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SMALL SUBSET OF THE PLANE WHICH ALMOST CONTAINS ALMOST ALL BOREL FUNCTIONS*

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A Borel subset B of the plane is constructed which is small from the Lebesgue measure point of view and large in the sense of the Baire category. All vertical sections of B have measure zero, and for each Borel function $f: R \to R$ for all but countably many y the set $\{x \in R: (x, f(x)+y) \in B\}$ is comeager.

1. Introduction. It is well known that the Fubini Theorem and the Kuratowski-Ulam Theorem cannot be mixed together. It is easy to find a Borel subset of the plane with all vertical sections having the Lebesgue measure zero and almost all in the sense of the Baire category horizontal sections being comeager. We can just take A^2 where A is the classical example of a measure zero dense G_δ subset of the real line.

In this paper we show that such antagonism between measure and category is much stronger. We give an example of a G_δ subset of the plane such that all its vertical sections have the Lebesgue measure zero and for each Borel function $f: R \to R$ all but countably many sections parallel to f are comeager. We also indicate why there is no such example in which the roles of measure and category are interchanged.

2. Notation. Throughout the paper I stands for the half open interval $[0, 1[$, R for the reals, $+_a$ for the addition modulo a, and λ for the Lebesgue measure. For subsets A and B of the reals let $A \mp B = \{a \mp b: a \in A \text{ and } b \in B\}$. Natural numbers are the sets of smaller natural numbers, ω is the set of all natural numbers, $\omega^{<\omega}$ and ω^ω are the sets of finite, resp. infinite, sequences of natural numbers. $[A]^{<\varkappa}$ and $[A]^\varkappa$ denote the sets of all subsets of A of cardinality $< \varkappa$, resp. of cardinality $\varkappa$. For $A \subseteq X \times Y$ and $x \in X$ let $A_x = \{y \in Y: (x, y) \in A\}$.

3. Erdös-Folklore Lemma. We start with the following Folklore Lemma.

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LEMMA 3.1. *For any $\varepsilon > 0$ there exists a natural number k with the property that for each $A \in [I]^k$ there is a finite union of rational intervals $B \subseteq I$ such that $A +_1 B = I$ and $\lambda(B) < \varepsilon$.*

We don't know whether the proof of this lemma was ever published. For the sake of completeness let us show how it follows from a lemma of Lorentz [L].

LEMMA 3.2 [*Lorentz*]. *For each natural number n and each nonempty set $A \subseteq n$ there is a set $B \subseteq n$ such that $A +_n B = n$ and* $\operatorname{card}(B) \leq n \circ \operatorname{card}(A)^{-1} \circ (1 + \log(\operatorname{card}(A)))$.

Proof of Lemma 3.1. We shall show that for any finite set $A \subseteq I$ there is a finite union of rational intervals $B \subseteq I$ such that

$$\lambda(B) \leq 2 \circ \operatorname{card}(A)^{-1} \circ (1 + \log(\operatorname{card}(A)))$$

and $A +_1 B = I$. To see this find n so large that in each interval $[i/n, (i+1)/n[$, $i = 0, \ldots, n-1$, there is at most one point of A. Let $A^* = \{i < n : [i/n, (i+1)/n[\cap A \neq \emptyset\}$. Then $\operatorname{card}(A) = \operatorname{card}(A^*)$. By the lemma there is $B^* \subseteq n$ such that $A^* + B^* = n$ and $\operatorname{card}(B^*) \leq n \circ \operatorname{card}(A^*)^{-1} \circ (1 + \log(\operatorname{card}(A^*)))$. Set

$$B' = \bigcup \{](i-1)/n, (i+1)/n[: i \in B^*\} \quad \text{and}$$
$$B = \{x \bmod 1 : x \in B'\}.$$

It is easy to see that B works. □

4. The construction. The main step in our construction is the following lemma.

LEMMA 4.1. *For any $\varepsilon > 0$ there exists an open set $B \subseteq R \times I$ such that each vertical section of B has measure less than ε and for any infinite set $T \subseteq I$ the set $\{x \in R : T +_1 B_x = I\}$ is comeager.*

Proof. For any $\tau \in \omega^{<\omega}$ let $[\tau] = \{t \in \omega^\omega : \tau \subseteq t\}$. The collection $\{[\tau] : \tau \in \omega^{<\omega}\}$ is a basis for the standard topology on ω^ω. ω^ω with this topology is homeomorphic to the irrationals. Thanks to this homeomorphism it is enough to find a subset B of $\omega^\omega \times I$ which enjoys the properties stated in the lemma.

