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MULTIPLIERS OF H^p AND BMOA

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We characterize the multipliers from H^1 to X, where X is BMOA, VMOA, $\mathscr{B}$ or $\mathscr{B}_0$ and from H^p to H^q $(p < \min(q, 1))$. Also we give short proofs of some results of Hardy and Littlewood and Fleet.

I. Introduction. For $0 < p \leq \infty$, by H^p we denote the space of functions $f(z)$ analytic in the unit disk U, for which

$$M_p^p(r, f) = \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p \, d\theta$$

or

$$M_\infty(r, f) = \max_{0 \leq \theta \leq 2\pi} |f(re^{i\theta})|$$

remains bounded as $r \to 1$. Duren's book **[4]** and Garnett's book **[11]** will be frequently cited as a reference to H^p theory and related subjects.

Let A and B be two vector spaces of sequences. A sequence $\lambda = \{\lambda_n\}$ is said to be a multiplier from A to B if $\{\lambda_n \alpha_n\} \in B$ whenever $\{\alpha_n\} \in A$. The set of all multipliers from A to B will be denoted by (A, B). We regard spaces of analytic functions in the disk as sequence spaces by identifying a function with its sequence of Taylor coefficients.

Hardy and Littlewood **[14]** have proved the following theorem: If $1 \leq p \leq 2 \leq q$ and $p^{-1} - q^{-1} = 1 - \sigma^{-1}$ and if

$$M_\sigma(r, g') \leq c(1 - r)^{-1}, \qquad 0 < r < 1, \tag{1.1}$$

then $g \in (H^p, H^q)$ (c will be used for a general constant, not necessarily the same at each occurrence). Stein and Zygmund **[21]** (see also Sledd **[20]**) have observed that the condition (1.1) is also necessary in the case $p = 1$, $q \geq 2$. Hence the following theorem holds.

THEOREM HL. *Let $2 \leq q < \infty$. Then $g \in (H^1, H^q)$ if and only if*

$$M_q(r, g') \leq c/(1 - r), \qquad 0 < r < 1. \tag{1.2}$$

This result does not hold for $q = \infty$. In fact, it follows from the Fefferman theorem that $(H^1, H^\infty) = \mathrm{BMOA}$, the space of analytic functions of bounded mean oscillation (for this and the other properties of BMOA see **[11]**, **[2]** and **[7]**).

Theorem HL and a duality argument give the following result:

PROPOSITION 1. *If* $1 < p \le 2$ *then a necessary and sufficient condition that* $g \in (H^p, \mathrm{BMOA})$ *is that* (1.2) *be true, where* $1/p + 1/q = 1$.

We will show that this result holds for $p = 1$.

THEOREM 1. $(H^1, \mathrm{BMOA}) = \mathscr{B}$, *where* $\mathscr{B}$ *is the class of Bloch functions: an analytic function* g *in the unit disc belongs to* $\mathscr{B}$ *iff*

$$\|f\|_{\mathscr{B}} = \sup_{z \in U}(1 - |z|)|g'(z)| < \infty .$$

Furthermore, it will be shown in Section IV, that $(H^1, X) = \mathscr{B}$, where X is BMOA, VMOA, $\mathscr{B}$ or $\mathscr{B}_0$.

It is interesting that results about multipliers of H^p functions are more complete in the case $0 < p < 1$ than in the case $p = 1$.

In **[12, 13]** Hardy and Littlewood stated, without proof, a sufficient condition for g to be a multiplier from H^p to H^q, where $p < 1$ and $q \ge 1$. A proof was found by Duren and Shields **[6]**. (Also see Duren's book **[4]** for further information on multipliers.) Here we extend the Duren-Shields proof to the case $0 < p < q \le 1$, using a result of Flett and the Lemma MP.

