) = \gamma(R, S)$ . In general, choose a path from  $U$  to  $V$  in  $P_A$  which is concatenation of edges  $\langle U_{i-1}, U_i \rangle^{\varepsilon(i)}$  for  $i = 1, \dots, n$  where  $\varepsilon(i) = \pm 1$ . Then define

$$(3.5) \quad \Gamma(U, V) = \Gamma(U_0, U_1)^{\varepsilon(1)} * \Gamma(U_1, U_2)^{\varepsilon(2)} * \dots * \Gamma(U_{n-1}, U_n)^{\varepsilon(n)}.$$

From the definition of  $\Gamma$ , we see that  $\Gamma(U, U) = \gamma(1, A) = 1$  by (a) of (3.3). It is also clear that  $\Gamma(U, V) * \Gamma(V, W) = \Gamma(U, W)$  provided  $\Gamma$  is independent of the path chosen in  $P_A$  from  $U$  to  $V$ . But this follows immediately from the RS Triangle Identities (1.2) and simple connectivity of  $P_A$  proved in [8]. The formula for  $\Gamma(\alpha(U), \alpha(V))$  follows from the diagram (3.1) and (d) of (3.3).

**PROPOSITION 3.6.** *There is a map  $\Phi = \Phi(A, B)$  from  $\text{Isom}(\sigma_A, \sigma_B)$  to  $\pi_1(\text{RS}(\mathcal{E}); A, B)$  such that if  $\alpha \in \text{Isom}(\sigma_A, \sigma_B)$  and  $\beta \in \text{Isom}(\sigma_B, \sigma_C)$  then*

$$\Phi(\beta\alpha) = \Phi(\alpha) * \Phi(\beta).$$

Considering  $\sigma_A \in \text{Aut}(\sigma_A) = \text{Isom}(\sigma_A, \sigma_A)$  we have

$$\Phi(\sigma_A) = \gamma(A, 1).$$

From this result we then obtain the homomorphism

$$(3.7) \quad \Phi_A: \text{Aut}(\sigma_A) \rightarrow \pi_1(\text{RS}(\mathcal{E}), A)$$

by taking  $A = B$  and setting  $\Phi_A(\alpha) = \Phi(\alpha^{-1})$ .

*Proof.* Let  $\alpha \in \text{Isom}(\alpha_A, \sigma_B)$ . We then have

$$(\alpha^{-1}, \alpha A): A = M(U^A) \rightarrow M(\alpha(U^A)) = \alpha A \alpha^{-1}.$$

Define

$$(3.8) \quad \Phi(\alpha) = \gamma(\alpha^{-1}, \alpha A) * \Gamma(\alpha(U^A), U^B).$$

Now let  $\alpha \in \text{Isom}(\sigma_A, \sigma_B)$  and  $\beta \in \text{Isom}(\sigma_B, \sigma_C)$ . Then

$$\begin{aligned}\Phi(\beta\alpha) &= \gamma(\alpha^{-1}\beta^{-1}, \alpha\beta A) * \Gamma(\beta\alpha(U^A), U^C) \\ &= \gamma(\alpha^{-1}, \alpha A) * \gamma(\beta^{-1}, \alpha\beta A\alpha^{-1}) * \Gamma(\beta\alpha(U^A), \beta(U^B)) \\ &\quad * \Gamma(\beta(U^B), U^C).\end{aligned}$$

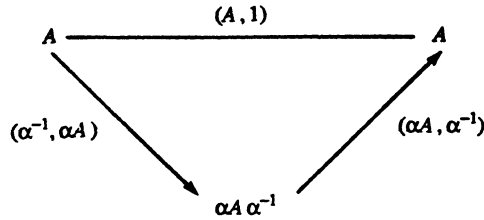
From (3.4) we see that

$$\begin{aligned}\Gamma(\beta\alpha(U^A), \beta(U^B)) \\ = \gamma(\beta^{-1}, \alpha\beta A\alpha^{-1})^{-1} * \Gamma(\alpha(U^A), U^B) * \gamma(\beta^{-1}, \beta B).\end{aligned}$$

