

INTEGRAL DOMAINS THAT LOSE IDEALS IN OVERRINGS

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We study domains with the property that, in any proper overring, some proper ideal extends to the unit ideal, and domains with the property that, in any proper overring, some proper ideal of a certain type (v -ideal, invertible ideal, or principal ideal) extends to the unit ideal. We characterize the Noetherian domains with these properties, and we show that even the strongest of these properties does not imply the QR-property.

0. Introduction. An integral domain D with field of fractions K is said to have the “QR-property” if every subring of K that properly contains D (i.e., every “proper overring” of D) is a ring of fractions of D with respect to some multiplicatively closed set. Robert Gilmer has asked whether a sufficient condition for D to have the QR-property is that, for every proper overring E of D , some nonunit of D is a unit in E . In the present paper we show that the answer is no, essentially because even if D satisfies the latter condition, a ring of fractions over it need not satisfy it. Another way of phrasing the latter condition is: For every proper overring E of D , there is a proper principal ideal of D that “blows up”, i.e., extends to the unit ideal, in E . In these terms, it is natural to extend the idea to classes of ideals other than the class of principal ideals; for instance, it is more likely that a maximal ideal will blow up than a principal ideal. Thus, we were led to the following definition:

DEFINITION. We call an integral domain D “plosive” if, in every proper overring E of D , some proper ideal of D extends to the unit ideal in E . Letting “blue” denote any of the adjectives “ v ”, “invertible”, or “principal”, we call D “blue-plosive” if, in every proper overring E of D , some proper blue ideal of D extends to the unit ideal of E .

In these terms, then, Gilmer’s question was whether a principal-plosive domain has the QR-property; and examples below show that this is not true and that a principal-plosive domain may have a ring of fractions that is not principal-plosive.

Gilmer's question was asked independently on page 9 of [AA]. In response, Dobbs [Do] has shown that D has the QR-property if and only if, for any overrings R_1 and R_2 of D with R_1 properly contained in R_2 , there are elements of D that are units in R_2 but not in R_1 .

Let us begin with some simple observations. First, since every maximal ideal in a domain is the contraction of a maximal ideal in the integral closure, a plosive domain is integrally closed. Next, clearly a principal-plosive domain is invertible-plosive, an invertible-plosive domain is v-plosive, and a v-plosive domain is plosive. Finally, if an ideal extends to the unit ideal in some overring, then so do all the ideals containing it and some finitely generated ideal contained in it. Thus, we have not defined "finitely generated-plosive" or "maximal-plosive", for these concepts are equivalent to "plosive"; nor have we defined "t-plosive" or "v-finite-plosive", for they are equivalent to "v-plosive".

In §1 below we consider plosive domains. We note that a plosive domain with only finitely many maximal ideals is Bezout. We show that, for a domain D that is the intersection of a locally finite family of quasilocal one-dimensional overrings (e.g., a Krull domain), D is plosive if and only if it is a generalized Krull domain in which each prime of height one is an intersection of maximal ideals. We note as corollaries that any normal Noetherian Hilbert domain and any polynomial ring in two indeterminates over a normal Noetherian domain is plosive. But an example shows that a normal Hilbert domain that is not Noetherian need not be plosive.

In §2 we consider v-plosive, invertible-plosive, and principal-plosive domains. We show that a Prüfer v-multiplication domain of finite character is v-plosive if and only if maximal t-ideals are maximal ideals of the domain; and that a Prüfer domain is principal-plosive if and only if every proper finitely generated ideal is contained in the radical of a proper principal ideal. (By [P, Theorem 5], a domain has the QR-property if and only if it is a Prüfer domain in which the radical of every finitely generated ideal is the radical of a principal ideal.) We show that, for a domain that is the intersection of a locally finite family of quasilocal one-dimensional overrings, the properties "v-plosive" and "invertible-plosive" are equivalent to each other and to the condition that the domain is one-dimensional Prüfer with Noetherian spectrum.

Section 3 provides two examples of principal-plosive domains that do not have the QR-property. One is the intersection of a family of

one-dimensional quasilocal overrings, and the other is the intersection of a locally finite family of rank two valuation domains.

