PACIFIC JOURNAL OF MATHEMATICS
Vol. 144, No. 2, 1990

ON THE PROJECTIVE NORMALITY OF SOME VARIETIES OF DEGREE 5

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We give some sufficient conditions for projective normality of complete non-singular varieties of degree five. And we prove that every complete non-singular surfaces of degree five embedded by a complete linear system is projectively normal.

Introduction. Let X be a complete non-singular variety over an algebraically closed field, and let L be an ample line bundle on X. The classification of some (X, L) is found in Fugita's papers (Fujita [**1**], [**2**], [**3**], [**4**]). In this paper, we consider the projective normality of (X, L) and the defining equations. This problem is trivial in the case of $(D^n) = 1$, 2 where $n = \dim X$ and $\mathscr{L} = \mathscr{O}(D)$. If $(D^n) = 3$, then (X, L) is projectively normal and the ideal is generated by degree 2 and 3 (X.X.X. [**11**]). If $(D^n) = 4$, then (X, L) is projectively normal and the ideal is generated by degree 2 and 3 (Swinnerton-Dyer [**10**]). So we consider the case of $(D^n) = 5$. In this paper we give some sufficient conditions for projective normality of varieties of degree 5 and give the generator of the defining ideal. The main part of this paper is the case of $(D^n) = 5$ and $\Delta(X, L) = 2$ (other cases are clearly obtained by Fujita's theory). This is a non-degenerate and non-singular variety of codimension 2 in some projective space $\mathbb{P}^N$. On the other hand, the following conjecture is known as a conjecture of Hartshorne.

Conjecture (*cf. Hartshorne* [**6**]). If $X \subset \mathbb{P}^N$ is a non-singular closed subvariety and $\dim X > 2N/3$, then X is a complete intersection.

If this conjecture is true, then we obtain that every non-degenerate and non-singular variety which is degree 5 and codimension 2 is not contained in $\mathbb{P}^N$ for $N \geq 7$. As every non-singular variety is projectively normal if it is a complete intersection, therefore the results in this paper are recognized as a step to prove the above conjecture. Throughout this paper, variety means a complete non-singular variety.

Notations.

$(D_1 \cdot \cdots \cdot D_n)$: The intersection number of divisors $D_1, \ldots, D_n$ on a variety X where $n = \dim X$.

O_X: The structure sheaf of a variety X.

L_Y: The restriction of a line bundle L to a subscheme Y.

$H^i(X, \mathcal{F})$: The ith cohomology group of a sheaf F.

$h^i(X, \mathcal{F})$: The dimension of $H^i(X, \mathcal{F})$ as a vector space.

$|D|$: The complete linear system defined by a divisor D.

$\phi_{|D|}$: The rational map defined by $|D|$.

$\mathcal{L}$: The invertible sheaf associated to a line bundle L.

$\mathcal{O}(D)$: The invertible sheaf associated to a divisor D.

$\mathbb{P}(E)$: The projective bundle defined by a vector bundle E.

K_X: The canonical divisor on a non-singular variety X.

$\mathcal{O}_X(k)$: The sheaf $\mathcal{O}_X \otimes \mathcal{O}_{\mathbb{P}^n}(k)$ for a projective variety X embedded in $\mathbb{P}^n$.

1. Preliminary. We give several theorems from Fujita's theory.

DEFINITION ([**2**]). Let X be a non-singular variety and let L be an ample line bundle. We define a Δ-genus of (X, L) by

$$\Delta(X, L) = (D^n) + n - h^0(X, L)$$

where $n = \dim X$ and $L = \mathcal{O}(D)$.

The above pair (X, L) is called a polarized non-singular variety.

DEFINITION ([**8**]). Let (X, L) be a polarized non-singular variety. We say that L is normally generated if

$$H^0(X, \mathcal{L})^{\otimes k} \to H^0(X, \mathcal{L}^{\otimes k})$$

is surjective for any positive integer k. And in this case, we call (X, L) projectively normal.

DEFINITION ([**2**]). Let (X, L) be a polarized non-singular variety and set $L = \mathcal{O}(D)$. Let V be a reduced irreducible non-singular member of $|D|$ (if there exists). We call V a regular member if

$$H^0(X, \mathcal{L}) \to H^0(V, \mathcal{L}_V)$$

is surjective.

DEFINITION ([**2**]). Let (X, L) be a polarized non-singular variety. We define $g(X, L)$ by

$$2g(X, L) - 2 = ((K_X + (n-1)D).D^{n-1})$$

where $L = \mathcal{O}(D)$ and $n = \dim X$. We call this $g(X, L)$ a sectional genus of (X, L).