Substituting and simplifying gives

$$\begin{aligned}\Phi(\beta\alpha) &= \gamma(\alpha^{-1}, \alpha A) * \Gamma(\alpha(U^A), U^B) * \gamma(\beta^{-1}, \beta B) * \Gamma(\beta(U^B), U^C) \\ &= \Phi(\alpha) * \Phi(\beta).\end{aligned}$$

To compute  $\Phi(\sigma_A)$ , recall that  $\sigma_A(U^A) \rightarrow U^A$ . We then have the triangle



where  $\alpha$  is the bijection between the states  $U^A$  and the states of  $\sigma_A(U^A)$  induced by  $\sigma_A$ . This shows

$$\Phi(\sigma_A) = \gamma(\alpha^{-1}, \alpha A) * \Gamma(\sigma_A(U^A), U^A) = \gamma(A, 1).$$

This completes the construction of  $\Phi_A$ . Next consider the homomorphism

$$\Theta_A: \pi_1(\text{RS}(\mathcal{E}), A) \rightarrow \text{Aut}(\sigma_A).$$

The formula for  $\Theta_A$  was given in (1.8), and as discussed in §1 it only remains to give the

*Proof of (1.9).* Suppose that  $(R, S): A \rightarrow B$ . The equation  $c(R, S)c(S, R) = \sigma_A$  implies  $c(R, S)^{-1} = \sigma_A^{-1}c(S, R)$ . If  $y = \{y_n\} \in X_B$ , we can therefore characterize  $x = c(R, S)^{-1}(y)$  to be the unique point  $x = \{x_n\} \in X_A$  such that

$$S(y_{n-1}, x_n)R(x_n, y_n) = 1$$

for all  $n$ .

First we will show that the RS Triangle Identities imply

$$c(R_2, S_2)c(R_1, S_1) = c(R_3, S_3).$$

Let  $x = \{x_n\} \in X_M$  with  $y = c(R_1, S_1)(x) = \{y_n\}$  and  $z = c(R_2, S_2)(y) = \{z_n\}$ . Let  $w = c(R_3, S_3)^{-1}(z) = \{w_n\}$ . We want to show that  $x_n = w_n$  for all  $n$ . We know that  $w_n$  is the unique state satisfying

$$S_3(z_{n-1}, w_n)R_3(w_n, z_n) = 1.$$

Let  $k$  be the unique state such that

$$S_3(z_{n-1}, w_n)R_1(w_n, k)R_2(k, z_n) = S_2(z_{n-1}, k)R_2(k, z_n) = 1.$$

Then  $k = y_n$ . We also have

$$R_3(x_{n-1}, z_{n-1}) = R_1(x_{n-1}, y_{n-1})R_2(y_{n-1}, z_{n-1}) = 1.$$

Thus

$$R_2(y_{n-1}, z_{n-1})S_3(z_{n-1}, w_n) = S_1(y_{n-1}, w_n) = 1$$

and

$$S_1(y_{n-1}, w_n)R_1(w_n, y_n) = 1.$$

We conclude that  $w_n = x_n$ .

For the converse, remember that each matrix in  $\mathcal{E}$  has at least one non-zero entry in each row and in each column.

The equation  $S_3R_1 = S_2$ : The left hand side LHS is a zero-one matrix, because  $S_3R_1R_2 = S_3R_3 = Q$  is zero-one. Fix a pair of indices  $(p, l)$  such that  $S_2(p, l) = 1$ . Choose points  $z = \{z_n\} \in X_Q$  and  $y = \{y_n\} \in X_P$  such that  $z_0 = p$ ,  $y_1 = l$ , and  $z = c(R_2, S_2)(y)$ . Then choose  $x = \{x_n\} \in X_M$  with  $y = c(R_1, S_1)(x)$ . In particular, we have  $z = c(R_2, S_2)c(R_1, S_1)(x) = c(R_3, S_3)(x)$ . Then