All rings are commutative with unity, and subrings share the unity of the larger ring. The symbols “ $<$ ” and “ $>$ ” between sets denote proper containment. We call a family of domains with the same field of fractions a “locally finite” family if any nonzero element of the field of fractions is a nonunit in at most finitely many members of the family. Such a family has the same intersection as an “irredundant” subfamily, i.e., a subfamily F such that, for every member of F , there is an element of the field of fractions that is not in that member but is in every other member of F . The property of a locally finite family that we use most frequently is that a ring of fractions of the intersection with respect to a multiplicatively closed subset is the intersection of the rings of fractions of the members of the family with respect to the same subset, and the family of localizations is still locally finite [Gi2, (43.5), page 527]. (The hypothesis that the members of the family are overrings of the intersection is not needed, but we will usually apply it in that case.) A consequence is that every prime in the intersection is a union of primes contracted from members of the family. We call a domain a “generalized Krull domain” if its localizations at all primes of height one are (rank one) valuation domains, these localizations form a locally finite family, and the domain is the intersection of these localizations [Gi2, page 524]. The “ v -closure” I_v of an ideal I in a domain is the intersection of the principal fractional ideals containing I ; if $I_v = I$, then I is a “ v -ideal”. An ideal of the form I_v for some finitely generated ideal I is a “ v -finite” ideal. The “ t -closure” I_t of an ideal I is the union of the v -closures of the ideals generated by finite subsets of I ; if $I_t = I$, then I is a “ t -ideal” [J, page 19]. Maximal t -ideals exist and are prime ideals, and a domain is the intersection of the localizations at its maximal t -ideals. A domain D is a “Prüfer v -multiplication domain”, abbreviated PVMD, if each finitely generated ideal is “ t -invertible”, i.e., if, for any finitely generated ideal I , $(II^{-1})_t = D$ [Gi2, page 427]. The localization of a PVMD at a maximal t -ideal is a valuation domain [Gr1, Theorem 5, page 715]. A PVMD for which the family of localizations at maximal t -ideals is locally finite is “of finite character”. Thus, a generalized Krull domain is a PVMD of finite character. A “GCD-domain” is a domain in which every pair of elements has a greatest common divisor. For elements a and b of a GCD-domain D , with greatest common divisor d , $(a, b)_v = (d)$ [Au, page 330];

thus, a GCD-domain is a PVMD. In a GCD-domain, the concepts of v-plosive, invertible-plosive, and principal-plosive domain agree.

1. Plosive domains. In trying to determine circumstances under which plosive domains have the QR-property, we asked whether a quasilocal plosive domain has to be a valuation domain and whether the plosive property localizes. The answer to the second question is negative, as we shall see in a moment; but the answer to the first is positive:

PROPOSITION 1.1. *A plosive domain with only finitely many maximal ideals is a Bezout domain.*

Proof. Let D be a plosive domain with only finitely many maximal ideals M , and for each M pick a valuation overring of D dominating D_M . Then no maximal ideal of D extends to the unit ideal in the intersection E of these valuation domains, so $D = E$. By [K, Theorem 107, page 78], D is a Bezout domain. \square

We next determine the Noetherian plosive domains; in fact, our result is somewhat more general. In the proof we make repeated use of the following: Let M be a maximal ideal in the domain D and u be a nonzero element of the field of fractions of D . By [K, Theorem 67, page 40], if neither u nor u^{-1} is in D_M , then the kernel of the natural surjection $D_M[X] \rightarrow D_M[u]$ is contained in $MD_M[X]$; so M does not extend to the unit ideal in $D[u]$.

THEOREM 1.2. *Suppose the domain D is the intersection of a locally finite family of quasilocal one-dimensional overrings. Then D is plosive if and only if it is a generalized Krull domain in which each height one prime is an intersection of maximal ideals. In this case, if the family of overrings is irredundant, then each is a localization of D and a valuation domain.*

Proof. We may assume that the family of overrings $\{V_\alpha\}$ is irredundant. Suppose first that D is plosive. We show first that the centers P_α of the V_α on D are not related by inclusion. Assume that, for different indices α and β , $P_\beta \subseteq P_\alpha$, and pick u in $(\bigcap_{\gamma \neq \alpha} V_\gamma) - V_\alpha$. Suppose that, for all maximal ideals M containing P_α , $u^{-1} \notin D_M$. Then we claim no maximal ideal of D extends to the unit ideal in $D[u]$, which is a contradiction. For, if the maximal ideal M contains