THEOREM 2. *Let* $0 < p < 1$ *and* $p < q \le \infty$, *and let* m *be integer* $\ge 1/p$. *Then* $g \in (H^p, H^q)$ *iff*

$$M_q(r, g^{(m)}) \le c(1 - r)^{1/p - m - 1}, \qquad 0 < r < 1. \tag{1.3}$$

It is interesting that we can regard this result as an extension of the well-known theorem of Duren, Romberg and Shields **[4, 5]** on the dual of H^p, $0 < p < 1$. Namely, if we identify a continuous linear functional $\Lambda \in (H^p)^*$ with the sequence $\{\lambda_n\}$, where $\lambda_n = \Lambda(z^n)$, then $(H^p) = (H^p, H^\infty) = (H^p, A(U))$, $0 < p < +\infty$, where $A(U)$ is the disc algebra. For details see **[18]**. Theorem 2 will be proved in Section III.

Our approach is based on results about containments between H^p, BMOA and generalized mean Lipschitz spaces which are mainly due to Hardy and Littlewood. New and short proofs of some of these results will be given in Section II.

II. Background and preliminary result. Let $g(z) = \sum \hat{g}(n)z^n$ be analytic on the unit disc U. We define the multiplier transformation $D^s g$ of g, where s is any real number, by

$$D^s g(z) = \sum_{n=0}^{\infty}(n+1)^s \hat{g}(n)z^n .$$

We shall also use the standard notation $\|g\|_p = \sup_{0\le r\le 1} M_p(r, g)$ $(0 < p \le +\infty)$ and $g_r(z) = g(rz)$ $(0 < r < 1, z \in U)$.

To avoid some technical difficulties, we work rather with $D^m g$ instead of $g^{(m)}$, where m is a non-negative integer. The following lemma shows that integral means of $D^m g$ and $g^{(m)}$ have the same behavior. For example, the condition (1.3) can be written in the form

$$M_q(r, D^m g) \le c(1-r)^{1/p-m-1}, \qquad 0 < r < 1. \tag{2.1}$$

LEMMA 1. *Let g be an analytic function on the unit disc U, let m be a positive integer, let $\hat{g}(j) = 0$ for $0 \le j \le m$, and let $0 < p \le +\infty$. Then*

$$\begin{aligned} c^{-1} r^m M_p(r, g^{(m)}) &\le M_p(r, D^m g) \\ &\le c r^m M_p(r, g^{(m)}), \qquad 0 < r < 1, \end{aligned} \tag{2.2}$$

where c does not depend on g.

Proof of Lemma 1. It is easily seen that $D^m g$ is a linear combination of $z^j g^{(j)}$, $0 \le j \le m$, and $z^m g^{(m)}$ is a linear combination of $D^j g$, $0 \le j \le m$. It follows that

$$\|D^m g\|_p \le c \sum_0^m \|g^{(j)}\|_p$$

and

$$\|g^{(m)}\|_p \le c \sum_0^m \|D^j g\|_p ,$$

where c depends only on m and p. Let $0 \le j \le m-1$. Then

$$D^j g(z) = \int_0^1 D^{j+1} g(rz)\, dr$$

and consequently

$$|D^j g(z)| \le \sup_{0<r<1} |D^{j+1} g(rz)|.$$

Hence, by the Hardy-Littlewood maximal theorem,

$$\|D^j g\|_p \le c\|D^{j+1} g\|_p.$$

This implies $\|g^{(m)}\|_p \le c\|D^m g\|_p$. To prove that $\|D^m g\|_p \le c\|g^{(m)}\|_p$ we use the inequality $\|g^{(j)}\|_p \le c\|g^{(j+1)}\|_p$, $0 \le j \le m-1$, which is a special case of the following.

LEMMA 2. *Let $0 < p \le \infty$, $s = \min(p, 1)$, f be an analytic function on the unit disc U and $0 \le \rho < r < 1$. Then*

$$M_p^s(r, f) - M_p^s(\rho, f) \le c(r-\rho)^s M_p^s(r, f'), \tag{2.3}$$

where c is independent of f, r, ρ.

