PACIFIC JOURNAL OF MATHEMATICS
Vol. 145, No. 2, 1990

RELATIONS AMONG GENERALIZED CHARACTERISTIC CLASSES

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In this paper, we extend Brown and Peterson's algebraic calculations, using methods of homotopy theory, to the consideration of manifolds with structure and to characteristic classes arising from generalized cohomology theories.

0. Introduction. In [**BP**], E. Brown and F. Peterson made the first calculation of relations among the Stiefel-Whitney classes of the stable normal bundles of manifolds. Specifically, they computed

$$I_n = \bigcap_{M^n} \operatorname{Ker} \nu_M^* \subset H^*(BO; \mathbb{Z}/2),$$

where $\nu_M \colon M^n \to BO$ classifies the stable normal bundle of M^n, and the intersection is taken over all compact differentiable manifolds of dimension n. These calculations have, via the Brown-Gitler spectra [**BG**], proven to be of considerable value. Although they arose in the context of the Immersion Conjecture for compact differentiable manifolds and were instrumental in its solution [**C1**], these spectra were also used by M. Mahowald [**Ma**], and subsequently at odd primes by R. Cohen [**C2**], to produce infinite families in the homotopy groups of spheres. G. Carlsson used the Spanier-Whitehead duals of these spectra to prove the Segal Conjecture for elementary abelian 2-groups [**Ca**], and H. Miller then used the algebra thus developed by Carlsson in his proof of the Sullivan Conjecture [**Mi**].

These theories should be related to the bordism theories coming from our chosen class of manifolds. We wish to calculate

$$I_n = \bigcap_{(M^n, \tilde{\nu})} \operatorname{Ker} \tilde{\nu}^* \subset E^*(B),$$

where B is the classifying space associated to a certain class of manifolds, denoted by pairs $(M^n, \tilde{\nu})$; $\tilde{\nu} \colon M^n \to B$ is a lifting of ν_M (B comes equipped with a map to BO); and E^* is the cohomology theory. We will place the following conditions on E^*, where TB is the Thom-spectrum associated to B.

2.1. (a) TB has an E-orientation, and

(b) Given a class $u \in E^q(M^n)$, there exists a class $v \in E^{n-q}(M^n)$ such that $\langle u \cdot v, [M^n] \rangle \neq 0$.

In Section 2 we define a map

$$\psi_q \colon E^q(TB) \to \pi_n(TB \wedge E_{n-q+})^*,$$

where E_{n-q} is the $(n-q)$th space in the Ω-spectrum E representing E^*, and for a group G, $G^* = \operatorname{Hom}(G, \pi_0 E)$. We show that if $J_n = \Phi(I_n) \subset E^*(TB)$, where Φ is the Thom isomorphism, then we have the following:

2.6. THEOREM. *If E^* satisfies 2.1, then $J_n \cap E^q(TB) = \operatorname{Ker} \psi_q$.*

Dualizing to homology, we obtain our main result:

2.7. THEOREM. *Under assumptions 2.1, the following diagram commutes:*

$$\begin{array}{ccc}
\pi_n(TB \wedge E_{n-q+}) & \overset{\psi_q^*}{\rightarrowtail} & (E^q(TB))^* \\
\downarrow (\iota_{n-q})_* & & \uparrow \eta_q \\
TB_q(E) & \overset{\chi}{\rightarrowtail} & E_q(TB)
\end{array}$$

Here, ι_{n-q} is the stabilization map, χ is induced by the switch-map $TB \wedge E \to E \wedge TB$, and η_q is evaluation. In those cases where η_q is an isomorphism, therefore, we have reduced our original calculation to that of the stabilization map and χ. At the end of Section 2, we show that these results reduce to those of Brown and Peterson, by setting $B = BO$ and $E^* = H^*(-\mathbb{Z}/2)$.

In the final sections of this paper, we apply this program to the case $B = BU$, where U is the infinite unitary group (thus, the manifolds under consideration are stably almost-complex). We use the Morava K-theories as our generalized (co)homology theories, since they are complex-oriented and satisfy the strong duality conditions which we need. The paper ends with a calculation of the image of the stabilization map, which we now summarize briefly. The Thom-spectrum MU, localized at the odd prime p, is made up of similar spectra BP. Let $m \geq 1$, and let $K(m)$ be the corresponding Morava K-theory at the prime p. We show that $BP_*K(m)$ is generated as a π_* BP-module by elements $v_m^r \tilde{\xi}^J$, where $J = (j_1, j_2, \dots)$ is a nonnegative finite sequence with each $j_k < p^m$. The image of the suspension map may be described as follows: given $\alpha \in BP_*K(m)$, let $d(\alpha)$ be the minimum q such that there is an $\alpha_q \in \pi_*(BP \wedge K(m)_{q+})$ with $(\iota_q)_*(\alpha_q) = \alpha$. Finally, let $|J| = \sum j_t \quad (< \infty)$.

4.9. THEOREM. $d(v_m^r \tilde{\xi}^J) = 2|J| - 2r(p^m - 1)$.

This paper consists primarily of work done on my Ph.D. thesis at Stanford University under the supervision of Ralph L. Cohen. It is with great pleasure that I thank Professor Cohen for his guidance, support and inspiration. In addition, I gratefully acknowledge the assistance of W. Stephen Wilson in realizing the results of Section 4 from his paper [W].

1. Preliminaries. Throughout this paper we shall be concerned with manifolds with structure in the sense of Stong [St]. In this section we recall briefly the definition and basic properties of such objects. Henceforth all manifolds are assumed to be compact and differentiable.