P_α , then neither u nor u^{-1} is in D_M ; while if M does not contain P_α , then D_M is an intersection of some of the V_γ not including V_α , so u is in D_M . In either case M does not extend to the unit ideal in $D[u]$. Thus, $u^{-1} \in D_M$ for some maximal ideal M containing P_α , and hence $u^{-1} \in V_\alpha$; and since $u \notin V_\alpha$, u^{-1} is in the maximal ideal of V_α . Take a nonzero element a of P_β , and use the fact that V_α is one-dimensional quasilocal to find a power of u^{-1} sufficiently large that $u^{-n}V_\alpha < aV_\alpha$. Then $t = au^n \notin V_\alpha$, but t is in the maximal ideal of V_β . The argument above shows that $t^{-1} \in D_M$ for some maximal ideal M containing P_α . But then t^{-1} is also in the localization of D at P_β and hence in V_β , contradicting the fact that t is in the maximal ideal of V_β .

It follows from the above and the fact that passage to a ring of fractions distributes over a locally finite intersection that the domains V_α are the localizations of D at its height one primes; and so every element of D is in only finitely many height one primes. It remains to show that the height one primes of D are intersections of maximal ideals and that the localizations at these primes are valuation domains. For the former, assume there is a height one prime P of D that is properly contained in the intersection I of the maximal ideals that contain it. Take a nonzero element b of P , and choose an element a in I and in all the (finitely many, by the local finiteness) height one primes containing b other than P , but not in P . Replace a by a power, if necessary, so that, for each minimal prime Q of the ideal (a, b) , $b \notin aD_Q$. (It suffices to find a power of a that does this in each of the D_Q for Q one of the finitely many height one primes containing a .) We claim that no maximal ideal M of D extends to the unit ideal of $D[a/b]$, a contradiction since a is not in P and hence is not a multiple of b . If $P \subseteq M$, then $a/b \notin D_P$, so $a/b \notin D_M$; and since M contains (a, b) and hence one of its minimal primes Q , and $b/a \notin D_Q$, we also have $b/a \notin D_M$. If $P \not\subseteq M$, then $a/b \in \bigcap \{D_Q : \text{ht}(Q) = 1, Q \subseteq M\} = D_M$. In either case M does not extend to the unit ideal in $D[a/b]$.

Finally, we must show that a localization of D at a height one prime P is a valuation domain. Suppose not; then there is a valuation domain W properly dominating D_P . Let E denote the intersection of W and the domains V_α other than D_P . Then $E > D$, for otherwise the locally finite family $\{W\} \cup \{V_\alpha : V_\alpha \neq D_P\}$ has an irredundant subfamily intersecting in D ; one must be W , and we saw in the first paragraph of the proof that this implies W is the localization of D at

its center, i.e., $W = D_P$. So take an element u of $E - D$; we contend no maximal ideal M of D extends to the unit ideal in $D[u]$, a contradiction: If $P \not\subseteq M$, then $u \in \bigcap \{D_Q : \text{ht}(Q) = 1, Q \subseteq M\} = D_M$. If $P \subseteq M$, then neither u nor u^{-1} is in D_M . For, if $u \in D_M$, then $u \in D_P$, contrary to our choice of u ; while if $u^{-1} \in D_M$, then $u^{-1} \in D_P$; it is a nonunit there and also in W , but $u \in W$.

For the converse, suppose D is a generalized Krull domain in which each height one prime is an intersection of maximal ideals. To show D is plosive, it suffices to take an element u of the field of fractions of D not in D and show that some maximal ideal of D extends to the unit ideal in $D[u]$. The hypotheses guarantee that, for some height one prime P of D , $u \notin D_P$, and D_P is a rank one valuation domain. Now if $PD[u] \cap D = P$, then $PD[u]$ is contracted from $D[u]_{D-P} \supseteq D_P[u]$; but the last ring is a field. So at least one of the maximal ideals M intersecting in P is such that $MD[u] \cap D > M$, i.e., M extends to the unit ideal in $D[u]$. \square

COROLLARY 1.3. *A Noetherian domain is plosive if and only if it is an integrally closed domain in which each height one prime ideal is an intersection of maximal ideals.*

Thus, the property of being plosive does not pass to rings of fractions: A polynomial ring in two indeterminates over a field has the property, but its localization at a maximal ideal does not. In fact:

COROLLARY 1.4. *A normal Noetherian Hilbert domain is plosive.*

But a normal Hilbert domain that is not Noetherian need not be plosive: Let k be a field and x and t be indeterminates over it. The domain $D = k + xk(t)[x]$ is integrally closed, and the contraction map $\text{Spec}(k(t)[x]) \rightarrow \text{Spec}(D)$ is an isomorphism. (Cf. [BR, MZ].) Thus, D is Hilbert, but it is not plosive.