Proof. Let $p \le 1$. Then $s = p$ and

$$M_p^p(r, f) - M_p^p(\rho, f) \le \frac{1}{2\pi}\int_0^{2\pi} |f(re^{it}) - f(\rho e^{it})|^p\, dt.$$

Since

$$f(re^{it}) - f(\rho e^{it}) = \int_\rho^r f'(ue^{it})e^{it}\, du,$$

we have

$$|f(re^{it}) - f(\rho e^{it})| \le (r-\rho)\sup\{|f'(ue^{it})| : \rho < u < r\},$$

and (2.3) follows from the maximal theorem. If $p \ge 1$ we use Minkowski's inequality to obtain

$$M_p(r, f) - M_p(\rho, f) \le \int_\rho^r M_p(u, f')\, du. \tag{2.4}$$

This gives (2.3) with $c = 1$.

The following lemma is due to Flett [**9**, **10**].

LEMMA F. *Let* $p < 1$ *and* f *be an analytic function on the unit disc* U. *If*

$$\int_0^1 (1-r)^{p-1} M_p^p(r, D^1 f)\,dr < \infty$$

then $f \in H^p$.

Proof. Let $r_n = 1 - 2^{-n}$, $n \geq 0$. Then

$$\|f\|_p^p = |f(0)|^p + \sum_0^\infty [M_p^p(r_{n+1}, f) - M_p^p(r_n, f)].$$

Hence, by Lemma 2,

$$\begin{aligned}\|f\|_p^p &\leq |f(0)|^p + c\sum_0^\infty 2^{-np} M_p^p(r_n, f') \\ &\leq |f(0)|^p + c\int_0^1 (1-r)^{p-1} M_p^p(r, f')\,dr.\end{aligned}$$

Now the desired result follows from (2.2).

Lemma 2 may be used to prove the following result of Hardy and Littlewood [**4**, **12**].

LEMMA HL. *Let* $\alpha > -1$, $0 < p \leq \infty$ *and* $0 < q < \infty$. *If*

$$\text{(i)} \qquad \int_0^1 (1-r)^{\alpha+q} M_p^p(r, D^1 f)\,dr < \infty$$

then

$$\text{(ii)} \qquad \int_0^1 (1-r)^{\alpha} M_p^q(r, f)\,dr < \infty.$$

Proof. We shall consider the case $p < 1$. Let $A_n = M_p^p(r_n, f)$ and $B_n = M_p^q(r_n, f')$, where $r_n = 1 - 2^{-n}$, $n \geq 0$. Let $\beta = q/p$. Then (ii) is equivalent to

$$\infty > \sum_{n=1}^\infty 2^{-n(\alpha+1)} A_n^\beta =: K_1,$$

and (i) is equivalent to

$$\infty > \sum_{n=1}^\infty 2^{-n(\alpha+q+1)} A_n^\beta =: K_2.$$

If $\beta \leq 1$ then

$$\begin{aligned} K_1 &= \sum_1^\infty 2^{-n(\alpha+1)}(A_n - A_{n-1} + A_{n-1})^\beta \\ &\leq \sum_1^\infty 2^{-n(\alpha+1)}(A_n - A_{n-1})^\beta + \sum_1^\infty 2^{-n(\alpha+1)} A_{n-1}^\beta \\ &= \sum_1^\infty 2^{-n(\alpha+1)}(A_n - A_{n-1})^\beta + 2^{-(\alpha+1)}(K_1 + A_0^\beta). \end{aligned}$$

On the other hand, by Lemma 2, $A_n - A_{n-1} \leq c2^{-np} B_n$. This gives

$$(1 - 2^{-(\alpha+1)})K_1 \leq cK_2 + 2^{-(\alpha+1)} A_0^\beta,$$

and the result follows. If $\beta > 1$ we use the Minkowski inequality to obtain

$$\begin{aligned} K_1^{1/\beta} &\leq \left\{ \sum_1^\infty 2^{-n(\alpha+1)}(A_n - A_{n-1})^\beta \right\}^{1/\beta} \\ &\quad + \left\{ \sum_1^\infty 2^{-n(\alpha+1)} A_{n-1}^\beta \right\}^{1/\beta}. \end{aligned}$$

The rest is similar to the case $\beta < 1$.