$$1 = S_3(z_0, x_1)R_3(x_1, z_1) = S_3(z_0, x_1)R_1(x_1, y_1)R_2(y_1, z_1)$$

and  $S_3(p, x_1)R_1(x_1, l) = 1$ . Thus  $\text{LHS} \neq 0$ . On the other hand, suppose for an index  $i$  that  $S_3(p, i)R_1(i, l) = 1$ . We must show  $S_2(p, l) = 1$ . Choose a point  $z$  in  $X_Q$  with  $z_0 = p$  and  $R_2(l, z_1) = 1$ . Then choose a point  $x$  in  $X_M$  with  $z = c(R_3, S_3)(x)$  and  $x_1 = i$ . Consider  $y = c(R_1, S_1)(x)$ . Then  $z = c(R_2, S_2)(y)$ . We claim that  $y_1 = l$ . Recall  $R_1(x_1, y_1)R_2(y_1, z_1) = 1$ . But  $l$  is the unique state with  $R_3(i, z_1) = R_1(i, l)R_2(l, z_1) = 1$ . Hence  $y_1 = l$ . Since  $y = c(R_2, S_2)^{-1}(z)$ , we must have  $S_2(p, l)R_2(l, z_1) = 1$  and so  $S_2(p, l) = 1$ .

The equation  $R_2S_3 = S_1$ : The left hand side LHS is a zero-one matrix, because  $R_1R_2S_3 = R_3S_3 = M$  is zero-one. Suppose  $RHS \neq 0$  and fix a pair of indices  $(k, i)$  such that  $S_1(k, i) = 1$ . Choose  $x$  in  $X_M$  and  $y$  in  $X_P$  with  $x_1 = i$ ,  $y_0 = k$ , and  $y = c(R_1, S_1)(x)$ . Let  $z = c(R_2, S_2)(y) = c(R_3, S_3)(x)$ . Then  $R_3(x_0, z_0) = R_1(x_0, y_0)R_2(y_0, z_0) = 1$  and  $S_3(z_0, x_1) = 1$ . Therefore  $R_2(k, z_0)S_3(z_0, i) = 1$ , which says  $LHS \neq 0$ . Conversely, assume  $LHS \neq 0$  and choose a state  $p$  with  $R_2(k, p)S_3(p, i) = 1$ . Choose an  $x$  in  $X_M$  such that  $x_1 = i$  and  $R_1(x_0, k) = 1$ . Then  $R_1(x_0, k)R_2(k, p)S_3(p, i) = R_3(x_0, p)S_3(p, i) = 1$ . Let  $y = c(R_1, S_1)(x)$  and  $z = c(R_2, S_2)(y) = c(R_3, S_3)(x)$ . Then  $z_0 = p$  and  $y_0$  must satisfy  $R_1(x_0, y_0)S_1(y_0, x_1) = 1$ . We also have  $R_1(x_0, y_0)R_2(y_0, z_0)S_3(z_0, x_1) = 1$ . Thus  $R_1(x_0, y_0)R_2(y_0, z_0) = 1$ , so  $y_0 = k$ . This gives  $S_1(k, i) = 1$ .

*Proof that  $\Phi_A$  and  $\Theta_A$  are isomorphisms.*

*Step 1.  $\Theta_A\Phi_A = \text{Id}$ .*

Let  $U = \{U_k\}$  be Markov partition in  $P_A$  with  $P = M(U)$ . We have the well-known isomorphism  $I = I(U, U^A)$  in  $\text{Isom}(\sigma_A, \sigma_P)$  defined on  $x$  in  $X_A$  by the condition

$$(3.9) \quad I(x)_n = k \quad \text{if and only if} \quad \sigma_A^n(x) \in U_k.$$

See [8] for example. Now consider an isomorphism  $\alpha \in \text{Isom}(\sigma_A, \sigma_B)$  and let  $U \in P_A$  and  $V \in P_B$  where  $\alpha(U) = V$ . Let  $U = \{U_e\}$  and  $V = \{V_f\}$  where  $e$  and  $f$  run through indexing sets  $E$  and  $F$  respectively. The homeomorphism  $\alpha$  induces a bijection from  $E$  to  $F$ . Let  $P = M(U)$  and  $Q = M(V)$ . Then  $Q = \alpha P \alpha^{-1}$  and  $(\alpha^{-1}, \alpha P): P \rightarrow Q$ .