On the other hand, a Noetherian plosive domain need not be Hilbert. In fact, if D is a normal Noetherian domain and x and y are indeterminates, then $D[x, y]$ is plosive. (For, let P be a height one prime in $D[x, y]$. If $P \cap D[x] \neq (0)$, then P is generated by this intersection, and forming the factor ring with respect to P yields a polynomial ring in the image of y over a domain, which has zero Jacobson radical. If $P \cap D[x] = (0)$, then the factor ring with respect to P is a domain extension of $D[x]$ finitely generated as an algebra, and

since $D[x]$ has zero Jacobson radical, so does the factor ring [AM, Exercise 22, page 70; Gi1, Corollary 1, page 282]. Thus in either case P is an intersection of maximal ideals.) But when D is not Hilbert, then neither is $D[x, y]$. In particular, if V is a discrete rank one valuation domain, then V is plosive, $V[x]$ is not plosive, and $V[x, y]$ is plosive. But under suitable hypotheses this is the only case of bad behavior of the plosive property in finitely generated normal domain extensions. The “dimension formula” of the next proposition appears in [M, Theorem 15.5, pages 118–119] and [ZS, Appendix 1, page 326].

PROPOSITION 1.5. *Let D be a Noetherian plosive domain that satisfies the dimension formula, and let E be a normal domain containing D and finitely generated as an algebra over D . Suppose that either E is algebraic over D or D is not one-dimensional semilocal. Then E is also plosive.*

Proof. Let Q be a height one prime in E . If the height of the prime $P = Q \cap D$ is zero, then E is not algebraic over D , so D has infinitely many height one primes (or D is a field). Since each height one prime is an intersection of maximal ideals, D has zero Jacobson radical. Now, E/Q is a finitely generated extension of D , so E/Q also has zero Jacobson radical [AM, Exercise 22, page 70; Gi1, Corollary 1, page 282]. If P has height one, then P is an intersection of maximal ideals, so D/P has zero Jacobson radical and hence so does E/Q . If P has height greater than one, then the dimension formula yields that the transcendence degree of E/Q over D/P is positive, so E/Q is finitely generated over a polynomial ring over D/P , and hence again E/Q has zero Jacobson radical. Thus, in each case, Q is an intersection of maximal ideals. \square

We do not know whether the dimension formula hypothesis in this proposition is necessary. Also, we do not know whether it is possible, in a normal Noetherian domain, for only a finite number of height one primes to fail to be intersections of maximal ideals, unless that number is zero.

2. V-, invertible-, and principal-plosive domains. In order to have plenty of v-ideals, the results of this section often include the hypothesis of PVMD, so that the v-finite ideals form a group under v-multiplication: $I \times_v J = (IJ)_v$. Given a PVMD D and a valuation overring W , it is tempting to believe that the group of v-finite ideals is embedded in the value group of W , by taking the minimum

value of an element of the ideal as the value of the ideal. This is true, of course, of the subgroup of principal fractional ideals. The trouble is that this mapping need not be a group homomorphism. For instance, let k be a field and x and y be indeterminates, and consider the domain $D = k[x^2, xy, y^2]$ (which can also be realized as $k[u, v, w]/(uw - v^2)$). Let G denote the product of two copies of the integers, lexicographically ordered left to right, and define the valuation overring W of D by assigning to x and y (in the extension $k[x, y]$ of D) the values $(1, 0)$ and $(0, 1)$ respectively. Then the inverse of the ideal $I = (x^2, xy)$ of D is $(1, y/x)$ (since the ideal (x^2, xy, y^2) has v -closure D), so the value $(-1, 1)$ of the inverse of I is not the negative $(0, -1)$ of the value of I . When a valuation overring is a localization of a domain, however, the value mapping on v -finite ideals of the domain is an ordered semigroup homomorphism [HO3, page 859].