For each $k \in \omega$ let $\{I_j^k : j \in \omega\}$ be an enumeration of all finite unions of open rational subintervals of I such that $\lambda(I_j^k) < \varepsilon \circ 2^{-k-1}$.

Let

$$B = \bigcup_{\tau \in \omega^{<\omega}} \left([\tau] \times \bigcup_{k=0}^{\operatorname{card}(\tau)-1} I^k_{\tau(k)} \right).$$

Clearly B is an open subset of $\omega^\omega \times I$ which has all vertical sections of measure less than ε. Further, if we fix an infinite set $T \subseteq I$, then for each $\tau \in \omega^{<\omega}$ we can find a natural number k such that the thesis of Lemma 3.1 holds with ε replaced by $\varepsilon \circ 2^{-\operatorname{card}(\tau)-1}$. This means that for each $S \in [T]^k$ we can find an index j such that $S +_1 I_j^{\operatorname{card}(\tau)} = I$. Therefore for each x which prolongs the concatenation $\tau * j$ we have $S +_1 B_x = I$. By the definition of B it follows that the set

$$\{x \in \omega^\omega : \exists S \in [T]^{<\omega}\ S +_1 B_x = I\}$$

is comeager. □

We are now ready to prove the main result.

Theorem 4.2. *There exists a G_δ set $B \subseteq R \times I$ such that all its vertical sections have measure zero and for each Borel function $f: R \to I$ for all but countably many y the set $\{x \in R : f(x) +_1 y \in B_x\}$ is comeager.*

Proof. Let B^n satisfy the thesis of Lemma 4.1 with $\varepsilon = 1/n$. Set $B = \bigcap_n B^n$. Let $f: R \to I$ be a given Borel function. Suppose that for uncountably many y the set $\{x \in R : f(x) +_1 y \notin B_x\}$ is nonmeager. Then for some n and for uncountably many y the set $W^y = \{x \in R : f(x) +_1 y \notin B^n_x\}$ is nonmeager and has the Baire property (because f is a Borel function). So each W^y is comeager in some rational interval and since there are only countably many rational intervals, there must be a rational interval J and an uncountable set T of y's such that W^y is comeager in J for each $y \in T$. Let $S \subseteq T$ be a countable infinite set. Then the set $\bigcap_{y \in S} W^y$ is comeager in J, so the set $\{x \in R : f(x) \notin B^n_x -_1 S\}$ is also comeager in J, which contradicts Lemma 4.1. □

To get the version of Theorem 4.2 stated in the abstract we just use the following version of Lemma 3.1.

Lemma 4.3. *For any reals $a < b$, $c < d$, $\varepsilon > 0$, there is a natural number k with the property that for each k-element subset A of $]c, d[$ there is a finite union of rational intervals $B \subset]a-d, b-c[$ such that $]a, b[\subset A + B$ and $\lambda(B) < \varepsilon$.* □

With this lemma we get

LEMMA 4.4. *For any reals $a < b$, $c < d$, $\varepsilon > 0$, there exists an open set $B \subseteq R \times]a-d, b-c[$ such that each vertical section of B has measure less than ε and for any infinite set $T \subseteq]c, d[$ the set*

$$\{x \in R: \forall y \in]a, b[\ (B_x + y) \cap T \neq \varnothing\}$$

is comeager. □

LEMMA 4.5. *For any $\varepsilon > 0$ there exists an open set $B \subseteq R \times R$ such that each vertical section of B has measure less than ε and for any set $T \subseteq R$ which is infinite in some interval the set*

$$\{x \in R: T + B_x = R\} = \{x \in R: \forall y \in R\ (B_x + y) \cap T \neq \varnothing\}$$

is comeager.

Proof. For each $n \in \omega$ we can find by Lemma 4.4 a set $B^n \subseteq R \times]-2n, 2n[$ such that each vertical section of B^n has measure less than $\varepsilon \circ 2^{-n-1}$ and for any infinite set $T \subseteq]-n, n[$ the set

$$\{x \in R: \forall y \in]-n, n[\ (B^n_x + y) \cap T \neq \varnothing\}$$

is comeager. Let $B = \bigcup_n B^n$. Then $\lambda(B_x) < \varepsilon$ for each $x \in R$ and if a set $T \subseteq R$ is infinite in some interval $]-n, n[$ then the set

$$\begin{aligned}\{x \in R: &\forall y \in R\ (B_x + y) \cap T \neq \varnothing\}\\ &= \bigcap_{m>n} \{x \in R: \forall y \in]-m, m[\ (B_x + y) \cap T \neq \varnothing\}\\ &\supseteq \bigcap_{m>n} \{x \in R: \forall y \in]-m, m[\ (B^m_x + y) \cap T \neq \varnothing\}\end{aligned}$$

is comeager because each of the intersected sets is comeager. □

Using Lemma 4.5 as we have used Lemma 4.1 we finally get

THEOREM 4.6. *There exists a G_δ set $B \subseteq R \times R$ such that all its vertical sections have measure zero and for each Borel function $f: R \to R$ for all but countably many y the set $\{x \in R: f(x) + y \in B_x\}$ is comeager.*