For the proof of Theorem 1 we need another result of Hardy and Littlewood **[13]**:

LEMMA HL1. *If* $0 < p \leq 2$ *and* $f \in H^p$ *then*

$$\text{(i)} \qquad \int_0^1 (1-r) M_p^2(r, D^1 f)\, dr < \infty.$$

If $2 \leq p < \infty$ *then* (i) *implies* $f \in H^p$.

Proof. By the Hardy-Stein identity **[15]**

$$r \frac{d}{dr} M_p^p(r, f) = p^2 \int_0^r F_p(\rho)\, d\rho,$$

where

$$F_p(\rho) = \frac{1}{2\pi} \int_0^{2\pi} |f(\rho e^{it})|^{p-2} |f'(\rho e^{it})|^2\, dt.$$

Let $0 < p \le 2$, $\|f\|_p = 1$ and, for a fixed ρ, $\varphi(t) = |f(\rho e^{it})|^p$. Since $\frac{1}{2\pi}\int_0^{2\pi} \varphi(t)\,dt \le 1$ and $p/2 \le 1$ we have, by Jensen's inequality,

$$\begin{aligned} F_p(\rho)^{p/2} &= \left\{\int_0^{2\pi} |f'(\rho e^{it})/f(\rho e^{it})|^2 \varphi(t)\,dt\right\}^{p/2} \\ &\ge \int_0^{2\pi} |f'(\rho e^{it})/f(\rho e^{it})|^p \varphi(t)\,dt \\ &= M_p^p(\rho, f'). \end{aligned}$$

Hence

$$\frac{d}{dr} M_p^p(r, f) \ge p^2 r^{-1} \int_0^r M_p^2(\rho, f')\rho\,d\rho.$$

Now integration yields

$$\|f\|_p^p - |f(0)|^p \ge p^2 \int_0^1 \rho \log\frac{1}{\rho} M_p^2(\rho, f')\,d\varphi,$$

and this proves the first implication of Lemma HL1. The case $p \ge 2$ is treated in a similar way.

III. Proof of Theorem 2 and Lemma MP. For the proof of Theorem 2 we need further lemmas. The first of them is a well-known result of Hardy and Littlewood [**4**, **12**].

LEMMA HL2. *If $f \in H^p$ and $p < q < \infty$ then*

$$\int_0^1 (1 - r)^{q/p-2} M_q^q(r, f)\,dr < \infty.$$

LEMMA MP. *Let $f, g \in H^q$, $0 < q \le 1$. Then*

$$M_q(r, f * g) \le (1 - r)^{1-1/q} \|f\|_q \|g\|_q \quad \textit{for } 0 < r < 1.$$

Proof. We may suppose that f, g are polynomials. Then

$$f * g(r^2 e^{it}) = \frac{1}{2\pi} \int_0^{2\pi} f(re^{it}e^{i\theta})\overline{h(re^{i\theta})}\,d\theta \qquad 0 < r < 1,$$

where

$$h(z) = \sum_0^\infty \overline{\hat{g}(n)} z^n.$$

Hence

$$|f * g(r^2 e^{it})| \le \frac{1}{2\pi} \int_0^{2\pi} |f(re^{it}e^{i\theta})|\,d\theta = M_1(r, h^2),$$

where $h^t(z) = f(e^{it}z)h(z)$. Using the familiar estimate

$$M_1(r, h^t) \le (1 - r^2)^{1-1/q} \|h^t\|_q,$$

we obtain

$$|f * g(r^2 e^{it})|^q \le (1 - r^2)^{q-1} \frac{1}{2\pi} \int_0^{2\pi} |f(e^{it}e^{i\theta})|^q |h(e^{i\theta})|^q \, d\theta.$$

Now integration gives

$$M_q^q(r^2, f * g) \le (1 - r^2)^{q-1} \|f\|_q^q \|h\|_q^q.$$

This completes the proof because $\|h\|_q = \|g\|_q$.