In preparation for the first result of the section, we note the following. Let D be a PVMD of finite character. Then the representation of D as the intersection of its localizations at maximal t -ideals is already irredundant. For, suppose P is a maximal t -ideal, and let E be the intersection of the localizations D_Q of D at all the maximal t -ideals Q different from P . If $E = D$, then PD_P is the union of primes contracted from rings of fractions of the domains D_Q . But D_P is a valuation domain, so these primes form a chain; and a nonzero element of PD_P is a unit in all but finitely many of the D_Q , so the members of the chain cannot all be proper. Since P is not contained in any Q , this is a contradiction. This argument shows more in the case that D is v -plosive: Since P is the only maximal t -ideal that can extend to the unit ideal in the intersection E , there is a finitely generated ideal of D that is contained in P but not in any other maximal t -ideal.

PROPOSITION 2.1. *A Prüfer v -multiplication domain in which each maximal t -ideal is a maximal ideal is v -plosive. If a PVMD is of finite character and v -plosive, then every maximal t -ideal is maximal.*

Proof. Let D be a PVMD in which all maximal t -ideals are maximal ideals, and let E be an overring of D . If no maximal t -ideal P of D extends to the unit ideal in E , then for each P there is a maximal ideal $N(P)$ of E lying over P ; and since $E_{N(P)}$ dominates the valuation domain D_P , they are equal. Thus, $E \subseteq \bigcap_P E_{N(P)} = \bigcap_P D_P = D$. So D is v -plosive.

Conversely suppose D is a v-plosive PVMD of finite character, and let P be a maximal t-ideal. If P is not maximal, it is properly contained in a maximal ideal M , which is a union of primes contained in maximal t-ideals. One of these, say Q , is not contained in P . Take a valuation overring W of D with primes contracting to Q and M . Since no maximal t-ideal of D extends to the unit ideal in $E = W \cap \bigcap_N D_N$, where N varies over the maximal t-ideals not contained in M , we have $E = D$. Localizing the (locally finite) expression for E at P , we see that PD_P is a union of primes, one of which, say P_1 , is contracted from W , while the rest are contracted from localizations of the domains D_N with $N \not\subseteq M$. Let I be a finitely generated ideal contained in P but not in any other maximal t-ideal. Since D_P is a valuation domain, ID_P is principal, and no prime contracted from a localization of a D_N contains its generator. Moreover, since $P_1 \subseteq PD_P$, the prime in W from which P_1 is contracted does not contain the one from which Q was contracted; so the latter contains the former. But then $P_1 \cap D \subseteq Q$, and since Q is contained in a maximal t-ideal other than P , $I \not\subseteq Q$, so $ID_P \not\subseteq P_1$. Since none of the primes contracted from W or the domains D_N contain the generator of ID_P , their union is not PD_P , a contradiction. \square

Example 3.2 below shows that a PVMD of finite character in which each maximal t-ideal is a maximal ideal need not be a Prüfer domain. But Theorem 2.4 shows that a PVMD of finite character in which every maximal t-ideal is a *height one* maximal ideal is Prüfer.

The following corollary is immediate from Proposition 2.1.

COROLLARY 2.2. *A GCD-domain in which each maximal t-ideal is a maximal ideal is principal-plosive. If the GCD-domain is of finite character, then the converse holds.*

Let D be a Prüfer domain. Then for any element u of the field of fractions of D , the denominator ideal of u , $I = D :_D u = \{a \in D : au \in D\}$, is a finitely generated ideal of D (since D is coherent) and hence is invertible. Now

$$D[u] = \bigcap \{D_P : P \in \text{Spec}(D), u \in D_P\},$$

since any overring of D is flat [LM, Proposition 4.14, page 91, and Theorem 6.10, page 132]; so for any maximal ideal M of $D[u]$, $ID[u]_M = (D : u)D_{M \cap D} = D_{M \cap D} : u = D_{M \cap D} = D[u]_M$, and hence I extends to the unit ideal in $D[u]$. Thus, a Prüfer domain is invertible-plosive. (In fact, $D[u]$ is just the ideal transform $\bigcup_n I^{-n}$ of I .)

PROPOSITION 2.3. *A Prüfer domain is principal-plosive if and only if each proper finitely generated ideal is contained in the radical of a proper principal ideal.*

Proof. Let D be a principal-plosive Prüfer domain, and let I be a proper finitely generated ideal. Then the ideal transform E of I is a proper overring of D , so there is a nonunit b of D that is a unit in E . That means $b^{-1}I^n \subseteq D$ for some n , and so I has a power contained in bD .