If L is very ample, then this $g(X, L)$ is the genus of the generic curve section of X in the projective embedding defined by L.

THEOREM A ([2]). *Let (X, L) be a polarized non-singular variety. If V is a reduced irreducible non-singular member of $|D|$ where $\mathscr{L} = \mathscr{O}(D)$, then $\Delta(V, L_V) \leq \Delta(X, L)$. Moreover the following conditions are equivalent:*

(a) $\Delta(X, L) = \Delta(V, L_V)$,
(b) *V is a regular member.*

Proof. As $0 \to \mathscr{O}_X \to \mathscr{L} \to \mathscr{L}_V \to 0$ is exact, therefore

$$h^0(V, \mathscr{L}_V) \geq h^0(X, \mathscr{L}) - 1.$$

Hence $\Delta(X, L) - \Delta(V, L_V) = h^0(V, \mathscr{L}_V) - h^0(X, \mathscr{L}) + 1 \geq 0$, because $(D^n) = (D|_{V^{n-1}})$ where $\mathscr{L} = \mathscr{O}(D)$. By the above equation, the last part of this theorem is clear.

THEOREM B. *If X is a variety and L is a very ample line bundle, then $\Delta(X, L) \geq 0$.*

Proof. It is a well-known fact (see Fujita [1]).

THEOREM C. *Let (X, L) be a polarized non-singular variety. If $\Delta(X, L) = 0$, then (X, L) is isomorphic to $(\mathbb{P}(E), H_E)$ or $(\mathbb{P}^2, H_{\mathbb{P}^2}(2))$ where E is a vector bundle on $\mathbb{P}^1$, H_E is a tautological bundle on $\mathbb{P}(E)$ and $H_{\mathbb{P}^2}(i) = \mathscr{O}(i)$ on $\mathbb{P}^2$ $(i \in \mathbb{Z})$.*

Proof. This is a well-known classical theorem (see Fujita [1]).

THEOREM D ([2]). *Let (X, L) be a polarized non-singular variety. If $g(X, L) = 0$ and L is very ample, then $\Delta(X, L) = 0$.*

Proof. We prove this theorem by the induction on $n = \dim X$. If $n = 1$, then this theorem is trivial. We may assume that $n \geq 2$. Let V be a reduced irreducible non-singular member of $|D|$ where $\mathscr{L} = \mathscr{O}(D)$. By the induction hypothesis, we assume $\Delta(V, L_V) = 0$ because $g(V, L_V) = g(X, L) = 0$. Hence $H^1(V, \mathscr{L}_V^{\otimes(-t)}) = 0$ for every $t \geq 0$ by Theorem C. Therefore the long exact sequence

$$\cdots \to H^1(X, \mathscr{L}^{\otimes(-(t+1))}) \to H^1(X, \mathscr{L}^{\otimes(-t)}) \to H^1(V, \mathscr{L}_V^{\otimes(-t)})$$

says that $h^1(X, \mathscr{L}^{\otimes(-(t+1))}) \geq h^1(X, \mathscr{L}^{\otimes(-t)})$ for any $t \geq 0$. As

$$H^1(X, \mathscr{L}^{\otimes(-s)}) = 0$$

for sufficiently large s, we obtain $H^1(X, \mathscr{O}_X) = 0$. Therefore V is a regular member. Hence we obtain this theorem.

THEOREM E. *Let (X, L) be a polarized non-singular variety and let $d = (D^n)$ where $\mathscr{L} = \mathscr{O}(D)$ and $n = \dim X$. Moreover we assume that $\Delta(X, L) \leq g(X, L)$ and L is very ample. In this case, the following are true:*

(a) *if $d \geq 2\Delta(X, L)-2$, then every reduced irreducible non-singular member $V \in |D|$ is a regular member;*

(b) *if $d \geq 2\Delta(X, L) + 1$, then (X, L) is projectively normal and $\Delta(X, L) = g(X, L)$;*

(c) *if $d \geq 2\Delta(X, L) + 2$, then the ideal of (X, L) is generated by degree 2.*

Proof. See Fujita **[2]**. As L is very ample, the proof is the same in the case of characteristic $p > 0$.