Suppose one is given a sequence of spaces B_k, fibrations $f_k\colon B_k \to BO(k)$, and maps $g_k\colon B_k \to B_{k+1}$ such that the diagram

$$\begin{array}{ccc} B_k & \overset{g_k}{\rightarrowtail} & B_{k+1} \\ \downarrow f_k & & \downarrow f_{k+1} \\ BO(k) & \overset{j_k}{\rightarrowtail} & BO(k+1) \end{array}$$

commutes, j_k being induced by the usual inclusion $O(k) \hookrightarrow O(k+1)$. Let $b = \varinjlim B_k$ and as usual $BO = \varinjlim \mathrm{BO}(k)$.

1.1. DEFINITION. A (B, f)-manifold is a pair $(M^n, \tilde{\nu})$, where M^n is an n-dimensional manifold and $\tilde{\nu}\colon M^n \to B$ is a lifting of the stable normal bundle classifying map of M^n.

For example, if we let $B_{2k} = B_{2k+1} = BU(k)$, the classifying space for the unitary group, let $f_{2k}\colon BU(k) \to BO(2k)$ be induced by the standard map $U(k) \to O(2k)$, and define $f_{2k+1} = j_{2k} \circ f_{2k}$, then we are considering stably almost-complex manifolds.

For a space X, we define as usual $\Omega_n(B, f; X)$ to be the set of equivalence classes of triples $(M^n, \tilde{\nu}, h)$, where $(M^n, \tilde{\nu})$ is a (B, f)-manifold and $h\colon M \to X$ is a map, under the relation of cobordism. Note that for our above example, $\Omega_n(B; f; \text{pt.}) = \Omega_n^U$, the complex cobordism groups.

Now let TB_k be the Thom space of the bundle $f_k^*(\gamma_k)$, for γ_k the universal bundle over $\mathrm{BO}(k)$. The spaces TB_k form a spectrum, which we denote TB. Then one has the Thom-Pontrjagin theorem for (B, f)-manifolds.

1.2. THEOREM. $\Omega_n(B; X) \cong \pi_n(TB \wedge X_+)$. □

The map $T: \Omega_n(B, f; X) \to \pi_n(TB\wedge X_+)$ is constructed as follows: there is a stable map $\Delta^*: T\nu_M \to T\nu_M \wedge M_+$, induced by the diagonal map $\Delta: M \to M \times M$. Let $r: S^n \to T\nu_M$ be the stable map given by the Thom-Pontrjagin construction. Then $T([M^n, \tilde{\nu}, h])$ is the composition

$$S_n \overset{r}{\rightarrowtail} T\nu_M \overset{\Delta^*}{\rightarrowtail} T\nu_M \wedge M_+ \overset{T\tilde{\nu}\wedge h}{\rightarrowtail} TB \wedge X_+.$$

In the remainder of this section we recall some facts about generalized cohomology theories and duality in manifolds that will be useful to us. The reference here is **[A2]**.

For what follows we shall let $E = \{E_q\}$ be the Ω-spectrum representing our homology theory E_* and cohomology theory E^*. We shall assume unless stated otherwise that E is a ring spectrum with multiplication μ, so that our theories E^* and E_* come equipped with the cup, cap and Kronecker products.

We recall that an E- **orientation** (or **Thom class**) for a k-plane bundle ξ over a space X with Thom space T_ξ is an element $u_E \in \widetilde{E}^k(T\xi)$ which restricts to a generator $\widetilde{E}_*(S^k)$ (as an $\widetilde{E}^*(S^0)$-module) on each fiber. If $(M^n, \tilde{\nu})$ is a (B, f)-manifold, then for k sufficiently large the normal bundle ν_M^k to any immersion of M^n in $\mathbb{R}^{n+k}$ has a TB-orientation. If each ν_M^k has an E-orientation for k sufficiently large, then one obtains a stable Thom class $u_E \in E^0(T\nu_M)$, which we call an E **-orientation for** M^n. In particular, every (B, f)-manifold has a TB-orientation. If M^n has an E-orientation, then we have a Thom isomorphism $\Phi_u: E^q(M^n) \to E^q(T\nu_M)$, given by cup-product with $u_E: E^{(M^n)} \to E^p(T\nu_M)$.

Poincaré duality holds for manifolds M^n with an E-orientation in the usual way: there exists a class $[M^n] \in E_n(M^n)$ such that the map $\frown [M^n]: E^{(M^n)} \to E_{n-p}(M^n)$ is an isomorphism for all p. As usual, we call such a class a **fundamental class** for M^n.

We recall that $T\nu_M$ and M_+^n are S-duals, with the duality isomorphism $s: E^q(T\nu_M) \to E_{n-q}(M_+^n)$ given by: $s(v)$ is the composition

$$S^n \overset{r}{\rightarrowtail} T\nu_M \overset{\Delta^*}{\rightarrowtail} T\nu_M \wedge M_+^n \overset{v\wedge 1}{\rightarrowtail} \Sigma^q E \wedge M_+^n.$$

Then Poincaré duality, the Thom isomorphism and S-duality are all related by the following result.

1.3. LEMMA [A2]. *Suppose $T\nu_M$ has a Thom class u_E. Then the following diagram commutes up to sign:*

$$\begin{tikzcd} E^q(M^n) \arrow[r, "\Phi_u"] \arrow[dr, "\frown(M^n)"'] & E^q(T\nu_M) \arrow[d, "s"] \\ & E_{n-q}(M^n) \end{tikzcd} \qquad \square$$

1.4. COROLLARY. $s(u_E) \in E_n(M^n)$ *is a fundamental class for* M^n. $\square$

In other words, we may take as a representative of $[M^n]$ the following composition:

$$S^n \xrightarrow{r} T\nu_M \xrightarrow{\Delta^*} T\nu_M \wedge M^n_+ \xrightarrow{u_E \wedge 1} E \wedge M^n_+.$$

2. The generalized Brown-Peterson process. As we stated earlier, our goal is to compute the ideal

$$I_n = \bigcap_{(M^n, \tilde{\nu})} \operatorname{Ker} \tilde{\nu}^* \subset E^*(B),$$

where the intersection is over all (B, f)-manifolds $(M^n, \tilde{\nu})$, for a judicious choice of E^*.

