**D-HARMONIC DISTRIBUTIONS AND GLOBAL HYPOELLIPTICITY ON NILMANIFOLDS**

Jacek M. Cygan and Leonard F. Richardson

Let \( M = \Gamma \backslash N \) be a compact nilmanifold. A system of differential operators \( D_1, \ldots, D_k \) on \( M \) is globally hypoelliptic (GH) if when \( D_1 f = g_1, \ldots, D_k f = g_k \) with \( f \in D'(M), \ g_1, \ldots, g_k \in C^\infty(M) \) then \( f \in C^\infty(M) \). Let \( X_1, \ldots, X_k \) be real vector fields on \( M \) induced by the Lie algebra \( \mathcal{N} \) of \( N \). We study the relationships between (GH) of the system \( X_1, \ldots, X_k \) on \( M \), (GH) of the operator \( D = X_1^2 + \cdots + X_k^2 \), the constancy of \( D \)-harmonic distributions on \( M \), and related algebraic conditions on \( X_1, \ldots, X_k \in \mathcal{N} \).

**0. Introduction.** Let \( M = \Gamma \backslash N \) be a compact nilmanifold, where \( N \) is a connected, simply connected real nilpotent Lie group with a discrete subgroup \( \Gamma \). There is a unique probability measure \( \mu \) defined on the Borel sets on \( M \) and invariant under the action of \( N \) on \( M \) by right translations. Every \( \mu \)-integrable function \( f \) on \( M \) defines a distribution by the formula \( (f, \phi) = \int_M f \phi \, d\mu, \ \phi \in C^\infty(M) \). Let \( \mathcal{N} \) be the Lie algebra of \( N \). If \( X \in \mathcal{N} \) then \( X \) induces a vector field (which we will denote also by \( X \)) on \( \Gamma \backslash N \) by \( (Xf)(\Gamma n) = (d/dt)|_{t=0} f(\Gamma n \exp tX) \). Consider the left-invariant sum of squares of such vector fields \( X_1, \ldots, X_k \in \mathcal{N} \). This second order differential operator \( D = X_1^2 + \cdots + X_k^2 \) can be regarded as acting on the right on distributions on \( \Gamma \backslash N \). A distribution \( u \in D'(M) \) is \( D \)-harmonic if \( Du = 0 \) on \( M \). The operator \( D \) is globally hypoelliptic (GH) if when \( Df = g \) with \( f \in D'(M), \ g \in C^\infty(M) \), then \( f \in C^\infty(M) \). The system of vector fields \( X_1, \ldots, X_k \) on \( M \) is (GH) if when \( X_1 f = g_1, \ldots, X_k f = g_k \) with \( f \in D'(M), \ g_1, \ldots, g_k \in C^\infty(M) \), then \( f \in C^\infty(M) \). In this paper we investigate relationships between (GH) of \( D \), (GH) of the corresponding system \( X_1, \ldots, X_k \) of vector fields, the constancy of \( D \)-harmonic distributions on \( M \), and related algebraic conditions on \( X_1, \ldots, X_k \in \mathcal{N} \).

Our results are summarized in the figure below. In this figure, functionals \( \Lambda \in \mathcal{N}_j^* \) are assumed to be integral, i.e. \( \Lambda(\log \Gamma \cap \mathcal{N}_j) \subseteq \mathbb{Z} \); \( \mathcal{N} = \mathcal{N}_1 \supset \mathcal{N}_2 \supset \cdots \supset \mathcal{N}_r \supset \mathcal{N}_{r+1} = \{0\} \) is the lower central series of \( \mathcal{N} \) (we say \( \mathcal{N} \) is of step \( r \)), and \( \mathcal{D} \) is the subalgebra of \( \mathcal{N} \) defined generated by \( X_1, \ldots, X_k \). Let \( \mathfrak{W}_\pi \) be an ideal in \( \ker(d\pi) \) such that
$\mathcal{N}/\mathcal{W}_\pi$ has one dimensional center on which $\pi$ is non-trivial. Then $\mathcal{F} := \mathcal{L} + \mathcal{W}_\pi$ and $\mathcal{F} := \mathcal{L} + \mathcal{W}_\pi$.