THEOREM F. *Let $X \subset \mathbb{P}^N$ be a closed non-singular subvariety which is not contained in any hyperplane. If the degree of X is 4, then X is of the following type:*

(a) *hypersurface,*

(b) *$(2, 2)$ complete intersection,*

(c) *Segre variety $\mathbb{P}^1 \times \mathbb{P}^3$ in $\mathbb{P}^7$,*

(d) *Veronese surface $\mathbb{P}^2$ in $\mathbb{P}^5$,*

(e) *the variety obtained by hyperplane section or projection of* (a), (b), (c), (d), (e).

Proof. See Swinnerton-Dyer **[10]**.

By the above theorems, we obtain that (X, L) is projectively normal for $(D^n) = 3, 4$ where $\mathscr{L} = \mathscr{O}(D)$ and $n = \dim X$. Moreover (X, L) is also projectively normal if $(D^n) = 5$ and the codimension of $\phi_{|D|}(X)$ is 1, 3, 4. So we consider the case that $(D^n) = 5$ and the codimension of $\phi_{|D|}(X)$ is 2.

2. Codimension 2 **case.** Throughout §2, we assume that $h^0(X, \mathscr{L}) = n + 3$ where $n = \dim X$, $\mathscr{L} = \mathscr{O}(D)$, $(D^n) = 5$ and L is very ample. In this case, $g(X, L) = 1$ or 2 because $g(X, L) = 0$ implies that $\Delta(X, L) = 0$ by the Theorem D. This contradicts $(D^n) = 5$ and $h^0(X, L) = n + 3$. If $g(X, L) \geq 2$, then $g(X, L) = 2$ by Theorem E in §1.

THEOREM 1. *If* $g(X, L) = 2$, *then* (X, L) *is projectively normal and the defining ideal of* (X, L) *is generated by degree* 2 *and* 3.

To prove this theorem, we prepare two lemmas.

LEMMA 1. *Let* (X, L) *be as above. Let* V *be a reduced irreducible non-singular member of* $|D|$. *If the homogeneous ideal of* (V, L_V) *is generated by degree* 2 *and* 3, *then the homogeneous ideal of* (X, L) *is generated by degree* 2 *and* 3.

Proof. Let $I(k)$ be the polynomials defined by

$$I(k) = \ker[S^k H^0(X, \mathscr{L}) \to H^0(X, \mathscr{L}^{\otimes k})]$$

where S^k is a kth symmetric product and let $I_V(k)$ be the polynomials defined by

$$I_V(k) = \ker[S^k H^0(V, \mathscr{L}_V) \to H^0(V, \mathscr{L}_V^{\otimes k})].$$

We prove this lemma by induction on k. In the case of $k = 2, 3$, this lemma is trivial. We assume that $I(k)$ is generated by $I(2)$ and $I(3)$. By Theorem E (a) in §1, V is a regular member. Moreover (X, L) and (V, L_V) are projectively normal by Theorem E(b) in §1. Therefore we obtain the following diagram:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & I(k) & \to & I(k+1) & \xrightarrow{\pi} & I_V(k+1) & & \\
 & & \downarrow & \hookrightarrow & \downarrow & \hookrightarrow & \downarrow & & \\
0 & \to & S^k H^0(X, \mathscr{L}) & \to & S^{k+1} H^0(X, \mathscr{L}) & \to & S^{k+1} H^0(V, \mathscr{L}_V) & \to & 0 \\
 & & \downarrow & \hookrightarrow & \downarrow & \hookrightarrow & \downarrow & & \\
0 & \to & H^0(X, \mathscr{L}^{\otimes k}) & \to & H^0(X, \mathscr{L}^{\otimes(k+1)}) & \to & H^0(V, \mathscr{L}_V^{\otimes(k+1)}) & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

By the snake lemma, π is a surjective map. By the assumption, $I_V(k+1)$ is generated by degree 2 and 3. Therefore $I(k+1)$ is generated by degree 2 and 3.

LEMMA 2. *If* C *is a non-singular curve and* L *is a very ample line bundle on* C *and* $\Delta(C, L) = 2$, *then* (C, L) *is projectively normal and its ideal is generated by degree* 2 *and* 3.

Proof. See Saint-Donat **[9]**.

Proof of Theorem 1. It is clear by Lemma 1 and Lemma 2.

Next we prepare the following notation.