We fix our choice (B, f) of structure for our manifolds, and demand the following two conditions of our cohomology theory E^*:

2.1. (a) TB has an E-orientation, and

(b) Given a class $u \in E^q(M^n)$, there exists a class $v \in E^{n-q}(M^n)$ such that $\langle u \cdot v, [M^n]\rangle \neq 0$.

Condition 2.1(a) means that for each k, the bundle $f_k^*(\gamma_k)$ over B_k has an E-orientation U_k such that the composition $E^k(TB_k) \to E^k(\Sigma TB_{k-1}) \to E^{k-1}(TB_{k-1})$ carries U_k to U_{k-1}.

Two further remarks are in order here. First, 2.1(a) implies that each (B, f)-manifold $(M^n, \tilde{\nu})$ has an E-orientation: from the preceding comment we see that there is a stable class $U \in E^0(TB)$, called the **stable Thom class** for TB; cup-product with U yields a stable Thom isomorphism

$$\Phi\colon E^q(B) \to E^q(TB).$$

The composition $T\nu_M \xrightarrow{T\tilde{\nu}} TB \xrightarrow{U} E$ then yields a stable Thom class for $T\nu_M$ over the (B, f)-manifold $(M^n, \tilde{\nu})$. Second, 2.1(b) is stronger than Poincaré duality as described in Section 1; indeed, if, for example $E^* = H^*(-; \mathbb{Z})$, then 2.1(b) needn't hold if u is a torsion class. Poincaré duality alone is insufficient to prove Lemma 2.2 below, which is a key step in our reduction of the calculation. One needs other methods, for example, to find the ideal of relations for $B = BSO$ and $E^* = H^*(-;\mathbb{Z})$ (see **[Sh]**).

Let G be an abelian group. In what follows we shall let $G^* = \operatorname{Hom}(G, \pi_0 E)$, where E is understood from context.

Let $\{E_q\}$ be the Ω-spectrum representing E^*, and define a map $\varphi_q \colon E^q(B) \to \Omega_n(B, f; E_{n-q})^*$ by the rule

$$\varphi_q(v)([M^n, \tilde{\nu}, h]) = \langle \tilde{\nu}^*(v) \cdot h^*(\iota_{n-q}), [M^n] \rangle,$$

where $\iota_{n-q} \in E^{n-q}(E_{n-q})$ is the fundamental class. We shall see in a moment that φ_q is well-defined.

2.2. Lemma. *If E^* satisfies conditions* 2.1, *then* $I_n \cap E^q(B) = \operatorname{Ker}\varphi_q$.

Proof. Let $v \in E^q(B)$. Then

$$\begin{array}{lll} v \in I_n & \text{iff } \tilde{\nu}^*(v) = 0 \quad \text{for all } (M^n, \tilde{\nu}) & \\ & \text{iff } \langle \tilde{\nu}^*(v) \cdot y, [M^n] \rangle = 0 & \text{for all } (M^n, \tilde{\nu}) \\ & & \text{and all } y \in E^{n-q}(M^n) \\ & \text{iff } \langle \tilde{v}^*(v) \cdot h^*(\iota_{n-q}), [M^n] \rangle & \text{for all } (M^n, \tilde{\nu}) \\ & & \text{and all } h \colon M^n \to E_{n-q} \\ & \text{iff } v \in \operatorname{Ker}\varphi_q. & \end{array}$$

□

We now define a second map, $\psi_q \colon E^q(B) \to \pi_n(TB \wedge E_{n-q^+})^*$, by the following: for $v \in E^q(B)$, $\alpha \in \pi_n(TB \wedge E_{n-q^+})$, $\psi_q(v)(\alpha)$ is the composition

$$S^n \overset{\alpha}{\rightarrowtail} TB \wedge E_{n-q^+} \overset{\Phi(v)\wedge\iota_{n-q}}{\rightarrowtail} \Sigma^q E \wedge \sigma^{n-q} E \rightarrowtail \Sigma^n E.$$

2.3. Proposition. *The following diagram commutes*:

$$\begin{array}{ccc} E^q(B) & \overset{\varphi_q}{\rightarrowtail} & \Omega_n(B, f; E_{n-q})^* \\ & {\scriptstyle\psi_q} \searrow & \uparrow T^* \\ & & \pi_n(TB \wedge E_{n-q^+})^* \end{array}$$

where $T \colon \Omega_n(B, f; E_{n-q}) \to \pi_n(TB \wedge E_{n-q^+})$ *is the Thom-Pontrjagin map defined in Section* 1.

Note as a corollary that φ_q is well-defined.

First we need the following

2.4. LEMMA. *Let u_M be the Thom class $T\nu_M \to TB \to E$ as above. Let $v \in E^q(M^n)$, and let $j_n \in E_n(S^n)$ be induced by the unit $1 \in \pi_0 E$. Then*

$$\langle v, [M^n]\rangle = \langle r^*(u_M \cdot v), j_n\rangle,$$

where $r\colon S^n \to T\nu_M$ is as before.