\[
\begin{array}{|c|c|}
\hline
\Lambda((\mathcal{L} \cap \mathcal{M}_j) + \mathcal{M}_{j+1}) \neq 0 & \Rightarrow \text{D-harmonic distributions are constant} \\
\forall \Lambda \in (\mathcal{M}_j/\mathcal{M}_{j+1})^* & \\
j = 1, \ldots, r & (2.2)
\end{array}
\]

\[\downarrow 2\]

\[
\downarrow \text{System is (GH)}
\]

\[\uparrow 2' \quad \downarrow 3\]

\[
\downarrow \text{dim ker(D) < } \infty \quad \uparrow 6
\]

\[
\downarrow 4 \quad \uparrow 5
\]

\[
\mathcal{F} \cap \mathcal{F} \neq \{0\}, \forall \pi \in (\Gamma \backslash N)_{\infty} \\
(4.1) \quad (2')
\]

We explain below the labeled implications in the above figure referring the reader to indicated sections of the paper for details.

1. This is Theorem (2.1). Condition (2.2) with $j = 1$ provides constancy of the $D$-harmonic distributions on the associated torus.

2. This holds with the necessary assumption that the system $X_1, \ldots, X_k$ is (GH) on the associated torus (proved in [C-R2], Theorem 1).

2'. This requires the assumption that the system $X_1, \ldots, X_k$ is (GH) on the associated torus (implicitly contained in [C-R2] and discussed here in §4).

3. This is proved in §4 for $N$ with exclusively flat coadjoint orbits (which includes step 2 groups), and also for any nilpotent semidirect product $\mathbb{R} \rtimes \mathbb{R}^n$.

4. This is always true. (If $X_1f, \ldots, X_kf$ are $C^\infty$, then so is $Df = (X_1^2 + \cdots + X_k^2)f$ and by (GH) of $D$, $f \in C^\infty$.)

5. We prove this converse to implication 4 for $N$ of step 2, if $D = X_1^2 + X_2^2$ with $X_1 \in \mathcal{N}$, $X_2 \in \mathcal{M}_2$ and with a necessary growth condition on $X_2$ in §1. A growth condition on $X_1$ follows from (GH) of the system $X_1, X_2$. Implication 5 is false for solvmanifolds, even if all the vector fields $X_1, \ldots, X_k$ are algebraic, and hence satisfy all growth conditions. Indeed, the example in §3 shows such a $D$ with a non-$L^2$ distribution in its kernel.

6. See e.g. [G-W3], Lemma 3, p. 161.
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1. 2-step nilmanifolds. In this section we show that global hypoellipticity of the system $X_1, \ldots, X_k$ is insufficient for (GH) of $D$, even if $N$ is step 2, (Example (1.4)). Growth conditions on all the vector fields are needed. Under such conditions $D$ can be proven to be (GH), at least on step 2 nilmanifolds (Theorem (1.1)). We'll see in §3 that this cannot happen in general solvmanifolds.

(1.1) Theorem. Let $\mathcal{N}$ be a step 2 rational nilpotent Lie algebra, $N$ the corresponding connected, simply connected group, and $\Gamma$ a co-compact discrete subgroup of $N$. Let $Y_1, \ldots, Y_n; Z_1, \ldots, Z_k$ be a linear basis for $\mathcal{N}$ selected from $\log \Gamma$ and such that $Y_1 + [\mathcal{N}, \mathcal{N}]$, $l = 1, \ldots, n$ is a basis of $\mathcal{N}/[\mathcal{N}, \mathcal{N}]$, and $Z_p, p = 1, \ldots, k$ is a basis of $[\mathcal{N}, \mathcal{N}]$. Then the operator

$$D = X_1^2 + X_2^2,$$

where $X_1 = \alpha_1 Y_1 + \cdots + \alpha_n Y_n$, $X_2 = \beta_1 Z_1 + \cdots + \beta_k Z_k$, is (GH) on the compact nilmanifold $\Gamma \backslash N$, provided both $\alpha_1, \ldots, \alpha_n$ and $\beta_1, \ldots, \beta_k$ satisfy the following growth condition (which we state for the $\alpha$'s only):

$$|\alpha_1 k_1 + \cdots + \alpha_n k_n| \geq C(k_1^2 + \cdots + k_n^2)^{-p},$$

for some $p, C > 0$ and all integers $k_1, \ldots, k_n$ not all zero.