LEMMA 3.10.  $I(V, U^B)\alpha = c(\alpha^{-1}, \alpha P)I(U, U^A)$ .

*Proof.* Let  $I_U = I(U, U^A)$ ,  $I_V = I(V, U^B)$ , and  $c = c(\alpha^{-1}, \alpha P)$ . Remember that  $c$  is the one-block map taking  $y = \{y_n\}$  with  $y_n \in E$  to  $\alpha(y) = \{\alpha(y_n)\}$  with  $\alpha(y_n) \in F$ . Since  $\alpha$ ,  $c$ ,  $I_U$ , and  $I_V$  are shift commuting, it suffices to show that

$$I_V(\alpha(x))_0 = c(I_U(x))_0$$

for all  $x \in X_A$ . Let  $y = I_U(x) = \{y_n\}$  with  $y_n \in E$ . Let  $e = y_0$ . Then  $c(I_U(x))_0 = f$  if and only if  $\alpha(e) = f$ . That is, if and only if  $\alpha(U_e) = V_f$  where  $x \in U_e$ . On the other hand, suppose

$h = I_V(\alpha(x))_0$ . Then  $\alpha(x) \in V_h$ . Let  $g \in E$  be the unique index where  $x \in U_g$  with  $\alpha(U_g) = V_h$ . Then  $e = g$  and so  $f = h$ .

Let  $U \rightarrow V$  in  $P_A$ . Let  $R = R(U, V)$  and  $S = S(V, U)$ .

LEMMA 3.11.  $I(V, U^A) = c(R, S)I(U, U^A)$ .

Since  $I(U^A, U^A) = 1$ , we get  $I(U, U^A) = c(R, S)$  whenever  $U^A \rightarrow U$  in  $P_A$ .

*Proof.* Let  $I = I(U, U^A)$ ,  $J = I(V, U^A)$ , and  $c = c(R, S)$ . It suffices to show that

$$J(x)_0 = cI(x)_0$$

for all  $x \in X_A$ . Let  $y = I(x) = \{y_n\}$  with  $y_0 = k$  and  $y_1 = l$ . This means  $x \in U_k$  and  $\sigma_A(x) \in U_l$ , and consequently  $U_k \cap \sigma_A^{-1}(U_l) \neq \emptyset$ . Using  $V < U \cap \sigma_A^{-1}(U)$ , let  $V_b$  be the unique set in  $V$  with  $V_b \supset U_k \cap \sigma_A^{-1}(U_l)$ . Then  $J(x_0) = b$ . But also we have  $U_k \cap V_b \neq \emptyset$ . Equivalently,  $R(k, b)S(b, l) = 1$ , which implies  $c(y)_0 = b$ .

We are now ready to prove  $\Theta_A \Phi_A = \text{Id}$ . Recall that  $\Phi_A(\alpha) = \Phi(\alpha^{-1})$ . We will show more generally for  $\alpha \in \text{Isom}(\sigma_A, \sigma_B)$  that  $\Theta\Phi(\alpha) = \alpha^{-1}$  where

$$\Phi: \text{Isom}(\sigma_A, \sigma_B) \rightarrow \pi_1(\text{RS}(\mathcal{E}); A, B)$$

is defined by (3.8), and

$$\Theta: \pi_1(\text{RS}(\mathcal{E}); A, B) \rightarrow \text{Isom}(\sigma_B, \sigma_A)$$