Proof of Theorem 2. The proof that $g \in (H^p, H^q)$ implies (1.3) (or equivalently, (2.1)), does not depend on the hypothesis $q > p$. Namely, if $g \in (H^p, H^q)$ then

$$M_q(r, D^m g) = \|g * f_r\|_q \le c\|f_r\|_p, \qquad 0 < r < 1,$$

where $f(z) = \sum_0^\infty (n+1)^m z^n$, and m is an integer $\ge 1/p$. It is easily seen that $f(z) = P_m(z)(1-z)^{-m-1}$, where P_m is a polynomial. Hence

$$\|f_r\|_p^p \le c(1 - r)^{1-(m+1)p}.$$

To prove the converse let $h = f * g$, where g satisfies (2.1) and $f \in H^p$. We have to prove that $h \in H^q$ $(p < \min(q, 1))$. Consider first the case $q \ge 1$. It follows from (2.4) (with q instead of p) that it suffices to prove that

$$\int_0^1 M_q(r, D^1 h) \, dr < \infty.$$

By Lemma HL, this is implied by

$$\int_0^1 (1 - r)^{m-1} M_q(r^2, D^m h) \, dr < \infty.$$

We have

$$\begin{aligned} M_q(r^2, D^m h) &= \|f_r * D^m g_r\|_q \le \|f_r\|_1 \|D^m g_r\|_q \\ &= M_1(r, f) M_q(r, D^m g) \le c(1 - r)^{1/p-m-1} M_1(r, f). \end{aligned}$$

Hence

$$\int_0^1 (1 - r)^{m-1} M_q(r^2, D^m h) \, dr \le c \int_0^1 (1 - r)^{1/p-2} M_1(r, f) \, dr.$$

The second integral is finite because of Lemma HL2 (with $q = 1$).

Next, let $p < q < 1$. Combining Lemmas F and HL we see that $f \in H^q$ if

$$\int_0^1 (1-r)^{mq-1} M_q^q(r^3, D^m h)\, dr < \infty.$$

Using Lemma MP and the condition (2.1) gives

$$\begin{aligned} M_1^q(r^3, D^m h) &= M_q^q(r, f_r * D^m g_r) \le (1-r)^{q-1} \|f_r\|_q^q \|D^m g_r\|_q^q \\ &= (1-r)^{q-1} M_q^q(r, f) M_q^q(r, D^m g) \\ &\le c(1-r)^{q/p-qm-1} M_q^q(r, f). \end{aligned}$$

Hence

$$\int_0^1 (1-r)^{qm-1} M_q^q(r^3, D^m h)\, dr \le c \int_0^1 (1-r)^{q/p-2} M_q^q(r, f)\, dr.$$

Now the desired result follows from Lemma HL2.

IV. Multipliers into BMOA. Although the following result is interesting in itself, the containment $(H^1, \mathscr{B}) \subset \mathscr{B}$ is also very useful in the proof of Theorem 1.

PROPOSITION 2. $(H^1, \mathscr{B}) = \mathscr{B}$.

Proof. Let $f \in H^1$, $g \in \mathscr{B}$ and $h = f * g$. Then we have

$$\begin{aligned} (4.1) \qquad |r^2 e^{it} h'(r^2 e^{it})| &= \left| \frac{1}{2\pi} \int_0^{2\pi} f(re^{i(t+\theta)}) g'(re^{-i\theta}) re^{-i\theta}\, d\theta \right| \\ &\le M_1(r, f) M_\infty(r, g'). \end{aligned}$$

Since $g \in \mathscr{B}$ implies $M_\infty(r, g') \le c(1-r)^{-1}$, $0 < r < 1$, it follows from (4.1) that $h \in \mathscr{B}$. So we have $\mathscr{B} \subset (H^1, \mathscr{B})$.