Proof. Let $Du = g \in C^\infty(\Gamma \backslash N)$ with $u \in \mathcal{D}'(\Gamma \backslash N)$. We use an irreducible (non-canonical) Fourier series decomposition of $u$, $u = u_0 + \sum_\pi \sum_{q=1}^{m(\pi)} u_{\pi,q}$, where $u_0 \in \mathcal{D}'(\Gamma[N, N]\backslash N)$. Thus $u_0$ lives on the associated torus. The sum is over all $\infty$-dimensional representations $\pi \in (\Gamma[N])^\sim$ (with multiplicities $m(\pi)$). Also, $g$ has a Fourier series decomposition with $g_0 \in C^\infty(\Gamma[N, N]\backslash N)$. Condition (1.2) assures that the operator $\overline{D} = (\alpha_1 \partial / \partial x_1 + \cdots + \alpha_n \partial / \partial x_n)^2$ on the associated torus is (GH) by the Theorem in [G-W1]. We conclude that $u_0$ is in fact smooth. The proof that the sum over $\infty$-dimensional $\pi$ is smooth is a modification of the proof of global regularity of a real vector field on a compact nilmanifold (Theorem 1, page 351 of [C-R3]). For each fixed $\infty$-dimensional $\pi$ we construct a suitable Schrödinger model. Since $\pi$ is $\infty$-dimensional, there exists $i$ such that $[Y_i, X_1] \notin \ker(\partial \pi)$. Let $\mathcal{M}_\pi$ be an ideal in $\ker(\partial \pi)$ such that $\mathcal{N}/\mathcal{M}_\pi$ has 1-dimensional center. Passage to this quotient does not affect $\partial \pi(D)$. Introducing a Kirillov subalgebra generated by the images
of $X := Y_i$ and $Y := X_1$ in $\mathcal{N}/\mathcal{W}_\pi$, we obtain a Schrödinger model for $\pi$. In that model $d\pi(D) = -\lambda^2 \xi_1^2 - \Lambda(X_2)^2$, where $\Lambda \in \mathcal{N}^*$ corresponds via Kirillov theory $[K]$ to $\pi$. Moreover, $\Lambda([\mathcal{N}, \mathcal{N}] \cap \log \Gamma) \subseteq \mathbb{Z}$ and $\lambda = \Lambda([Y_1, X_1])$. We use the formula (1.8) on page 353 of $[C-R1]$ to write for any $U \in \mathcal{U}(\mathcal{N})$, the universal enveloping algebra of $\mathcal{N}$:

$$(1.3) \quad (Uf)_\pi = \pi\{[D[D \ldots [D, U] \ldots]g + D[D \ldots [D, U] \ldots]g + \ldots + D^{m-2}[D, U]g + D^{m-1}Ug\}P^{-m}_{\pi}$$

$\equiv h_m P^{-m}_{\pi}.$

Here $P_{\pi}(\xi_1, \ldots, \xi_k) = -\lambda^2 \xi_1^2 - \Lambda(X_2)^2$ and (instead of (1.9) on page 353 of $[C-R1]$) we use the estimate

$$|h_m P^{-m}_{\pi}| \leq |h_m| \Lambda(X_2)^{-2m}$$

$$\leq C^{-2} \sum \Lambda(Z_1)^2 + \ldots + \Lambda(Z_k)^2 |h_m| \equiv |\pi(V) h_m|$$

for some $V \in \mathcal{U}(\mathcal{N})$.