Conversely let D be a Prüfer domain in which every proper finitely generated ideal is contained in the radical of a proper principal ideal, and let u be in the field of fractions of D but not in D . Then $D[u]$ is the ideal transform of the proper finitely generated ideal $I = D :_D u$, and in this ideal transform the generator of the proper principal ideal that contains a power of I becomes a unit. \square

THEOREM 2.4. *Let D be a domain that is the intersection of a locally finite family of quasilocal one-dimensional overrings. Then the following conditions are equivalent:*

- (1) D is v -plosive.
- (2) D is invertible-plosive.
- (3) D is a one-dimensional Prüfer domain with Noetherian spectrum.

When these conditions hold, D is principal-plosive if and only if each maximal ideal is the radical of a principal ideal.

Proof. We have seen that (3) \Rightarrow (2), and (2) \Rightarrow (1) is clear, so suppose D is v -plosive. By Theorem 1.2, D is a generalized Krull domain, so it is enough to show that each maximal ideal of D has height one. Assume the maximal ideal M has height greater than one. Since M is the union of height one primes, D_M is not a valuation domain. So there is an element u of the field of fractions of D for which neither u nor u^{-1} is in D_M ; and multiplying by an element of $D - M$, we may assume that u is in every D_Q for Q a height one prime not contained in M . No height one prime in D extends to the unit ideal in $D[u]$; and every proper v -ideal in the generalized Krull domain D is contained in some height one prime, so no proper v -ideal extends to the unit ideal, a contradiction. \square

The last assertion follows from (3) and Proposition 2.3. \square

We would like to remove some of the adjectives in the hypothesis that the domain D is the intersection of a locally finite family

of quasilocal one-dimensional overrings (though Example 3.1 shows that some such hypothesis is necessary). Removing the word “one-dimensional” but adding “valuation”, we see that, for a domain D that is the intersection of an irredundant locally finite family of valuation domains V_α , a localization of D at a height one prime is a valuation domain, a localization of one of the V_α (since a quasilocal domain that is an intersection of finitely many valuation domains is equal to one of them). If we also assume that D is invertible-plosive, then each of the centers of the V_α on D contains an invertible ideal contained in no other such center, and we can almost repeat the second part of the proof of Proposition 2.1 to see that each such center is maximal. Removing the words “locally finite” and “one-dimensional” but adding “valuation” and “localization at a maximal ideal” allows us to mimic the first part of the proof of Proposition 2.1.

PROPOSITION 2.5. *Suppose a domain D has a family of maximal ideals P for which (1) each P is the radical of an invertible (respectively principal) ideal, (2) each D_P is a valuation domain, and (3) $D = \bigcap_P D_P$. Then D is invertible-plosive (respectively principal-plosive).*

Proof. Let E be a proper overring of D . If none of the maximal ideals P in the family of the hypothesis extend to the unit ideal of E , then for each P there is a maximal ideal $N(P)$ of E lying over P . Since $E_{N(P)}$ dominates the valuation domain D_P , they are equal; so $E \subseteq \bigcap_P E_{N(P)} = \bigcap_P D_P = D$, a contradiction. Thus one of the maximal ideals P in the family extends to the unit ideal in E , and hence so does some finitely generated ideal I contained in P , and hence so does the power of I contained in the invertible (respectively principal) ideal J with radical P , and hence so does J . \square

In view of Theorem 2.4, we wonder whether a v-plosive domain is always invertible-plosive. We do not know this is true even in every PVMD of finite character, but it is true if the PVMD also has Noetherian spectrum. Recall that a domain D is called “quasicohherent” if the inverse of each finitely generated nonzero fractional ideal of D is again finitely generated.

PROPOSITION 2.6. *Let D be a v-plosive Prüfer v-multiplication domain of finite character. Suppose that either (a) D is quasicohherent and no maximal t -ideal is the union of the prime t -ideals properly contained within it, or (b) D has Noetherian spectrum. Then D is invertible-plosive.*

Proof. Let P be a maximal t-ideal of D . We want to show that there is a finitely generated ideal I with radical P . In case (b) such an I clearly exists. In case (a) we note that P contains a finitely generated ideal J that is not contained in any other maximal t-ideal Q (since it extends to the unit ideal in $\bigcap_Q D_Q$). The primes contained in P are totally ordered by inclusion (since D_P is a valuation domain), so the hypothesis on prime t-ideals assures that we can add a generator to J so that J is not contained in any prime t-ideal properly contained in P . Let I denote the v-closure of J ; since P is a t-ideal, $I \subseteq P$. By quasicoherence I is finitely generated, and I has radical P because minimal primes of a t-ideal are themselves t-ideals [HH, Proposition 1.1(5), page 38; J, Théorème 9, page 30].