DEFINITION. Let (X, L) be a polarized non-singular variety and let L be a very ample line bundle. We define $c(X, L)$ by

$$\begin{aligned} c(X, L) = \text{minimum}\{i;\ & X = X_n \supset X_{n-1} \supset \cdots \supset X_i \supset \cdots \supset X_1 \text{ with} \\ & X_i \text{ being a reduced irreducible non-singular} \\ & \text{member of } |D_{t+1}| \text{ where } L_{X_T} = \mathscr{O}(D_t) \text{ and} \\ & \Delta(X_n, L_{X_n}) = \cdots = \Delta(X_{i+1}, L_{X_{i+1}}) > \Delta(X_i, L_{X_i})\}. \end{aligned}$$

where $n = \dim X$. In the case of $\Delta(X_1, L_{X_1}) = \Delta(X, L)$, we put $c(X, L) = 0$.

If $\Delta(X, L) = 2$ and $g(X, L) = 2$, then $c(X, L) = 0$. If $\Delta(X, L) = 2$ and $g(X, L) = 1$, then $1 \leq c(X, L) \leq \dim X - 1$. Therefore Theorem 1 is in the case of $c(X, L) = 0$.

THEOREM 2. *If $c(X, L) = 1$, then (X, L) is projectively normal and the ideal defining (X, L) is generated by degree* 3.

We prepare the following two lemmas.

LEMMA 3. *If $C \subset \mathbb{P}^3$ is a non-singular elliptic curve of degree* 5 *which is not contained in any hyperplane, then*

$$H^0(\mathbb{P}^3, \mathscr{O}_{\mathbb{P}^3}(k)) \to H^0(C, \mathscr{O}_C(k))$$

is surjective for every $k \geq 2$.

Proof. Let $\mathscr{O}_C(1) = \mathscr{O}(D)$. We obtain the following diagram:

$$\begin{tikzcd} & \mathbb{P}^4 \arrow[d, dotted, "\text{projection}"] \\ C \arrow[ur, "\phi_{|D|}"] \arrow[r, hook] & \mathbb{P}^3 \end{tikzcd}$$

As $(C, \mathscr{O}(D))$ is projectively normal, hence

$$H^0(C, \mathscr{O}_C(k)) \otimes H^0(C, \mathscr{O}_C(m)) \to H^0(C, \mathscr{O}_C(k+m))$$

is surjective for every k, $m \geq 1$. By the assumption, the canonical map

$$H^0(\mathbb{P}^3, \mathscr{O}_{\mathbb{P}^3}(1)) \to H^0(C, \mathscr{O}_C(1))$$

is injective. Now we show that

$$H^0(\mathbb{P}^3, \mathscr{O}_{\mathbb{P}^3}(2)) \to H^0(C, \mathscr{O}_C(2))$$

is an isomorphism. As $h^0(\mathbb{P}^3, \mathscr{O}_{\mathbb{P}^3}(2)) = h^0(C, \mathscr{O}_C(2)) = 10$, therefore we may show that

$$H^0(\mathbb{P}^3, \mathscr{O}_{\mathbb{P}^3}(2)) \to H^0(C, \mathscr{O}_C(2))$$

is injective. If this is not true, then there exists some quadratic surface Q in $\mathbb{P}^3$ with $Q \supset C$. If Q is non-singular, then the degree of $C = a+b$ and the genus of $C = ab - a - b + 1$ for some integers a, b. This cannot occur because the degree of $C = 5$ and the genus of $c = 1$. If Q is singular, then the genus of $C = a^2 - a$ for odd degree $2a+1$ of C. Hence degree of $C = 5$ and genus of $C = 1$ does not occur. Therefore the above map is injective, hence is an isomorphism. Next we show that $H^0(\mathbb{P}^3, \mathscr{O}_{\mathbb{P}^3}(3)) \to H^0(C, \mathscr{O}(3))$ is surjective. We take the basis of $H^0(C, \mathscr{O}_C(1))$ with

$$H^0(\mathbb{P}^3, \mathscr{O}_{\mathbb{P}^3}(1)) = [x_0, x_1, x_2, x_3],$$
$$H^0(C, \mathscr{O}_C(1)) = [x_0, x_1, x_2, x_3, x_4]$$

where $[x_0, \dots, x_N]$ means that $x_1, \dots, x_N$ are bases of a vector space. As

$$H^0(\mathbb{P}^3, \mathscr{O}_{\mathbb{P}^3}(2)) = [x_0^2, x_1^2, x_2^2, x_3^2, x_0x_1, x_0x_2, x_0x_3, x_1x_2, x_1x_3, x_2x_3]$$

and $H^0(\mathbb{P}^3, \mathscr{O}_{\mathbb{P}^3}(2)) \cong H^0(C, \mathscr{O}_C(2))$, therefore $H^0(C, \mathscr{O}_C(2))$ has the above basis. But x_ix_4 $(i = 0, \dots, 4)$ are contained in $H^0(C, \mathscr{O}_C(2))$, and therefore we obtain the following relations:

$$(*) \qquad x_ix_4 = f_i(x_0, x_1, x_2, x_3)$$

where $i = 0, 1, 2, 3, 4$ and f_i $(i = 1, 2, 3, 4)$ are homogeneous polynomials of degree 2. As $(C, \mathscr{O}_C(1))$ is projectively normal, hence

$$H^0(C, \mathscr{O}_C(1))^{\otimes 3} \to H^0(C, \mathscr{O}_C(3))$$

is surjective. Therefore we obtain the generators of $H^0(C, \mathscr{O}_C(3))$ as follows,

$$(1) \qquad \begin{cases} x_0^3, x_1^3, x_2^3, x_3^3 \\ x_0^2x_1, x_0^2x_2, x_0^2x_3, x_1^2x_0, x_1^2x_2, x_1^2x_3 \\ x_2^2x_0, x_2^2x_1, x_2^2x_3, x_3^2x_0, x_3^2x_1, x_3^2x_2 \\ x_0x_1x_2, x_0x_1x_3, x_0x_2x_3, x_1x_2x_3 \end{cases}$$

$$(2) \qquad \begin{cases} x_4^3, x_4^2x_0, x_4^2x_1, x_4^2x_2, x_4^2x_3 \\ x_4x_0^2, x_4x_1^2, x_4x_2^2, x_4x_3^2 \\ x_4x_0x_1, x_4x_0x_2, x_4x_0x_3, x_4x_1x_2, x_4x_1x_3, x_4x_2x_3. \end{cases}$$

The part (1) is clearly the image of $H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(1))$. And the relation $(*)$ says that the part (2) is also in the image of $H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(3))$. Because

$$\begin{aligned} x_4x_ix_j &= f_i(x_0, x_1, x_2, x_3)x_j \qquad (i, j \neq 4), \\ x_4^2x_i &= f_4(x_0, x_1, x_2, x_3)x_i \quad (i = 0, 1, 2, 3), \\ x_4^3 &= f_4(x_0, x_1, x_2, x_3)x_4 \end{aligned}$$

by the relation $(*)$; moreover the relation $(*)$ says f_4x_4 is in the image of $H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(3))$. Hence

$$H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(3)) \to H^0(C, \mathcal{O}_C(3))$$

is surjective. Finally we prove this lemma. If $k = 2, 3$, then this lemma is true by the above argument. We consider the case in which $k \geq 4$. First, we show this lemma in the case that k is even. Let $k = 2m$. We show in this case by the induction on m. In this, we give the following diagram:

$$\begin{array}{ccc} H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(2m)) & \to & H^0(C, \mathcal{O}_C(2m)) \\ \uparrow & \hookrightarrow & \uparrow \\ H^0(\mathbb{P}^3, \mathcal{O}(2(m-1))) \otimes H^0(\mathbb{P}^3, \mathcal{O}(2)) & \to & H^0(C, \mathcal{O}(2(m-1))) \otimes H^0(C, \mathcal{O}(2)) \end{array}$$

By the hypothesis of induction and projective normality of $(C, \mathcal{O}_C(1))$, we obtain

$$H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(2m)) \to H^0(C, \mathcal{O}_C(2m))$$

is surjective. Next we consider the case in which k is odd and $k \geq 5$. But this case is clear by the same argument. Therefore we obtain this lemma.

LEMMA 4. *If $C \subset \mathbb{P}^3$ is as in Lemma 3, then the homogeneous ideal of $C \subset \mathbb{P}^3$ is generated by degree 3.*

Proof. Let I_k be the kernel of $H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(k)) \to H^0(C, \mathcal{O}_C(k))$. We show that

$$I_k \otimes H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(1)) \to I_{k+1}$$

is surjective for every $k \geq 3$. We take a divisor D with $\mathcal{O}_C(1) \cong \mathcal{O}(D)$ and support of D consists of 5 distinct points. As $D \subset \mathbb{P}^2$, we define I'_k $(k = 1, 2, \ldots)$ by

$$0 \to I'_k \to H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(k)) \to H^0(D, \mathcal{O}_D(k)) \cong H^0(D, \mathcal{O}_D).$$