The second inequality is what we need the assumption (1.2) about the coefficients $\beta_1, \ldots, \beta_k$ of $X_2$. Also, (1.3) works only if $(\text{ad } D)^m U = 0$ for some $m$ (depending of course on $U$). Since $D = X_1^2 + X_2$ with $X_2$ central in $\mathcal{N}$, this is the same as $(\text{ad } X_1^2)^m U = 0$. The latter condition is true for any nilpotent Lie algebra $\mathcal{N}$, any $X_1 \in \mathcal{N}$ and $U \in \mathcal{U}(\mathcal{N})$. To see this, wlog we assume that $U = U_1 U_2 \ldots U_p$ with $U_i \in \mathcal{N}, i = 1, \ldots, p$. Note that $\text{ad}(X_1^2)$ is a derivation of the associative algebra $\mathcal{U}(\mathcal{N})$. By Leibnitz’s rule $\text{ad}(X_1^2)^m U = a \text{ linear combination of the terms of the form of } \text{ad}(X_1^2)^l U_1 \ldots \text{ad}(X_1^2)^r U_p$, where $l_1 + \ldots + l_p = m$. Thus it suffices to show that there exists a number $l$ such that $\text{ad}(X_1^2)^l$ maps $\mathcal{N}$ into $0$. This last statement is contained in Lemma 5.1 on page 230 of $[G]$.

(1.4) Example. Let $N$ be a direct product of the 3 dimensional Heisenberg group and $\mathbb{R}$. Let $X, Y, Z,$ and $Z_1$ with $[X, Y] = Z$ be a rational basis of $\mathcal{N}$. Consider $D = (X + \alpha Y)^2 + (Z + \beta Z_1)^2$ with $\alpha$ irrational non-Liouville and $\beta$ a Liouville number. As in the proof of Theorem (1.1), for $\pi \in (\Gamma \setminus N)^\wedge$, pick a Schrödinger model with Kirillov subalgebra generated by $Y$ and $X + \alpha Y$. In that model $d\pi(D) = -\lambda^2 \xi_1^2 - (\lambda + \beta \lambda_1^2)$ with $\lambda = \Lambda(Z), \lambda_1 = \Lambda(Z_1)$, where $\Lambda \in \mathcal{N}^*$ corresponds to $\pi$. Computations similar to those of Example 1 on page 355 of $[C-R3]$ show that $D$ cannot be (GH) on $\Gamma \setminus N$.

2. $D$-harmonic distributions on nilmanifolds. The following Theorem (2.1) does not require any growth assumptions on $X_1, \ldots, X_k$. (Whether $D$ in the Theorem is (GH), even with $X_1, \ldots, X_k$ and
their commutators satisfying (1.2), is still an open problem. This problem is still open even if \( X_1, \ldots, X_k \) are algebraic.) Consider the case \( k = 1 \), with \( \Gamma \backslash N \) being the torus, say two dimensional, and \( D = (\alpha_1 Y_1 + \alpha_2 Y_2)^2 = X_1^2 \). Then Theorem (1.1) corresponds to the statement that \( D \) is (GH) provided \( \alpha_2/\alpha_1 \) is an irrational non-Liouville number. On the other hand, the 2-torus version of Theorem (2.1) says that \( \ker D = \mathbb{C} \cdot 1 \) provided \( \alpha_2/\alpha_1 \) is irrational.

**Theorem (2.1).** Let \( \mathcal{N} \) be a rational nilpotent Lie algebra of step \( r \), \( N \) the corresponding connected, simply connected group, and \( \Gamma \) a cocompact discrete subgroup of \( N \). Let \( X_1, \ldots, X_k \) generate a Lie subalgebra \( \mathcal{L} \) of \( \mathcal{N} \). Suppose that \( \mathcal{L} \) has the property

\[
\text{(2.2)} \quad \text{For each non-zero integral linear functional } \Lambda \in (\mathcal{N}_j/\mathcal{N}_{j+1})^*, \\
\Lambda((\mathcal{L} \cap \mathcal{N}_j) + \mathcal{N}_{j+1}) \neq 0, \; j = 1, \ldots, r.
\]

\([\Lambda \in \mathcal{N}_j^* \text{ is called integral if } \Lambda(\log \Gamma \cap \mathcal{N}_j) \subset \mathbb{Z}.\]

If \( u \in \mathcal{D}'(\Gamma \backslash N) \) and \((X_1^2 + \cdots + X_k^2)u = 0\), then \( u \) can be identified with a constant function.