To prove the converse we suppose that g is an analytic function on the unit disc U and that $f * g \in \mathscr{B}$, whenever $f \in H^1$. Applying the closed graph theorem in the standard way, we conclude

$$\|g * f\|_{\mathscr{B}} \le c\|f\|_1, \qquad f \in H^1.$$

If we substitute $\kappa_r(z) = (1-rz)^{-2}$, $0 < r < 1$, for f in the last inequality, we get

$$\|D^1 g_r\|_{\mathscr{B}} = \|g * \kappa_r\|_{\mathscr{B}} \le c\|\kappa_r\|_1 = c(1-r^2)^{-1}.$$

Since

$$\|g_r'\|_{\mathscr{B}} = \sup_\rho (1-\rho^2) M_\infty(r\rho, g''),$$

it follows from Lemma 1,

$$M_\infty(r\rho, g'') \le c(1-\rho^2)^{-1}(1-r^2)^{-1}, \qquad 0<\rho, r<1.$$

Hence

$$M_\infty(\rho^2, g'') \le c(1-\rho^2)^{-2}.$$

Now Theorem 5.5 [4] shows $g \in \mathscr{B}$.

Using the estimate (4.1) we have just proved that $h = f * g \in \mathscr{B}$ whenever $f \in H^1$ and $g \in \mathscr{B}$. Theorem 1 below shows that we can prove more, i.e., $h \in \mathrm{BMOA}$.

Our proof is based on the Lemma HL1 (the case $p=1$) and the following result.

LEMMA 3. *If h is an analytic function in the unit disc U, the following implication holds:*

$$\int_0^1 (1-r)M_\infty^2(r, h')\,dr < \infty \Rightarrow h \in \mathrm{BMOA}.$$

Proof of Theorem 1. Suppose that $f \in H^1$ and $g \in \mathscr{B}$. Although (4.1) does not work in this setting, a similar estimate for the second derivative gives the desired result. Indeed,

$$\begin{aligned} |D^2h(r^2e^{it})| &= \left|\frac{1}{2\pi}\int_0^{2\pi} D^1f(re^{i\theta})D^1g(re^{i(t-\theta)})\,d\theta\right| \\ &\le cM_1(r, D^1f)(1-r)^{-1}, \qquad \text{i.e.} \\ (1-r)^3M_\infty^2(r^2, D^2h) &\le cM_1^2(r, D^1f)(1-r). \end{aligned}$$

Now, by Lemma HL1, we conclude that $\int_0^1(1-r)^3M_\infty^2(r, D^2h)\,dr < +\infty$.

We can use the Lemma HL to show that h satisfies the condition of Lemma 3 and we further conclude that $h \in \mathrm{BMOA}$. So, we have $\mathscr{B} \subset (H^1, \mathrm{BMOA})$.

The converse follows from the fact $(H^1, \mathrm{BMOA}) \subset (H^1, \mathscr{B})$ and Proposition 2.

Proof of Lemma 3. Let h be an analytic function in the unit disc and

$$\|h\|_* = \sup_{\lambda\in U}\iint_U |h'(z)|^2(1-|z|^2)(1-|\lambda|^2)|1-\bar{\lambda}z|^{-2}\,dx\,dy.$$

It is implicit in Lemma 3.2 **[11]**, p. 238, that $h \in \mathrm{BMOA}$ iff $\|h\|_* < +\infty$. If we use polar coordinates and the fact

$$(1/2\pi)\int_0^{2\pi} |1-\bar{\lambda} r e^{i\theta}|^{-2}\, d\theta = (1-|\lambda r|^2)^{-1},$$

we find

$$\|h\|_* \le \sup_{\lambda \in U} 2\pi(1-|\lambda|^2)\int_0^1 (1-|\lambda r|^2)^{-1}(1-r^2)M_\infty(r, h^1)\, dr.$$

Now the desired conclusion follows from the simple estimate $1-|\lambda|^2 \le 1-|\lambda r|^2$.

COROLLARY 1. $(H^1, \mathrm{VMOA}) = \mathscr{B}$, *where by* VMOA *we denote the space of analytic functions of Vanishing Mean Oscillation (see, for example,* **[11]**).