Since I extends to the unit ideal in any extension of D in which P does so, it is enough to show that I is invertible, i.e., that its localization at each maximal ideal is principal. At the maximal ideals not containing I this is clear, so suppose I is contained in the maximal ideal M . Then $P \subseteq M$ and so by Proposition 2.1 $P = M$. Since D_P is a valuation domain, ID_P is principal. \square

It would be interesting to know whether the “locally finite” hypothesis is necessary in the above results. In this connection, we pose three questions, which by Theorem 2.4 have positive answers when the spectrum is Noetherian: (1) Must a one-dimensional, invertible-plosive domain be Prüfer? (2) Must a one-dimensional, principal-plosive Prüfer domain have the QR-property? (3) More generally, must a one-dimensional, principal-plosive domain have the QR-property?

3. Examples. In this section we provide two examples of principal-plosive domains that do not have the QR-property. The first has already appeared in the literature in a different context; we recall its essential features.

EXAMPLE 3.1. The domain D in [H2] is the intersection of a family (not locally finite) of quasilocal one-dimensional overrings and is principal-plosive, but it is not a Prüfer domain. This D was constructed as follows: Let k be a field and y, z, x_1, x_2, \dots be indeterminates. The valuation domain $k(z, x_1, x_2, \dots)[y]_{(y)}$ has the form $k(z, x_1, x_2, \dots) + M$ where M is its maximal ideal; set $V \cong k(x_1, x_2, \dots) + M$, $K_n = k(y, z, x_1, x_2, \dots, x_n)$, and $V_n = V \cap K_n$. For each n starting with $D_1 = V_1$, let R_{n+1} be the Kronecker function ring of D_n with respect to the element $r = (1 + yx_{n+1})/y$ and $D_{n+1} = R_{n+1} \cap V_{n+1}$. Then the desired D is the union of all the D_n .

Since the localization V of D is not a valuation domain (and hence is not plosive, by Proposition 1.1), D is not Prüfer and hence not QR; so it remains to show that it is principal-plosive. For this it suffices to take any u in the field of fractions K of D , not in D , and show there is a nonunit in D that is a unit in $D[u]$.

For some n , $u \in K_{n-1} - D_{n-1}$ (note that K_i is the field of fractions of D_i), and hence $u \in K_n - R_n$ since R_n is a Kronecker function ring of D_{n-1} . Since R_n is Bezout, $R_n[u]$ is a localization of R_n and hence also of D_n (since $R_n = D_n[1/s]$ where $s = y/(1+yx_n)$). Let $a \in D_n$ be a nonunit of R_n that is a unit in $R_n[u]$, and pick m so that $s^m V_n \subseteq aV_n$. (Such an m exists because V_n is one-dimensional quasilocal.) Then since $V_n = (D_n)_P$ where P is the center of V_n on D_n , $(aR_n \cap D_n)(D_n)_P \supseteq (aD_n : s^m)V_n = aV_n : s^m V_n = (D_n)_P$, so $aR_n \cap D_n \not\subseteq P$. Since $aV_n \cap D_n$ is P -primary, the ideals $aR_n \cap D_n$, $aV_n \cap D_n$ are relatively prime, so their intersection is their product: $aD_n = aD_n \cap D_n = a(R_n \cap V_n) \cap D_n = (aR_n \cap aV_n) \cap D_n = (aR_n \cap D_n) \cap (aV_n \cap D_n) = (aR_n \cap D_n)(aV_n \cap D_n)$. Thus $aR_n \cap D_n$ is invertible, hence finitely generated, hence its extension to the Kronecker function ring R_{n+1} of D_n is principal, generated by b , say. Since $aR_n \neq R_n$, b is not a unit of R_{n+1} . Moreover, since R_{n+1} is the Kronecker function ring of D_n with respect to $r = (1+yx_{n+1})/y$ and hence also with respect to $r^{-1} \in \max(V)$, we can choose b to be of the form $a_0 + a_1 r^{-1} + \cdots + a_k r^{-k}$ where $(a_0, \dots, a_k)D_n = aR_n \cap D_n$ and $a_0 \notin \max(V)$. Then $b \in D_{n+1}$ but $b \notin \max(V)$, so b is a unit in V_{n+1} and hence in $V_{n+1}[u]$. To see b is a unit in $R_{n+1}[u]$ (a localization of R_{n+1} and hence integrally closed), it is enough to note that, if W is a valuation overring of $R_{n+1}[u]$, then either W is centered on P in D_n , in which case b must be a unit in W since it has multiples (in $aR_n \cap D_n$) that are units in W ; or W is centered on another prime in D_n , in which case $R_n[u] = D_n[1/s][u] \subseteq W$, so a and hence b are units in W .