**Remark 1.** \( X_1, \ldots, X_k \) satisfying condition (2.2) in general do not generate the whole tangent space of \( \Gamma \backslash N \). Consequently, \( D \)-harmonic distributions a priori need not even be continuous functions. Therefore, compactness of \( \Gamma \backslash N \) alone cannot guarantee such ‘harmonic’ distributions to be constants.

**Remark 2.** If \( r = 1 \), then Theorem (2.2) is about a torus. (Recall that \( \mathcal{N} = \mathcal{N}_1 \) and \( \mathcal{N}_2 \) is the commutator of \( \mathcal{N} \).)

We start the proof of Theorem (2.1) with the following proposition.

**Proposition (2.3).** The condition (2.2) above and the following condition (2.4) are equivalent for every compact nilmanifold \( \Gamma \backslash N \).

\[
\text{(2.4)} \quad \text{For each } \pi \in (\Gamma \backslash N)^\sim \sim \{1\}, \text{ if } 1 \leq j \leq r \text{ is such that } \\
\pi(N_{j+1}) \equiv I, \text{ but } \pi(N_j) \neq I, \text{ then } d\pi(\mathcal{L} \cap \mathcal{N}_j) \neq 0,
\]

where \( N_j = \exp \mathcal{N}_j \), and \( (\Gamma \backslash N)^\sim \) denotes the irreducible unitary representations of \( N \) contained in the quasi-regular representation of \( N \) on \( L^2(\Gamma \backslash N) \).

**Proof** (2.2) \( \iff \) (2.4). Each \( \pi \in (\Gamma \backslash N)^\sim \) corresponds to some \( \tilde{\Lambda} \in \mathcal{N}^* \) integral on a rational maximal subordinate subalgebra \( \mathcal{M} \) of \( \mathcal{N} \).
In particular, for \( j \) as in (2.4) \( \mathcal{N}_j \subseteq \mathcal{M} \), and
\[
d\pi|_{\mathcal{N}_j} = i\tilde{\Lambda}|_{\mathcal{N}_j} = i\Lambda \quad \text{with} \quad \Lambda \in (\mathcal{N}_j/\mathcal{N}_{j+1})^* \quad \text{integral.}
\]
Conversely, any integral \( \Lambda \in (\mathcal{N}_j/\mathcal{N}_{j+1})^* \) can be extended by 0 on a rational basis of \( \mathcal{N} \) to an integral \( \tilde{\Lambda} \in \tilde{\mathcal{N}}^* \). \( \pi \in \tilde{\mathcal{N}} \) corresponding via Kirillov theory to \( \tilde{\Lambda} \) is in the spectrum of \( \Gamma \backslash \tilde{\mathcal{N}} \) ([M]). Equation (2.5) holds as before. Thus (2.2) and (2.4) are equivalent.

In view of the Proposition (2.3), all we need to prove Theorem (2.1) is the following:

(2.6) **Lemma.** Let \( X_1, \ldots, X_k \) generate a Lie subalgebra \( \mathcal{L} \) of a nilpotent Lie algebra \( \mathcal{N} \). Let \( \pi \in \tilde{\mathcal{N}} \) be such that \( d\pi(\mathcal{L} \cap \mathcal{N}_{r}) \neq 0 \). Then for every \( u_{\pi} \in (\mathcal{H}^\infty_{\pi})^* \), \( d\pi(X_1^2 + \cdots + X_k^2)u_{\pi} = 0 \) implies \( u_{\pi} = 0 \). Here \( \mathcal{N} = \exp \mathcal{N} \) and \( \mathcal{N}_{r} \) is the lowest non-zero term of the lower central series of \( \mathcal{N} \).