Proof. By definition $\langle v, [M^n]\rangle$ is the composition

$$S^n \xrightarrow{[M^n]} E \wedge M_+^n \xrightarrow{1\wedge v} E \wedge \Sigma^q E \xrightarrow{\mu} \Sigma^q E.$$

By Corollary 1.4, this is the same as

$$S^n \xrightarrow{r} T\nu_M \xrightarrow{\Delta^*} T\nu_m \wedge M_+^n \xrightarrow{u_M\wedge 1} E \wedge M_+^n \xrightarrow{1\wedge v} E \wedge \Sigma^q E \xrightarrow{\mu} \Sigma^q E.$$

But the above composition $T\nu_M \to E$ is equal to $u_E \cdot v$, by definition. The result follows immediately. □

Proof of Proposition 2.3. Let

$$v \in E^q(B), \quad [M^n, \tilde{\nu}, h] \in \Omega_n(B, f; E_{n-q+}).$$

Then if we let $\alpha \in \pi_n(TB \wedge E_{n-q+})$ be the composition

$$S^n \xrightarrow{r} T\nu_M \xrightarrow{\Delta^*} T\nu_M \wedge M_+^n \xrightarrow{T\tilde{\nu}\wedge h} TB \wedge E_{n-q+},$$

we have that $((T^* \circ \psi_q(v))([M^n, \tilde{\nu}, h]) = \langle \alpha^*(\Phi(v) \cdot \imath_{n-q}), j_n\rangle$, where $\Phi(v) \cdot \imath_{n-q}$ is given by the composition

$$TB \wedge E_{n-q+} \xrightarrow{\Phi(v)\wedge\imath_{n-q}} \Sigma^q E \wedge \Sigma^{n-q} E \xrightarrow{\mu} \Sigma^n E.$$

By Lemma 2.4,

$$\begin{aligned}\varphi_q([M^n, \tilde{\nu}, h]) &= \langle \tilde{v}^*(v) \cdot h^*(\imath_{n-q}), [M^n]\rangle \\ &= \langle r^*((u_M \cdot \tilde{\nu}^*(v)) \cdot h^*(\imath_{n-q})), j_n\rangle.\end{aligned}$$

Now $\Phi(v)$ is the composition

$$TB \xrightarrow{\Delta^*} TB \wedge B_+ \xrightarrow{U\wedge v} E \wedge \Sigma^q E \xrightarrow{\mu} \Sigma^q E.$$

Hence $\alpha^*(\Phi(v) \cdot \imath_{n-q}) = r^*(T\tilde{\nu}^*(U \cdot v) \cdot h^*(\imath_{n-q})$. Thus in order to complete the proof of Proposition 2.3 it remains to show that

$$T\tilde{\nu}^*(U \cdot v) = u_M \cdot \tilde{v}^*(v) \in E^q(T\nu_M).$$

Since u_M is the composition $T\nu_M \overset{T\tilde{\nu}}{\rightarrowtail} TB \overset{U}{\rightarrowtail} E$, $u_M \cdot \tilde{\nu}^*(v)$ is given by

$$T\nu_M \overset{\Delta^*}{\rightarrowtail} T\nu_M \wedge M^n_+ \overset{T\tilde{\nu}\wedge\tilde{\nu}}{\rightarrowtail} TB \wedge B_+ \overset{U\wedge v}{\rightarrowtail} E \wedge \Sigma^q E \overset{\mu}{\rightarrowtail} \Sigma^q E$$

and $\tilde{\nu}^*(U \cdot v)$ is given by

$$T\nu_M \overset{T\tilde{\nu}}{\rightarrowtail} TB \overset{\Delta^*}{\rightarrowtail} TB \wedge B_+ \overset{U\wedge v}{\rightarrowtail} E \wedge \Sigma^q E \overset{\mu}{\rightarrowtail} \wedge \Sigma^q E.$$

So we need only show that the following commutes:

$$\begin{array}{ccccc} T\nu_M & \overset{T\tilde{\nu}}{\rightarrowtail} & TB & \overset{\Delta^*}{\rightarrowtail} & TB \wedge B_+ \\ \updownarrow \| & & & & \updownarrow \| \\ T\nu_m & \overset{\Delta^*}{\rightarrowtail} & T\nu_M \wedge M^n_+ & \overset{t\tilde{\nu}\wedge\tilde{\nu}}{\rightarrowtail} & TB \wedge B_+ \end{array}$$

But the fist line is induced by

$$M^n \rightarrowtail B \rightarrowtail B \times B,$$

and the second line by

$$M^n \overset{\Delta}{\rightarrowtail} M^n \times M^n \overset{\tilde{\nu}\times\tilde{\nu}}{\rightarrowtail} B \times B.$$

Since these two compositions are equal, the proposition follows. □

2.5. Corollary. $I_n \cap E^q(B) = \operatorname{Ker} \psi_q$. □

Making use of the Thom isomorphism, we now study instead $J_n = \Phi(I_n) \subset E^*(TB)$. Define a map, which by abuse of notation we still call ψ_q, from $E_q(TB)$ to $\pi_n(TB \wedge E_{n-q^+})^*$ by the rule: $\psi_q(v)(\alpha)$ is the composition

$$S^n \overset{\alpha}{\rightarrowtail} TB \wedge E_{n-q^+} \overset{v\wedge\iota_{n-q}}{\rightarrowtail} \Sigma^q E \wedge \Sigma^{n-q} E \overset{\mu}{\rightarrowtail} \Sigma^n E.$$

We have the following immediate consequence:

2.6. Corollary. $J_n \cap E^q(TB) = \operatorname{Ker} \psi_q$. □

Using the fact that $\pi_n(TB \wedge E_{n-q^+}) = TB_n(E_{n-q^+})$, we have a map

$$\psi_q^* \colon TB_n(E_{n-q^+} \to E^q(TB)^*.$$

We last have a map $\eta_q \colon E_q(TB) \to E^q(TB)^*$ given by

$$\eta_q(x)(v) = \langle v, x \rangle$$

for $x \in E_q(TB)$, $v \in E^q(TB)$. Then the main result of this generalized Brown-Peterson process is the following, which is proven simply by checking the two composites on a homotopy level:

2.7. THEOREM. *The following diagram commutes*:

$$\begin{array}{ccc} TB_n(E_{n-q^+}) & \xrightarrow{\psi_q} & E^q(TB)^* \\ \downarrow (\iota_{n-q})_* & & \uparrow \eta_q \\ TB_n(\Sigma^{n-q}E) = TB_q(E) & \xrightarrow{\chi} & E_q(TB) \end{array}$$

where χ *is induced by the switch-map* $TB \wedge E \rightarrowtail E \wedge TB$. □

2.8. COROLLARY. *If* η_q *is an isomorphism and* $E = TB$, *then*

$$J_n \cap E^q E = \mathrm{Ker}((\iota_{n-q})^* \circ \chi^*),$$

where $\chi\colon E_*E \to E_*E$ *is the canonical anti-automorphism associated to the Hopf algebra* E_*E. □

In particular, for $TB = MO$, i.e., for unoriented cobordism, the above calculation reduces to that of Brown and Peterson. In fact, since MO splits as a wedge of Eilenberg-Mac Lane spectra $K\mathbb{Z}/2$, if we restrict our attention to the $K\mathbb{Z}/2$ summand containing the Thom class, one may easily verify that $(T\nu_M)^*\colon H^*(K\mathbb{Z}/2) \to H^*(T\nu_M)$ (with $\mathbb{Z}/2$ coefficients) is given by $(T\nu_M)^*(a) = a \cdot u_M$, where $a \in A$, the Steenrod algebra, and $u_M \in H^0(T\nu_M)$ is the Thom class. Then if

$$J_n(0) = \bigcap_{M^n} \mathrm{Ker}(T\nu_M)^*,$$

using the fact that $a \in \mathrm{Ker}(\iota_p)^*\colon A \to H^*(K(\mathbb{Z}/2, p))$ if and only if the element a has excess $e(a) > p$, we obtain the following result of Brown and Peterson's **[BP]**:

2.9. COROLLARY. $J_n(0) = \{a \in A \mid \dim(\chi(a)) + e(\chi(a)) > n\}$. □

3. MU, BP, **and the Morava** K**-Theories.** In the remainder of this paper we restrict ourselves to the study of stably almost-complex manifolds, where $B_{2k} = B_{2k+1} = BU(k)$, and the resulting Thom spectrum is MU. Now MU localized at a prime p splits into a wedge of suspensions of BP summands. Unfortunately, neither MU nor BP satisfies condition 2.1(b) in general. Thus we are led to use $E = K(m)$, the Morava K-theories, as our generalized (co)homology theories. In this section we collect some facts about $K(m)$ and related spectra.

Fix a prime p. For the BP spectrum associated to p we have

$$\pi_*BP \cong \mathbb{Z}_{(p)}[v_1, v_2, \dots], \qquad \dim(v_i) = 2(p^i - 1),$$

where $\mathbb{Z}_{(p)}$ represents the integers localized at p. The Morava K-theories are BP-module spectra related to BP by maps $K_m\colon BP \to K(m)$. We collect their basic properties in the following (see, for example, **[RW2]**):

3.1. PROPOSITION. (a) *For $p \neq 2$, $K(m)$ is a commutative ring spectrum.*

(b) $\pi_*K(m) \cong (\mathbb{Z}/p)[v_m, v_m^{-1}]$.

(c) $K(m)_*(X\times Y) \cong K(m)_*(X)\otimes_{\pi_*K(m)}K(m)_*(Y)$ *for spaces* X, Y.

(d) *As a map of coefficient rings,*

$$(K_m)_*(v_m) = v_m, \quad \text{and} \quad (K_m)_*(v_q) = 0, \quad q \neq m.$$

(e) $K(m)_*(X) \cong (K(m)^*(X))^*$ *for X a space or a spectrum.*

(f) *Let $\underline{K(m)}_q$ be the qth space in the Ω-spectrum for $K(m)$. Then there are homotopy equivalences for each q, $\underline{K(m)}_{q+2(p^m-1)} \to \underline{K(m)}_q$.* □

Note that 3.1(e) follows from the Universal Coefficient Theorem spectral sequence (see **[A1]**), since $\pi_*K(m)$ is a "graded field" and hence $K(m)_*(X)$ is free over $\pi_*K(m)$.

Next we introduce some intermediate theories lying between BP_* and $K(m)_*$ which will be of use to us. Let E be a ring spectrum, and let $x \in \pi_n E$. Then multiplication on the left by x induces a map $x\colon \Sigma^n E \to E$. Let $I(m) \subset \pi_*BP$ be the ideal defined by

$$I(0) = 0, \quad I(1) = (p), \quad I(m) = (p, v_1, \dots, v_{m-1}) \text{ for } m > 1.$$

3.2. PROPOSITION **[JW1]**. *There exist spectra $P(m)$, $m = 0, 1, 2, \dots$, such that*

(a) $P(0) = BP$;

(b) $\pi_*P(m) \cong BP/I(m) \cong \mathbb{Z}_{(p)}[v_m, v_{m+1}, \dots]$ *for* $m \geq 1$;

(c) $P(m)$ *is a left BP-module spectrum;*

(d) $P(m+1)$ *is related to $P(m)$ by a stable cofibration*

$$\Sigma^{2(p^m-1)}P(m) \overset{v_m}{\rightarrowtail} P(m) \overset{g_m}{\rightarrowtail} P(m+1);$$

(e) $(g_m)_*\colon \pi_*P(m) \to \pi_*P(m+1)$ *is given on generators by*

$$(g_m)_*(v_i) = 0 \quad \text{if } i \le m, \qquad = v_i \quad \text{if } i > m.$$

(f) *for* $p > 2$, $P(m)$ *is a commutative ring spectrum.* □

Thus $P(m+1)$ may be obtained from $P(m)$ by "killing" the element v_m via the cofibration of 3.2(d). Proceeding in this manner, we may start from $P(m)$ and kill the generators $v_{m+1}, v_{m+2}, \dots$ of $\pi_*P(m)$ to obtain in the limit the BP-module spectrum $k(m)$. We have then that $\pi_*k(m) \cong (\mathbb{Z}/p)[v_m]$. If we let $T_m = \{1, v_m, v_m^2, \dots\}$ be the multiplicative set of nonnegative powers of the element $v_m \in \pi_*k(m)$, then we may obtain $K(m)_*$ by localizing the homology theory $k(m)_*$ with respect to T_m via the techniques described in **[JW2]**.