If $k \geq 2$, then we give the following diagram:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & I_k & \to & I_{k+1} & \to & I'_{k+1} & & \\
 & & \downarrow & \hookrightarrow & \downarrow & \hookrightarrow & \downarrow & & \\
0 & \to & H^0(\mathbb{P}^3, \mathscr{O}(k)) & \to & H^0(\mathbb{P}^3, \mathscr{O}(k+1)) & \to & H^0(\mathbb{P}^2, \mathscr{O}(k+1)) & \to & 0 \\
 & & \downarrow & \hookrightarrow & \downarrow & \hookrightarrow & \downarrow & & \\
0 & \to & H^0(C, \mathscr{O}(k)) & \to & H^0(C, \mathscr{O}(k+1)) & \to & H^0(D, \mathscr{O}(k+1)) & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

By the snake lemma,

$$0 \to I_k \to I_{k+1} \to I'_{k+1} \to 0$$

is exact for every $k \geq 2$. Moreover we define λ so the following diagram commutes:

$$\begin{tikzcd}
I_k \otimes H^0(\mathbb{P}^3, \mathscr{O}_{\mathbb{P}^3}(1)) \arrow[r] \arrow[dr, "\lambda"'] & I'_k \otimes H^0(\mathbb{P}^3, \mathscr{O}_{\mathbb{P}^3}(1)) \arrow[d] \\
 & I'_k \otimes H^0(\mathbb{P}^2, \mathscr{O}_{\mathbb{P}^2}(1))
\end{tikzcd}$$

As $I_k \to I'_k$ is surjective if $k \geq 3$ and $H^0(\mathbb{P}^3, \mathscr{O}_{\mathbb{P}^3}(1)) \to H^0(\mathbb{P}^2, \mathscr{O}_{\mathbb{P}^2}(1))$ is surjective, therefore λ is surjective for $k \geq 3$. Next we define $\psi\colon I_k \to I_k \otimes H^0(\mathbb{P}^3, \mathscr{O}_{\mathbb{P}^3}(1))$ with $\psi(s) = s \otimes \delta$ where δ is a section of $H^0(\mathbb{P}^3, \mathscr{O}_{\mathbb{P}^3}(1))$ which is defining $\mathbb{P}^2$. This shows that the following diagram

$$\begin{tikzcd}
 & & I_k \otimes H^0(\mathbb{P}^3, \mathscr{O}_{\mathbb{P}^3}(1)) \arrow[r, "\lambda"] \arrow[d] & I'_k \otimes H^0(\mathbb{P}^2, \mathscr{O}_{\mathbb{P}^2}(1)) \arrow[d] & \\
0 \arrow[r] & I_k \arrow[ur, "\psi"] \arrow[r] & I_{k+1} \arrow[r] & I'_{k+1} \arrow[r] & 0
\end{tikzcd}$$

is commutative for $k \geq 2$. Therefore if $I'_k \otimes H^0(\mathbb{P}^2, \mathscr{O}_{\mathbb{P}^2}(1)) \to I'_{k+1}$ is surjective for every $k \geq 3$, then this lemma is proved. So we show that

$$I'_k \otimes H^0(\mathbb{P}^2, \mathscr{O}_{\mathbb{P}^2}(1)) \to I'_{k+1}$$

is surjective for $k \geq 3$. Let $V = H^0(\mathbb{P}^2, \mathscr{O}_{\mathbb{P}^2}(1))$ and let $V^k =$ the image of $H^0(\mathbb{P}^2, \mathscr{O}_{\mathbb{P}^2}(k)) \to H^0(D, \mathscr{O}_D(K))$. As the support of D is not collinear, $V \to H^0(D, \mathscr{O}_D(1))$ is injective. We show that $V^k = H^0(D, \mathscr{O}_D(k))$ for $k \geq 2$. If $V \neq H^0(D, \mathscr{O}_D(2))$, then the dimension of $\ker[H^0(\mathbb{P}^2, \mathscr{O}_{\mathbb{P}^2}(2)) \to H^0(D, \mathscr{O}_D(2))]$ is at least 2. Therefore there exist distinct quadratics Q_1 and Q_2 with $Q_i \supset D$ $(i = 1, 2)$. Q_1 and Q_2 satisfy $Q_1 \cap Q_2 =$ finite points. Because if $Q_1 \cap Q_2$ has component, then there exist distinct lines l_1, l_2, l_3 with