COROLLARY 2. $(H^1, \mathscr{B}_0) = \mathscr{B}$, *where the little Bloch space* $\mathscr{B}_0$ *is the set of analytic functions* f *on* U *for which* $(1-|\lambda|^2)|f'(\lambda)| \to 0$ *as* $|\lambda| \to 1$.

Proof. Let $f \in H^1$ and $g \in \mathscr{B}$. By Theorem 1,

$$\|F * g\|_* \le c\|F\|_1, \qquad F \in H^1, \tag{4.2}$$

where the constant c does not depend on F. If we substitute $F = f_r - f$ in (4.2), we get $\|f * g_r - f * g\|_* \le c\|f_r - f\|_1$. Since the term on the right-hand side of the last inequality approaches 0 when $r \to 1$ (see Theorem 2.6, **[4]**), it follows from Theorem 5.1, **[11]**, p. 250, that $f * g \in \mathrm{VMOA}$.

Let $n_1, n_2, \ldots$ be a lacunary sequence of integers in the sense that

$$n_{\kappa+1}/n_\kappa \ge q > 1.$$

Since $g(z) = \sum_{\kappa=1}^\infty z^{n_\kappa} \in \mathscr{B}$, the following corollary follows from Theorem 1.

COROLLARY 3. *If* $f(z) = \sum a_n z^n \in H^1$, *then for every lacunary sequence* $\{n_\kappa\}$,

$$F(z) = \sum_{\kappa=1}^\infty a_{n_\kappa} z^{n_\kappa} \in \mathrm{VMOA}.$$

Since VMOA $\subset$ BMOA $\subset H^2$, we get two corollaries from this result:

(a) Paley's theorem (see for example **[4]**, p. 104).

(b) the interesting fact: If $\{n_\kappa\}$ is lacunary sequence then a function $F(z) = \sum a_{n_\kappa} z^{n_\kappa} \in \text{VMOA}$ iff $F \in H^2$, i.e. $\sum_{\kappa=1}^{\infty} |a_{n_\kappa}|^2 < +\infty$.

If A is a sequence space, A^a (the Abel dual) is defined to be the set of sequence $\{\lambda_n\}$ such that $\lim_{r\to 1} \sum_{n=0}^{\infty} \lambda_n \alpha_n r^n$ exists for all $\{\alpha_n\} \in A$.

Proof of Proposition 1. Suppose that $2 \le q < \infty$ and $p^{-1} + q^{-1} = 1$. Since $(H^1)^a = \text{BMOA}$ and $(H^q)^a = H^p$ (see, for example, **[4]**, **[11]**), it follows from Lemma 1.1, **[1]** that

$$(H^1, H^q) \subset (H^p, \text{BMOA}) \subset ((\text{BMOA})^a, H^q).$$

Combining this relation with $H^1 \subset (\text{BMOA})^a$, we have $(H^1, H^q) = (H^p, \text{BMOA})$. Now, Theorem HL completes the proof.

The best known case $q = 2$ of Theorem HL can be rewritten in the form:

THEOREM HL*. *A sequence* $\{\lambda_\kappa\}$ *is a multiplier of* H^1 *into* H^2 (*alias* l^2) *iff*

$$\sum_{\kappa=1}^{n} |\kappa\lambda_\kappa|^2 = O(n^2).$$

As a corollary of this, Duren and Shields **[6]** (see also **[16]**) observe, more generally, $\{\lambda_\kappa\}$ is a multiplier of H^1 into l^q $(2 \le q < +\infty)$ iff

$$\sum_{k=1}^{n} |\kappa\lambda_\kappa|^q = O(n^q). \tag{4.3}$$

Using a similar procedure as in the proof of Proposition 1, we can prove

$$(H^1, l^q) = (l^p, \text{BMOA}) \tag{4.4}$$

where $1 \le q \le +\infty$, $p^{-1} + q^{-1} = 1$.