Finally, we note that the maps

$$MU \to BP \to P(1) \to \cdots \to P(m) \to k(m) \to K(m)$$

give MU an orientation with respect to the cohomology theories BP^*, $P(m)^*$, $k(m)^*$, and $K(m)^*$.

4. Calculation of relations for stably almost-complex manifolds. We now return to the generalized Brown-Peterson process and apply it to the case $B_{2k} = B_{2k+1} = BU$ as before. By 3.1(e) and the remark at the end of the last section, the cohomology theory $K(m)^*$ satisfies conditions 2.1(a′) and 2.1(b) for stably almost-complex manifolds. By Corollary 2.6, we need to determine the kernel of the map $\psi_q\colon K(m)^q(MU) \to MU_n(\underline{K(m)}_{n-q^+})^*$. Dually, we need to determine the cokernel (and hence the image) of the map

$$\psi_q^*\colon MU_n\underline{K(m)}_{n-q^+} \to K(m)_qMU.$$

Here we are making use of 3.1(e). By Theorem 2.7, then, since η_q is an isomorphism, we need to calculate the image of

$$MU_n\underline{K(m)}_{n-q^+} \overset{(\iota_{n-q})_*}{\rightarrowtail} MU_n\Sigma^{n-q}K(m) = MU_qK(m) \overset{\chi}{\rightarrowtail} K(m)_qMU.$$

Since MU localized at p is made up of BP-summands, it suffices, modulo χ, to calculate the image of the stabilization map

$$BP_*\underline{K(m)}_s \overset{(\iota_s)_*}{\rightarrowtail} BP_*K(m).$$

We make use of the following, where $E(x_1, \dots, x_t)$ is the exterior algebra on the generators $x_1, \dots, x_t$.

4.1. LEMMA. *$K(m)_*P(m) \cong K(m)_*BP \otimes E(\tau_0, \dots, \tau_{m-1})$ as modules over $\pi_*K(m)$, where $\dim(\tau_j) = 2p^j - 1$.*

Proof. The Atiyah-Hirzebruch spectral sequence for $k(m)_*BP$ collapses, yielding

$$k(m)_*BP \cong H_*BP \otimes \pi_*k(m) \cong (\mathbb{Z}/p)[v_m; c_1, c_2, \dots]$$

as $\mathbf{Z}/p$-algebras, where $\dim(c_j) = 2(p^j - 1)$.

If we apply $k(m)_*(\ )$ to the cofibration of 3.2(d), we obtain an exact sequence for $q < m$:

$$\cdots \to k(m)_sP(q) \overset{v_q}{\rightarrowtail} k(m)_{s+r}P(q) \to k(m)_{s+r}P(q+1) \\ \to k(m)_{s-1}P(q) \to \cdots$$

where $r = 2(p^q - 1)$.

But multiplication by v_q is zero in $k(m)_*(\ )$. Hence $k(m)_*P(q)$ injects in $k(m)_*P(q+1)$. Furthermore, when $s = 1$ we obtain a new element $\tau_q \in k(m)_{2q^q-1}P(q+1)$ which is external, as one easily checks inductively by using our knowledge of $k(m)_*BP$ (recall that $P(0) = BP$). Thus for $q < m$, $k(m)_*P(q+1) \cong k(m)_*P(q) \otimes E(\tau_q) \cong k(m)_*BP \otimes E(\tau_0, \dots, \tau_q)$. Localizing now with respect to $\{1, v_m, v_m^2, \dots\}$ gives the desired result. □

4.2. COROLLARY. *The map $BP_*K(m) \to P(m)_*K(m)$ is injective.*□

With 4.2 in mind, we shall make use of the following commutative diagram:

$$\text{(4.3)} \qquad \begin{array}{ccc} BP_*\underline{K(m)}_q & \overset{(\iota_q)_*}{\rightarrowtail} & BP_*K(m) \\ \downarrow & & \downarrow \\ P(m)_*\underline{K(m)}_q & \overset{(\iota_q)_*}{\rightarrowtail} & P(m)_*K(m), \end{array}$$

and calculate the image of $(\iota_q)_*$ on $P(m)_*$-homology.

4.4. REMARK. Because of problems with the multiplication in the spectra $P(m)$, $k(m)$, and $K(m)$ at the prime 2 **[R]**, we restrict our attention to p odd from now on.

Wilson has calculated $P(m)_*\underline{K(m)}_q$ for each q, by considering $P(m)_*\underline{K(m)}_* = \{P(m)_*\underline{K(m)}_q\}$ as a Hopf ring. The general reference for Hopf rings is **[RW1]**; here we recall only that there are structure maps

$*\colon P(m)_*\underline{K(m)}_k \otimes P(m)_*\underline{K(m)}_k \to P(m)_*\underline{K(m)}_k$ (for each k), and

$\circ\colon P(m)_*\underline{K(m)}_k \otimes P(m)_*\underline{K(m)}_n \to P(m)_*\underline{K(m)}_{k+n}$ (for all n, k)

satisfying certain properties (associativity, distributivity, having a unit, etc.) The map $*$ is induced by the loop-space multiplication on $\underline{K(m)}_k$, and $\circ$ is induced by the multiplication

$\mu\colon P(m) \wedge P(m) \to P(m)$ and $m_{k,n}\colon \underline{K(m)}_k \wedge \underline{K(m)}_n \to \underline{K(m)}_{k+n}$.