5. Applications. The above-constructed set B can be for each $\varepsilon > 0$ covered by an open set with all vertical sections of measure less than ε. The collection of subsets of the plane with this property is a σ-ideal. In **[CP]** in an answer to a question of Mokobodzki it is shown that there exist ω_1 sets from this ideal with the union outside the ideal. The example in **[CP]** is rather complicated. Theorem 4.6 provides a much easier example where all ω_1 sets are just translates of a single set. Precisely

THEOREM 5.1. *Let B be a set claimed to exist by Theorem* 4.6. *For any uncountable $T \subseteq R$, the union of sets $B(t) = \{(x, y): y \in B_x - t\}$, $t \in T$, cannot be covered by a Borel set with all vertical sections of measure zero.*

Proof (cf. **[CP]**). Otherwise we could choose a Borel selection $f: R \to R$ such that $f(x) \notin B(t)_x$ for each $t \in T$ and $x \in R$. Let T_f be the countable set of exceptional reals associated with the function f by Theorem 4.6. Let $t \in T \setminus T_f$. The set $\{x \in R: f(x) \in B_x - t\}$ is nonempty by Theorem 4.6, which contradicts the choice of f. □

It is well known that most statements about Borel functions and sets which involve Baire category say something interesting about properties of the reals when a Cohen real is added to the universe. Proofs of the following corollaries are obtained by taking sections at a Cohen real of appropriate versions of Theorem 4.6 and Lemma 4.5 (cf. **[CP]** and **[P]** for more results by this method).

COROLLARY 5.2. *Let M be a transitive model of ZFC set theory and let c be a Cohen real over M.*

(a) *In $M[c]$ for any $\varepsilon > 0$ there is an open set B such that $\lambda(B) < \varepsilon$ and for any countable infinite set $A \in M$ such that $A \subseteq M \cap I$, there is a finite set A' with the property that $A' +_1 B = I \cap M[c]$. (Thus for any $t \in I \cap M[c]$ we have $(B + t) \cap A' \neq \varnothing$.)*

(b) [*Carlson*] *In $M[c]$ there is a measure zero set of reals B such that for any $t \in R \cap M[c]$ the set $(R \cap M) \setminus (B + t)$ is countable.* □

COROLLARY 5.3 (*cf.* **[CP]** *Corollary* 2 *and* **[M]** *Problem* 9). *Suppose that $M \subseteq N$ are transitive models of ZFC such that there exists $x \in R \cap (N \setminus M)$. Let c be a Cohen real over N. Then there exists a Borel set B coded in $M[c]$ such that $R \cap M[c] \subseteq x + B$.* □

6. Note. The roles of measure and category in Theorem 4.6 cannot be reversed. In fact for any Borel subset B of the plane which has all vertical sections meager there is a Borel function $f: R \to R$ and a meager set C such that the set $\{x \in R: f(x) + c \supseteq B_x\}$ has full measure. This is due to the fact that in $M[r]$, where M is a transitive model of ZFC and r is a Solovay real over M, every meager set is covered by a translation of a meager set coded in M.

7. Acknowledgment. This paper was inspired by an unpublished paper of Carlson [C], which contains what appears here as Corollary 5.2(b). Carlson's paper was shown to me by David Fremlin during my visit to him in the spring of 1986. I am very grateful to David and his wife Maria for making my stay a pleasant and fruitful one.

References

[C] T. Carlson, *Dual Borel conjecture*, unpublished.

[CP] J. Cichoń and J. Pawlikowski, *On subsets of the plane and on Cohen reals*, J. Symbolic Logic, **51** (1986), 560–569.

[L] G. G. Lorentz, *On a problem of additive number theory*, Proc. Amer. Math. Soc., **5** (1954), 838–841.

[M] A. W. Miller, *Some properties of measure and category*, Trans. Amer. Math. Soc., **266** (1981), 93–114.

[P] J. Pawlikowski, *Why Solovay real produces Cohen real*, J. Symbolic Logic, **51** (1986), 957–968.

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