$$l_1 \cap D = 4 \text{ points}$$

and

$$Q_1 = l_1 + l_2, \qquad Q_2 = l_1 + l_3.$$

Hence $\mathbb{P}^3 - l_1 \to \mathbb{P}^1$ be a projection with center l_1, and let $C \cdots \to \mathbb{P}^1$ be a restriction map to C. Let $f\colon C \to \mathbb{P}^1$ be an associated morphism defined by the above map $C \cdots \to \mathbb{P}^1$. As $l_1 \cap D = 4$ points, therefore f is a bijective morphism. Hence the genus of $C =$ the genus of $\mathbb{P}^1 = 0$. This is a contradiction. So $Q_1 \cap Q_2 =$ finite points. As Q_1 and Q_2 are conics, $Q_1 \cap Q_2$ contains at most 4 points by Bezout's theorem. But $Q_1 \cap Q_2$ contains D with degree 5; this is a contradiction. Hence $V^2 = H^0(D, \mathscr{O}_D(2))$. We take $s \in V$ with

$$\begin{array}{ccc}
H^0(D, \mathscr{O}_D(k)) & \xrightarrow{\sim} & H^0(D, \mathscr{O}_D(k+1)). \\
\downarrow & & \downarrow \\
t & \mapsto & ts
\end{array}$$

In this, we obtain the following commutative diagram:

$$\begin{array}{ccc}
H^0(\mathbb{P}^2, \mathscr{O}_{\mathbb{P}^2}(k)) & \to & H^0(D, \mathscr{O}_D(k)) \\
\sigma \downarrow & \hookrightarrow & \downarrow \zeta \\
H^0(\mathbb{P}^2, \mathscr{O}_{\mathbb{P}^2}(k+1)) & \to & H^0(D, \mathscr{O}_D(k+1))
\end{array}$$

where σ, ζ, are defined by $f \mapsto fs$. Therefore we obtain

$$H^0(\mathbb{P}^2, \mathscr{O}_{\mathbb{P}^2}(k)) \to H^0(D, \mathscr{O}_D(k))$$

is surjective if $k \geq 2$. Hence

$$V^k = H^0(D, \mathscr{O}_D(k))$$

for every $k \geq 2$. Let $K(V^k, V)$ be $\ker[V^k \otimes V \to V^{k+1}]$ and $K(V, s)$ be $\ker[V^{\otimes s} \to V^s]$ where k and s are positive integers. We consider the following commutative diagram:

$$\begin{array}{ccccc}
0 & & 0 & & 0 \\
\downarrow & & \downarrow & & \downarrow \\
K(V^{k-1}, V) \otimes V & \xrightarrow{\alpha} & K(V^k, V) & \xleftarrow[\sim]{\xi} & K(V^{k-1}, V) \\
\downarrow & \hookrightarrow & \downarrow & \hookrightarrow & \downarrow \\
V^{k-1} \otimes V \otimes V & \xrightarrow{\beta} & V^k \otimes V & \xleftarrow[\sim]{\rho} & V^{k-1} \otimes V \\
\downarrow & \hookrightarrow & \downarrow & \hookrightarrow & \downarrow \\
V^k \otimes V & \to & V^{k+1} & \xleftarrow[\sim]{\zeta} & V^k \\
\downarrow & & \downarrow & & \downarrow \\
0 & & 0 & & 0
\end{array}$$

where $\beta(a\otimes b\otimes c) = ab\otimes c$, α is induced by β, $\zeta(f) = fs$, $\rho(f\otimes g) = fs\otimes g$, ξ is induced by ρ and s is an element of V defined as above. If $k \geq 3$, ρ and ζ are isomorphisms. Hence we obtain that α is a surjective map. Next we consider the following commutative diagram:

$$\begin{array}{ccccc} 0 \to K(V, k)\otimes V \xrightarrow{u} & K(V, k+1) & \xrightarrow{v} & K(V^k, V) \to 0 \\ & w\uparrow & \hookrightarrow & \alpha\uparrow \\ & K(V, k)\otimes V & \xrightarrow{v'} & K(V^{k-1}, V)\otimes V \to 0 \end{array}$$

where u, v, v' and w are canonical maps and the surjectivity of v and v' is induced by the following commutative diagram and the snake lemma:

$$\begin{array}{ccccccccc} & & 0 & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & K(V, k)\otimes V & \xrightarrow{u} & K(V, k+1) & \xrightarrow{v} & K(V^k, V) & & \\ & & \mathrm{id}\downarrow & \hookrightarrow & \downarrow & \hookrightarrow & \downarrow & & \\ 0 & \to & K(V, k)\otimes V & \to & V^{\otimes(k+1)} & \to & V^k\otimes V & \to & 0 \\ & & \downarrow & \hookrightarrow & \downarrow & \hookrightarrow & \downarrow & & \\ 0 & \to & 0 & \to & V^{k+1} & \xrightarrow{\mathrm{id}} & V^{k+1} & \to & 0 \\ & & \downarrow & & \downarrow & & & & \\ & & 0 & & 0 & & 0 & & \end{array}$$

Therefore $K(V, k+1) = \mathrm{im}(w) + \mathrm{im}(u)$ if $k \geq 3$. Hence we obtain that $I'_k \otimes H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1)) \to I'_{k+1}$ is surjective for $k \geq 3$. Hence we prove this lemma.