**Proof of Theorem (2.1).** We write an irreducible Fourier series expansion
\[
u = \sum_{\pi \in (\Gamma \backslash \mathcal{N})^*} \sum_{q=1}^{m_\pi} u_{\pi, q} = u_0 + \sum_{j=1}^{r} \sum_{\pi \in \Pi_j} \sum_{q=1}^{m_\pi} u_{\pi, q},
\]
where
\[
\Pi_j = \{ \pi \in (\Gamma \backslash \mathcal{N})^* : \pi(\mathcal{N}_{j+1}) = I, \pi(\mathcal{N}_j) \neq I \}, \quad j = 1, \ldots, r.
\]
Note that \( \Pi_1 \) consists of all 1-dimensional non-trivial representations in \( (\Gamma \backslash \mathcal{N})^* \). We apply Lemma (2.6) to \( u_{\pi, q} \) with \( \pi \in \Pi_r \), then again apply Lemma (2.6) to \( \mathcal{N}/\mathcal{N}_{r} \) which takes care of \( u_{\pi, q} \) with \( \pi \in \Pi_{r-1} \) in the above sum, etc. We are left with \( u_0 \) which corresponds to trivial \( \pi \), i.e. \( u = u_0 \) is a constant function on \( \mathcal{M} \).

The proof of Lemma (2.6) will follow from Lemma (2.7) below, but first we need some definitions (cf. [F-S]).

A Lie algebra \( \mathcal{L} \) is called **graded** if it has a direct sum decomposition \( \mathcal{L} = \sum_{j=1}^{r} \bigoplus V^j \) with the property that \( [V^j, V^k] \subseteq V^{k+j} \) if \( k + j \leq r \) and \( [V^k, V^j] = 0 \) if \( k + j > r \). A graded algebra is always nilpotent. A connected simply connected nilpotent Lie group \( L \) is called graded if its Lie algebra \( \mathcal{L} \) is graded.

Any graded (nilpotent) Lie algebra \( \mathcal{L} \) has a natural family of **dilations** \( \{ \alpha_\lambda \}_{\lambda > 0} \) (one parameter group of automorphisms of \( \mathcal{L} \)) defined on each \( V^j \) by \( \alpha_\lambda(Y) = \lambda^j Y, \ Y \in V^j, \ \lambda > 0 \). By the exponential
map \( \alpha_l \) corresponds to a one-parameter group of automorphisms of \( L \), the simply connected nilpotent Lie group corresponding to \( \mathcal{L} \).

A linear differential operator \( P \) on a graded group \( L \) is homoge-
neous of degree \( d \) if \( P(f \circ \alpha_l) = \lambda^d(Pf) \circ \alpha_l \) for any \( f \in C^\infty(L) \).

We call a differential operator \( P \) on a graded group \( L \) a Rockland operator if (i) \( P \) is left-invariant and homogeneous, and (ii) \( d\pi(P) \) is injective on \( H^\infty_\pi \) for every \( \pi \in \hat{\mathcal{L}} \) except the trivial representation. By a theorem of Helffer and Nourrigat [H-N], a Rockland operator (on a graded group \( L \)) is hypoelliptic: i.e. if \( u \) is a distribution on \( L \) such that \( Pu \) is \( C^\infty \) on an open \( \Omega \subset L \), then \( u \) is \( C^\infty \) on \( \Omega \).

**Lemma 2.7.** Let \( \mathcal{L} \) be a graded Lie subalgebra of a nilpotent Lie algebra \( \mathcal{N} \), and let \( P \in \mathcal{U}(\mathcal{L}) \), the universal enveloping algebra of \( \mathcal{L} \), be a Rockland operator on the graded group \( L \) corresponding to \( \mathcal{L} \). If \( \pi \in \hat{\mathcal{N}} \) is such that \( d\pi(L \cap \mathcal{N}_\ell) \neq 0 \), then \( d\pi(P)u_\pi = 0 \) for \( u_\pi \in (H^\infty_\pi)' \) implies \( u_\pi = 0 \). (\( \mathcal{N}_\ell \) is the lowest non-zero term of the lower central series of \( \mathcal{N} \).)