Using these two maps, the Hopf ring $P(m)_*\underline{K(m)}_*$ is generated by elements $e_1 \in P(m)_1\underline{K(m)}_1$, $a_{(i)} \in P(m)_{2p^i}\underline{K(m)}_1$ for $i < m$, and $b_{(i)} \in P(m)_{2p^i}\underline{K(m)}_2$, which we now describe. For $q < 2p^m - 1$, $\widetilde{P}(m)_q\underline{K(m)}_1 \cong \widetilde{H}_q(K(\mathbb{Z}/p, 1); \mathbb{Z}/p)$ since $\underline{K(m)}_1 \simeq K(\mathbb{Z}/p, 1)$ through dimension $2(p^m - 1)$, and $P(m) \simeq K\mathbb{Z}/p$ in stable dimensions less than $2(p^m - 1)$. $H_1(K(\mathbb{Z}/p, 1); \mathbb{Z}/p)$ and $H_{2p^i}(K(\mathbb{Z}/p, 1); \mathbb{Z}/p)$ are isomorphic to $\mathbb{Z}/p$; use this isomorphism on the canonical generators to define e_1 and $a_{(i)}$. $P(m)_*\mathbb{C}P^\infty$ is free over $\pi_*P(m)$ on generators $\beta_i \in P(m)_{2i}CP^\infty$. Using these elements and the $K(M)$-orientation for CP^∞, represented by a map $CP^\infty \to \underline{K(m)}_2$, one defines $b_{(i)} \in P(m)_{2p^i}\underline{K(m)}_2$.

For $I = (i_0, i_1, \ldots, i_{m-1})$ and $J = (j_0, j_2, \ldots)$ nonnegative finite sequences with $i_k = 0$ or 1 and $j_k < p^m$, define

$$a^I b^J = a_{(0)}^{\circ i_0} \circ \cdots \circ a_{(m-1)}^{\circ i_{m-1}} \circ b_{(0)}^{\circ j_0} \circ b_{(0)}^{\circ j_1} \circ \cdots$$

Then Wilson's theorem states that, as a $\pi_*P(m)$-algebra, $P(m)_*\underline{K(m)}_*$ is described in terms of the above elements as follows. For $j_0 < p^m - 1$, each $a^I b^J \circ e_1$ is an exterior generator; and depending on I and J each $a^I b^J$ is either a polynomial or a truncated polynomial generator, all using the $*$ product. Here, $P(m)_*\underline{K(m)}_*$ is considered as graded over $\mathbb{Z}/2(p^m - 1)$ instead of over $\mathbb{Z}$, by use of 3.1(f). The homotopy equivalence of 3.1(f) is given by the "periodicity operator" $[v_m] \in \pi_0\underline{K(m)}_{-2(p^m-1)}$ as:

$$\underline{K(m)}_{q+r} \approx S^0 \wedge \underline{K(m)}_{q+r} \overset{[v_m]\wedge 1}{\rightarrowtail} \underline{K(m)}_{-r} \wedge \underline{K(m)}_{q+2r} \overset{\mu}{\rightarrowtail} \underline{K(m)}_q,$$

where $r = 2(p^m - 1)$.

4.5. PROPOSITION (*Wilson* [W]). *The following relations hold in* $P(m)_*\underline{K(m)}_*$, *where* $\lambda\colon BP \to P(m)$ *is the induced map from* 3.2:

(a) $e_1\circ$*—is the homology suspension map.*

(b) $e_1 \circ e_1 = b_{(0)}$.

(c) $a_{(i)} \circ a_{(j)} = -a_{(j)} \circ a_{(i)}$.
(d) $\lambda_*(v_m)e_1 = [v_m] \circ b_{(0)}^{\circ p^m - 1} \circ e_1$.
(e) $[v_m] \circ b_{(k)}^{\circ p^m} = \sum_{i=0}^k \lambda_*(v_{m+i}^{p^{k-i}})b_{(k-i)} \bmod *,\ k > 0$. □

Our goal is to determine the image of the stabilization map $P(m)_*\underline{K(m)}_q \to P(m)_*\underline{K(m)}$. First we calculate the stable object $P(m)_*K(m)$. Let $R_m = \pi_*P(m)[v_m^{-1}] \equiv (\mathbb{Z}/p)[v_m, v_m^{-1}, v_{m+1}, \ldots]$, and let $E(x)$ and $P(x)$ denote, respectively, the exterior and polynomial algebras on the generator x.

4.6. THEOREM. *As $\pi_*P(m)$-modules,*

$$P(m)_*K(m) \cong E_{R_m}(\tilde{\tau}_0, \tilde{\tau}_1, \ldots \tilde{\tau}_{m-1}) \otimes P_{R_m}(\xi_1, \xi_2, \ldots)$$

modulo the relations

$$\xi_k^{p^k} = v_m^{-1} \sum_{i=0}^{k} v_{m+i}^{p^{k-i}} \xi_{k+i},$$

where $\dim(\tilde{\tau}_i) = 2p^i - 1$ *and* $\dim(\xi_i) = 2(p^i - 1)$.