is defined on

$$\gamma = \prod_{i=1}^n \gamma(R_i, S_i)^{\varepsilon_i}$$

by the formula

$$(3.12) \quad \Theta(\gamma) = \prod_{i=1}^n c(R_i, S_i)^{-\varepsilon_i}.$$

Let  $\alpha \in \text{Isom}(\sigma_A, \sigma_B)$  and let

$$\alpha(U^A) = V_0 \rightarrow V_1 \leftarrow \cdots \rightarrow V_{n-1} \leftarrow V_n = U^B$$

be the path in  $P_B$  as in the definition (3.8). For  $0 \leq k \leq n$ , let  $A_k = M(V_k)$ . For  $1 \leq k \leq n$ ,  $R_k = R(V_{k-1}, V_k)$ ,  $S_k = S(V_k, V_{k-1})$ ,

and  $\varepsilon_k = +1$  when  $V_{k-1} \rightarrow V_k$ , and let  $R_k = R(V_k, V_{k-1})$ ,  $S_k = S(V_{k-1}, V_k)$ , and  $\varepsilon_k = -1$  when  $V_k \rightarrow V_{k-1}$ . Then (3.10) and (3.11) show that

$$\begin{aligned}\alpha^{-1} &= c(\alpha^{-1}, \alpha A)^{-1} \prod_{k=1}^n c(R_k, S_k)^{-\varepsilon_k} \\ &= \Theta \left( \gamma(\alpha^{-1}, \alpha A) \prod_{k=1}^n \gamma(R_k, S_k)^{\varepsilon_k} \right) = \Theta \Phi(\alpha).\end{aligned}$$

*Step II:*  $\Phi_A \Theta_A = \text{Id}$ .

Assume  $(R, S): A \rightarrow B$  and let  $c = c(R, S): X_A \rightarrow X_B$ . Let  $U^A = \{U_i^A\}$  and  $U^B = \{U_k^B\}$  be the standard Markov partitions for  $\sigma_A$  and  $\sigma_B$  respectively. Let  $U = \{U_k\} = \{c^{-1}(U_k^B)\} = c^{-1}(U^B)$  and  $V = \{V_i\} = \{c(U_i^A)\} = c(U^A)$ .

LEMMA 3.13.  $U^A \rightarrow U$  in  $P_A$  and  $V \rightarrow U^B$  in  $P_B$ .

*Proof.* It clearly suffices to verify  $U^A \rightarrow U$ . The definition of  $c(R, S)$  shows that  $U_k$  consists of those points  $x$  in  $X_A$  such that  $R(x_0, k)S(k, x_1) = 1$ . In other words,

$$U_k = \bigcup U_i^A \cap \sigma_A^{-1}(U_j^A)$$

where the pairs  $(i, j)$  run over those states with  $R(i, k)S(k, j) = 1$ . For a given set  $U_i^A \cap \sigma_A^{-1}(U_j^A)$  in  $U^A \cap \sigma_A^{-1}(U^A)$ , let  $k$  be the unique state such that  $R(i, k)S(k, j) = 1$ . Then  $U_i^A \cap \sigma_A^{-1}(U_j^A) \subset U_k$ . This shows that  $U^A \xrightarrow{+} U^A \cap U$ . Next we verify  $U \xrightarrow{-} U^A \cap U$ . Fix a pair of states  $(k, l)$  and write

$$U_k = \bigcup U_a^A \cap \sigma_A^{-1}(U_b^A), \quad U_l = \bigcup U_c^A \cap \sigma_A^{-1}(U_d^A).$$

Then

$$\sigma_A(U_k) \cap U_l = \bigcup \sigma_A(U_a^A) \cap U_b^A \cap U_c^A \cap \sigma_A^{-1}(U_d^A)$$

where each non-empty term must have  $b = c$ ,  $R(a, k)S(k, b) = 1$ , and  $R(c, l)S(l, d) = 1$ . Then  $R(k, b)S(b, l) = 1$ , and so only one

$b$  can occur. The expression becomes

$$\sigma_A(U_k) \cap U_l = \bigcup \sigma_A(U_a^A) \cap U_b^A \cap \sigma_A^{-1}(U_d^A) \subset U_b^A \cap U_l.$$

This completes the proof of (3.13).