Proof of Theorem 2. First we show that

$$H^0(X, \mathscr{L})^{\otimes k} \to H^0(X, \mathscr{L}^{\otimes k})$$

is surjective for $k \geq 1$. If $k = 1$, then this is clear. Now we can take

$$X = X_n \supset X_{n-1} \supset \cdots \supset X_2 \supset X_1$$

such that X_i is a reduced irreducible non-singular member of $|D_{i+1}|$ where $\mathscr{L}_{X_i} = \mathcal{O}(D_i)$ $(i = 1, 2, \ldots, n = \dim X)$ and

$$2 = \Delta(X_n, L_{X_n}) = \cdots = \Delta(X_2, L_{X_2}) > \Delta(X_1, L_{X_1}) = 1$$

because $c(X, L) = 1$. As X_1 is an elliptic curve of degree 5 in $\mathbb{P}^3$, therefore $H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(k)) \to H^0(X_1, L_{X_1}^{\otimes k})$ is surjective for $k \geq 2$ by Lemma 3. We consider the following diagram:

$$\begin{array}{ccccccccc} 0 & \to & H^0(\mathbb{P}^4, \mathcal{O}(k-1)) & \to & H^0(\mathbb{P}^4, \mathcal{O}(k)) & \to & H^0(\mathbb{P}^3, \mathcal{O}(k)) & \to & 0 \\ & & \downarrow & \hookrightarrow & \downarrow & \hookrightarrow & \downarrow & & \\ 0 & \to & H^0(X_2, \mathscr{L}_{X_2}^{\otimes(k-1)}) & \to & H^0(X_2, \mathscr{L}_{X_2}^{\otimes k}) & \to & H^0(X_1, \mathscr{L}_{X_1}^{\otimes k}) & & \end{array}$$

By induction on k, L_{X_2} is projective normal. So it is clear that L is projectively normal because $\Delta(X_n, L_{X_n}) = \cdots = \Delta(X_2, L_{X_2})$. The last part of this theorem is obtained by Lemma 4 and the same argument.

COROLLARY. *If* (X, L) *is a polarized non-singular surface,* $(D^2) = 5$ *where* $\mathscr{L} = \mathscr{O}(D)$ *and* L *is very ample, then* (X, L) *is projectively normal.*

To conclude this section, we give two examples of varieties of degree 5 and codimension 2.

EXAMPLE 1. Let $f: S \to \mathbb{P}^2$ be a blowing up with center $p_1, \ldots, p_8 \in \mathbb{P}^2$ where $p_1, \ldots, p_8$ are in general position. We put $f^{-1}(p_i) = E_i$ $(i = 1, \ldots, 8)$ and $D = f^*(4l) - 2E_1 - E_2 - \quad -E_8$ where $l \subset \mathbb{P}^2$ is a line. This D is very ample, $(D^2) = 5$ and $g(S, \mathscr{O}(D)) = 2$ (see Hartshorne [5]). Therefore $c(S, \mathscr{O}(D)) = 0$.

EXAMPLE 2. Let $f: S = \mathbb{P}(\mathscr{E}) \to C$ be a ruled surface over an elliptic curve C where $\mathscr{E}$ is an indecomposable locally free sheaf of rank 2 on C. Let $\deg(\mathscr{E}) = 1$. Let C_0 be a section of f with $\operatorname{Pic}(S) = \mathbb{Z}C_0 \oplus f^* \operatorname{Pic}(C)$. Let D be a divisor in $\operatorname{Pic}(S)$ with $D = C_0 + f^*(T)$ and $\deg(T) = 2$. This D is very ample (see Hartshorne [5]). Let l be a fiber of f. As D is numerically equivalent to $C_0 + 2l$, therefore $(D^2) = 5$ and $(D.(D + K_S)) = 0$. Therefore $g(S, \mathscr{O}(D)) = 1$. This is an example of $c(S, \mathscr{O}(D)) = 1$.

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Received March 3, 1987.

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