Proof. The stabilization map $P(m)_*\underline{K(m)}_q \to P(m)_*K(m)$ is given, from 4.5(a), by $\circ$-multiplication with e_1 infinitely often. Stabilization kills $*$-products and e_1 stabilizes to $1 \in P(m)_0K(m)$, so we need only concern ourselves with elements of the form $[v_n]^r \circ a^I b^J$, where $r \in \mathbb{Z}$ and I and J are as before, with the additional property that $j_0 = 0$ (by 4.5(b)). By 4.5(d), all of these elements survive to $P(m)_*K(m)$.

In particular, let $\tilde{\tau}_i$ and ξ_j be the stable images of $a_{(i)}$ and $b_{(j)}$ respectively, for $0 \le i \le m-1$ and $j > 0$. One may easily verify that for $\alpha \in P(m)_*\underline{K(m)}_r$, $\beta \in P(m)_*\underline{K(m)}_s$,

$$\iota_{r+s}(\alpha \circ \beta) = \iota_r(\alpha)\iota_s(\beta)$$

in $P(m)_*K(m)$. Using this result we have that $\tilde{\tau}_i\tilde{\tau}_j = -\tilde{\tau}_j\tilde{\tau}_i$ (from (4.5(c)), and $a^I b^J$ stabilizes to $\tilde{\tau}^I \xi^J$, defined analogously.

By 4.5(d) we have that $[v_m] \in P(m)_0\underline{K(m)}_{2-2p^m}$ stabilizes to the same element as the image

$$v_m \in \pi_{2(p^m-1)}P(m) \to P(m)_{2(p^m-1)}K(m).$$

(That is to say, multiplication by v_m is the same on the left and on the right in $P(m)_*K(m)$.) Hence the coefficient ring for $P(m)_*K(m)$ becomes $\pi_*P(m)[v_m^{-1}] = R_m$.

Finally, since stabilization is a $\pi_*P(m)$-module map, 4.5(e) stabilizes to the relation

$$v_m\xi_k^{p^k} = \sum_{i=0}^{k} v_{m+i}^{p^{k-i}}\xi_{k-i}, \quad \text{or} \quad \xi_k^{p^k} = v_m^{-1}\sum_{i=0}^{k} v_{m+i}^{p^{k-i}}\xi_{k+i}.$$

This finishes the proof of 4.6. $\square$

From 4.6 we can tell how far each element of $P(m)_*K(m)$ desuspends. Given $\alpha \in P(m)_*K(m)$, let $d(\alpha)$ be the minimum q such that there is an $\alpha_q \in P(m)_*\underline{K(m)}_q$ with $(\iota_q)_*(\alpha_q) = \alpha$. Define $\tilde{\tau}^I\xi^J$ in analogy with a^Ib^J (except that there is no ξ_0).

4.7. Corollary. *$d(v_m^r\tilde{\tau}^I\xi^J) = |I| + 2|J| - 2r(p^m - 1)$, where*

$$|I| = \sum_{s=0}^{m-1} i_s \quad \textit{and} \quad |J| = \sum_{t=1}^{\infty} j_t \ (< \infty).$$

Proof. Since $a^I \in P(M)_*\underline{K(m)}_{|I|}$ and $b^J \in P(m)_*\underline{K(m)}_{|J|}$, we need note only that $d(v_m^r) = -2r(p^m - 1)$. $\square$

We now return our attention to $BP_*\underline{K(m)}_*$ and $BP_*K(m)$, by making use of 4.3. First we prove the following:

4.8. Lemma. (a) $\tilde{\tau}_1 \notin \operatorname{Im}\lambda_* \subset P(m)_*K(m)$.
(b) $b_{(i)} \in \operatorname{Im}\lambda_* \subset P(m)_*\underline{K(m)}_2$.

Proof. (a) Since $\lambda_*(xy) = \lambda_*(x)\lambda_*(y)$ (see, for example, **[Wü]**), we have that if $\lambda_*(\alpha) = \tilde{\tau}_i$, then $\alpha^2 = 0$. But by the proof of 4.1, $BP_*k(m)$ has no exterior elements, and the same holds, after localization, for $BP_*K(m)$. Hence no such α exists.

(b) We have that BP_*CP^∞ is free over π_*BP on generators $\tilde{\beta}_i \in BP_{2i}\mathbb{C}P^\infty$. By the commutativity of

$$\begin{array}{ccc} BP_*\mathbb{C}P^\infty & \overset{\theta_*}{\rightarrowtail} & BP_*\underline{K(m)}_2 \\ \downarrow \lambda_* & & \downarrow \lambda_* \\ P(m)_*\mathbb{C}P^\infty & \overset{\theta_*}{\rightarrowtail} & P(m)_*\underline{K(m)}_2 \end{array}$$

where $\theta\colon \mathbb{C}P^\infty \to \underline{K(m)}_2$ is the orientation, since $\lambda_*(\tilde{\beta}_i) = \beta_i$ and $(\theta\circ\lambda)_*(\tilde{\beta}_i) = b_{(i)}$, there is an element $\tilde{b}_i \in BP_{2i}\underline{K(m)}_2$ with $\lambda_*(\tilde{b}_{(i)}) = b_{(i)}$. $\square$

By the commutativity of λ_* with the stabilization map, we conclude that there is an element $\tilde{\xi}_i \in BP_{2(p^i-1)}K(m)$ with $\lambda_*(\tilde{\xi}_i) = \xi_i$.

By defining the function d in analogy with 4.7, the following is a consequence of 4.7 and 4.8:

4.9. COROLLARY. $d(v_m^r \tilde{\xi}^J) = 2|J| - 2r(p^m - 1)$. □

Using the fact that stabilization about is a $\pi_* BP$-module map, by 4.8 this completes the description of $\operatorname{Im} \iota_*\colon BP_* \underline{K(m)}_* \to BP_* K(m)$.

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Received September 18, 1987 and in revised form May 30, 1989.

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