PROPOSITION 3. (a) *A sequence* $\{\lambda_\kappa\}$ *is a multiplier from* l^p $(1 < p \le 2)$ *to* BMOA *iff it satisfies*

$$\sum_{k=2^n}^{2^{n+1}} |\lambda_\kappa|^q = O(1), \tag{4.5}$$

where $p^{-1} + q^{-1} = 1$.

(b) $(l^1, \mathrm{BMOA}) = l^\infty$.
(c) *Items* (a) *and* (b) *hold if we replace* BMOA *with* VMOA.

Proof. Part (a) follows from (4.4), the above-mentioned Duren-Shields observation, and the fact that conditions (4.3) and (4.5) are equivalent. To prove (b) we can combine the Duren-Shields result **[6]**, $(H^1, l^\infty) = l^\infty$, with (4.4).

A similar procedure as in the proof of Corollary 1 shows that (c) is true.

After this paper was prepared for publication, Professor B. Korenblum found an interesting proof of the main part of Theorem 1, using duality and Coifman's atomic decomposition.

References

[1] J. M. Anderson and A. L. Shields, *Coefficient multipliers of Bloch functions*, Trans. Amer. Math. Soc., **224** (1976), 255–265.

[2] A. Baernstein, *Analytic functions of bounded mean oscillation*, Aspects of Contemporary Complex Analysis, (1980), Academic Press, 3–36.

[3] R. R. Coifman, *A real variable characterization of H^p*, Studia Math., **51** (1974), 269–274.

[4] P. L. Duren, *Theory of H^p Spaces*, (1970), Academic Press.

[5] P. L. Duren, B. W. Romberg and A. L. Shields, *Linear functionals on H^p spaces with $0 < p < 1$*, J. Reine Angew. Math., **238** (1969), 32–60.

[6] P. L. Duren and A. L. Shields, *Coefficient multipliers of H^p and B^p spaces*, Pacific J. Math., **32** (1970), 69–78.

[7] Ch. Fefferman, *Characterizations of bounded mean oscillation*, Bull. Amer. Math. Soc., **77** (1971), 587–588.

[8] T. M. Flett, *On the rate of growth of mean values of holomorphic and harmonic functions*, Proc. London Math. Soc., (3) **20** (1970), 749–768.

[9] ——, *Lipschitz spaces of functions on the circle and the disc*, J. Math. Anal. Appl., **39** (1972), 125–158.

[10] ——, *The dual of an inequality of Hardy and Littlewood and some related inequalities*, ibid., **38** (1972), 746–765.

[11] J. Garnett, *Bounded Analytic Functions*, 1981, Academic Press.

[12] G. H. Hardy and J. E. Littlewood, *Some properties of fractional integrals*, II, Math. Z., **34** (1932), 403–439.

[13] ——, *Notes on the theory of series* (XX) *Generalizations of a theorem of Paley*, Quart. J. Math., Oxford Ser., **8** (1937), 161–171.

[14] ——, *Theorems concerning mean values of analytic or harmonic functions*, Quart. J. Math., Oxford Ser., **12** (1941), 221–256.

[15] W. K. Hayman, *Multivalent Functions*, Cambridge Univ. Press, London and New York, 1958.

[16] J. M. Hedlund, *Multipliers of H^p spaces*, J. Math. Mech., **18** (1969), 1067–1074.

[17] M. Mateljevic and M. Pavlovic, *L^p-behavior of the integral means of analytic functions*, Studia Math., **77** (1983), 219–237.
[18] M. Pavlovic, *Mixed norm spaces of analytic and harmonic functions*, I, Publ. Inst. Math. (Beograd), **40** (54), (1986), 117–141.
[19] ——, *An inequality for the integral means of Hadamard product*, Proc. Amer. Math. Soc., **103** (1988), 404–406.
[20] W. T. Sledd, *On multipliers of H^p spaces*, Indiana Univ. Math. J., **27** (1978), 797–803.
[21] E. M. Stein and A. Zygmund, *Boundedness of translation invariant operators on Hölder spaces and L^p-spaces*, Ann. of Math., **85** (1967), 337–349.

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