Let  $\alpha \in \text{Isom}(\sigma_A, \sigma_B)$  be of the form  $\alpha = c(R, S)$  corresponding to  $(R, S): A \rightarrow B$  and let  $f \in \text{Isom}(\sigma_B, \sigma_C)$ . Let  $U = f\alpha(U^A)$  and  $V = f(U^B)$ . From (3.13) we see that  $U \rightarrow V$  in  $P_B$ . Let  $A' = M(U)$ ,  $B' = M(V)$ ,  $R' = R(U, V)$ , and  $S' = S(V, U)$ . Then

$$\begin{aligned} A' &= f\alpha A\alpha^{-1}f^{-1}, & B' &= fBf^{-1}, \\ R' &= f\alpha Rf^{-1}, & S' &= fS\alpha^{-1}f^{-1}, \end{aligned}$$

and moreover there is the following diagram of triangles in  $\text{RS}(\mathcal{E})$ :

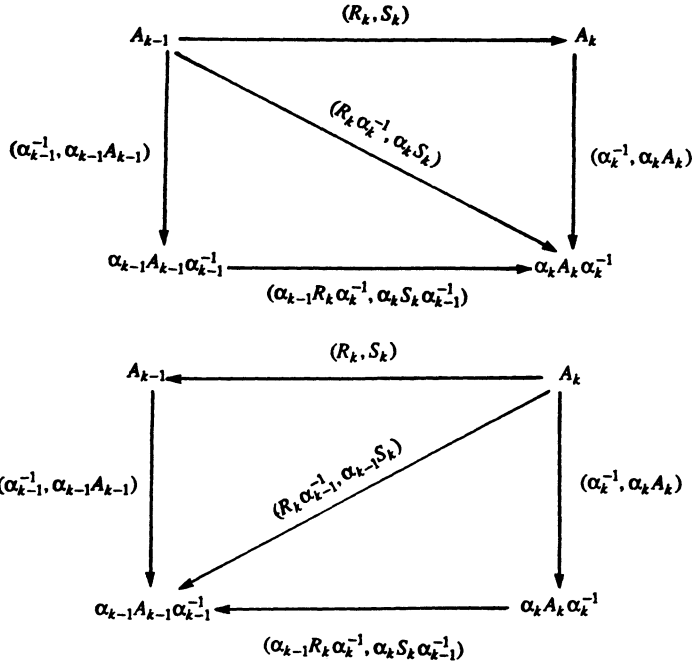
$$(3.14) \quad \begin{array}{ccc} A & \xrightarrow{(R, S)} & B \\ \downarrow (\alpha^{-1}f^{-1}, f\alpha A) & \searrow (Rf^{-1}, fS) & \downarrow (f^{-1}, fB) \\ f\alpha A\alpha^{-1}f^{-1} & \xrightarrow{(f\alpha Rf^{-1}, fS\alpha^{-1}f^{-1})} & fBf^{-1} \end{array}$$

Similarly to Step I we will show that  $\Phi\Theta(\gamma^{-1}) = \gamma$  for any path  $\gamma = \prod \gamma(R_k, S_k)^{\varepsilon_k}$  from  $A = A_0$  to  $B = A_n$  in  $\pi_i(\text{RS}(\mathcal{E}); A, B)$ . For  $0 \leq k \leq n-1$ , let  $\alpha_k \in \text{Isom}(\sigma_{A_k}, \sigma_B)$  be given by the formula

$$\alpha_k = c(R_n, S_n)^{\varepsilon_n} \cdots c(R_{k+1}, S_{k+1})^{\varepsilon_{k+1}}.$$

Let  $\alpha_n = \text{id}$ . Remember that our convention is to read composition of homeomorphisms between spaces from right to left. Let  $V_k = \alpha_k(U^{A_k})$  in  $P_B$  for  $0 \leq k \leq n-1$  and let  $V_n = U^B$ . From (3.1) we know that  $V_{k-1} \rightarrow V_k$  if  $\varepsilon_k = +1$  and  $V_k \rightarrow V_{k-1}$  if  $\varepsilon_k = -1$ . Let  $B_k = M(V_k)$ . Then  $B_k = \alpha_k A_k \alpha_k^{-1}$ . From (3.14) we have the

following two diagrams corresponding respectively to the parities  $\varepsilon_k = +1$  and  $\varepsilon_k = -1$ :