Now to show b is a unit in $D_{n+1}[u]$, it is enough to show it is a unit in the integral closure of $D_{n+1}[u]$, i.e., that it is a unit in every valuation overring W of $D_{n+1}[u]$: If the center of W on D_{n+1} is the same as that of V_{n+1} , then W contains the localization V_{n+1} of D_{n+1} at that center, and so b is a unit in W . Otherwise r is a unit in W , so $R_{n+1}[u] \subseteq W$, giving the same result. \square

EXAMPLE 3.2. A principal-plosive GCD-domain that is the intersection of a locally finite family of rank two valuation domains and has a

localization that is not plosive: Let k denote a countable algebraically closed field and x and y be indeterminates. Let h_1, h_2, \dots be the nonassociate irreducible polynomials in $k[x, y]$, and let q_1, q_2, \dots be the subsequence of h_1, h_2, \dots for which $q_i(0, 0) = 0$. (Thus, the h_i —or rather $h_i = 0$ —are the irreducible curves in the plane k^2 , and the q_i are the ones through the origin.) For each positive integer i , pick a point $z_i = (x_i, y_i)$ and a linear polynomial l_i for which:

- (1) $z_i \neq (0, 0)$;
- (2) z_i is a regular point on q_i , i.e., $q_i(z_i) = 0$ and the localization of $k[x, y]/(q_i)$ at the maximal ideal $(x - x_i, y - y_i)/(q_i)$ is a discrete rank one valuation domain;
- (3) $h_j(z_i) \neq 0$ for all $j < i$;
- (4) $l_j(z_i) \neq 0$ for all $j < i$;
- (5) $l_i(z_i) = 0$;
- (6) $l_i(0, 0) \neq 0$;
- (7) $l_i(z_j) \neq 0$ for all $j < i$; and
- (8) l_i is not tangent to q_i at z_i .

(Thus, there is one z for each curve q , no more than a finite number on each curve h , and the lines l hit only one z and not the origin, and the local ring of the plane k^2 at z_i is regular, with regular parameters q_i, l_i .) Let W_i denote the localization of $k[x, y]$ at the prime (q_i) , and let V_i be the inverse image in W_i of the valuation domain at z_i on the curve q_i from (2). Thus, V_i is a rank two valuation domain.

Set $D = \bigcap_i V_i$, and let Q_i and P_i denote the contractions to D of the height one and height two primes of V_i , respectively. Note that a rational function f is in P_i if and only if the denominator of f has no component through the origin and the restriction of f to the curve q_i has positive value at z_i , i.e., is in the maximal ideal of the valuation domain of (2). Since $k[x, y] \subseteq V_i \subseteq W_i$, $k[x, y] \subseteq D \subseteq \bigcap_i W_i = k[x, y]_{(x, y)}$. Since no h_i contains infinitely many z_j , the family $\{V_i\}$ is locally finite. Let $f \in P_i$. Since l_i has value 1 at z_i and neither the origin nor any other z_j is on l_i , $f/l_i \in V_j$ for all j , i.e., $f/l_i \in D$. Thus, $P_i = l_i D$. Moreover, for $j \neq i$, let n be the value at z_i of the restriction of the function q_j to the curve q_i ; then q_j/l_i^n is an element of Q_j not in P_i . Thus, V_i is the localization of D at P_i . By Proposition 2.5, D is principal-plosive. But $k[x, y]_{(x, y)}$ is a localization of D that is not plosive. \square

This example shows that the one-dimensional hypothesis in [H1, Proposition 1.2] is necessary, despite a remark to the contrary in [GW].

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