These fit together to provide a homotopy in  $RS(\mathcal{E})$  with end points fixed between the original path  $\gamma$  and the path

$$\gamma(\alpha_0^{-1}, \alpha_0A) * \Gamma(\alpha_0(U^A), U^B) = \Phi(\alpha_0).$$

But  $\alpha_0 = \Theta(\gamma^{-1})$ . So  $\Phi\Theta(\gamma^{-1}) = \gamma$ . This finally completes the proof that  $\Phi$  and  $\Theta$  are isomorphisms.

**REMARK 3.15.** The CW-complex  $RS(\mathcal{E})$  is locally compact. This follows from (3.13), the observation in [8] that  $P_A$  is a locally finite simplicial complex, and the fact that there are only finitely many  $(R, S): P \rightarrow P$  for a given square zero-one matrix  $P$ .

**4.  $RS(\mathcal{E})$  is aspherical.** We now give the proof of (1.13). Let  $RS(\mathcal{E})_A$  denote the component of  $RS(\mathcal{E})$  containing  $A$ . The universal cover  $\widetilde{RS}_A$  of  $RS(\mathcal{E})_A$  is the realization of the following simplicial set: The  $k$ -simplices are pairs  $(\gamma, \Delta)$  where

- (i)  $\Delta$  is a  $k$ -simplex of  $RS(\mathcal{E})_A$  given by the data  $\langle A_0, \dots, A_n \rangle$  and  $(R_{ij}, S_{ji}): A_i \rightarrow A_j$  as in (1.4) and
- (ii)  $\gamma$  is a homotopy class of paths from the base point  $A$  to  $A_0$ .

The  $i$ th face operator of  $\widetilde{RS}$  acts on  $\Delta$  just as it does in  $RS$ . For  $1 \leq i \leq n$ , it leaves  $\gamma$  unchanged, and for  $i = 0$  it changes  $\gamma$  to



$\gamma * \gamma(R_1, S_1)$ . Similarly for the degeneracy operators. The covering map

$$\widetilde{\text{RS}}(\mathcal{E})_A \rightarrow \text{RS}(\mathcal{E})_A$$

is induced by the map of simplicial sets taking  $(\gamma, \Delta)$  to  $\Delta$ .

For each  $k$ -simplex  $(\gamma, \Delta)$ , let  $\alpha_0 = \Theta(\gamma)$  and for  $1 \leq k \leq n$ , let  $\alpha_k = \Theta(\gamma * \gamma(R_{0k}, S_{k0}))$ . Let  $U_k = \alpha_k(U^{A_k}) \in P_A$ . The discussion in §3 shows  $\langle U_0, \dots, U_k \rangle$  is a  $k$ -simplex in  $P_A$ . Moreover, the correspondence taking  $(\gamma, \Delta)$  to  $\langle U_0, \dots, U_k \rangle$  is map of simplicial sets. Using (3.14) we obtain a homotopy commutative diagram

$$\begin{array}{ccc} \widetilde{\text{RS}}(\mathcal{E})_A & \longrightarrow & P_A \\ & \searrow & \swarrow \\ & \text{RS}(\mathcal{E})_A & \end{array}$$

Since  $P_A$  is contractible [8], we see that

$$\pi_n(\widetilde{\text{RS}}(\mathcal{E})_A, A) \rightarrow \pi_n(\text{RS}(\mathcal{E})_A, A)$$

is the zero homomorphism. On the other hand, it is also an isomorphism for  $n \geq 2$  because  $\widetilde{\text{RS}}(\mathcal{E})_A$  is the universal cover.

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UNIVERSITY OF CALIFORNIA  
BERKELEY, CA 94720