**Proof of Lemma 2.7.** Suppose \( d\pi(P)u = 0 \) for some \( 0 \neq u \in (H^\infty_\pi)' \). We are going to show then there is a non-smooth function \( \tilde{u} \) on \( L \) such that \( P\tilde{u} = 0 \). That would contradict the hypoellipticity of \( P \) on \( L \). We adapt the proof of Lemma (4.6) of Rothschild and Stein [R-S] to our situation. Let \( \psi \in H^\infty_\pi \) be such that \( (u, \psi) \neq 0 \), and let \( \{\alpha_\lambda\}_{\lambda > 0} \) be the one-parameter group of dilations of \( L \). For each dilation \( l \to \alpha_\lambda(l) \) \( (\lambda \in \mathbb{R}^+) \) of \( L \) define the representation \( \pi_\lambda \) of \( L \) by \( \pi_\lambda(l) := \pi(\alpha_\lambda(l)) \). Observe that if \( \pi(P)u = 0 \), it follows from the homogeneity of \( P \) that \( \pi_\lambda(P)u = 0 \) too. Let

\[
(2.8) \quad \tilde{u}(l) = \int_1^\infty (\pi_\lambda(l)u, \psi)\lambda^Q d\lambda, \quad l \in L,
\]

where the exponent \( Q \) is to be specified later. First we check that the integral in (2.8) converges for each fixed \( l \in L \). Since \( u \in (H^\infty_\pi)' \), we have

\[
(2.9) \quad |(\pi_\lambda(l)u, \psi)| = |(u, \pi(\alpha_\lambda(l^{-1}))\psi)| \leq C|||\pi(\alpha_\lambda(l^{-1}))\psi|||.
\]

By [K], \( H_\pi \) can be identified with \( L^2(\mathbb{R}^P) \), \( H^\infty_\pi \) with \( \mathcal{S}(\mathbb{R}^P) \), the Schwartz space of rapidly decreasing functions, and we can think of \( ||| \cdot ||| \) in (2.9) as being a combination of \( \mathcal{S}(\mathbb{R}^P) \) seminorms of the form \( |||\phi||| = \|x^\beta D_\alpha \phi(x)\| \), where \( || \) is the \( L^2(\mathbb{R}^P) \) norm. In our
case

\begin{equation}
\phi = \pi(\alpha_i(l^{-1}))\psi = \pi \left( \exp \left( \sum_j \lambda^j Y_j \right) \right) \psi,
\end{equation}

where

\begin{equation}
\log l = Y = \sum_{j=1}^r Y_j \in \bigoplus_{j} V_j = \mathcal{L}.
\end{equation}

The representation $\pi \in \tilde{N}$ acting on $\phi \in L^2(\mathbb{R}^P)$ can be written (cf. [H-N], page 904)

\begin{equation}
\pi(\exp Y)\phi(x) = \exp(i(\Lambda, v(Y, x)))\phi(\sigma(x, Y)),
\end{equation}

where $\sigma \in \mathbb{R}^P \,(= \mathcal{M} \setminus \mathcal{N})$ and $v \in \mathcal{N}$ are polynomials in $x \in \mathbb{R}^P$ and $Y \in \mathcal{L}$, and $\mathcal{M} \subset \mathcal{N}$ is a maximal subordinate subalgebra for $\Lambda \in \mathcal{N}^*$. Combining (2.10) and (2.12) we see that $|||\phi|||$ can be estimated by a combination of expressions of the form

\begin{equation}
||v_1(\cdot, Y)(D_\alpha \psi)\sigma(\cdot, Y)||,
\end{equation}

with $\sigma$ as in (2.12) and some polynomial $v_1$ of $x \in \mathbb{R}^P$ and $Y \in \mathcal{L}$, the norm $|||$ being the $L^2(\mathbb{R}^P)$ norm with respect to the $x \in \mathbb{R}^P$ variable marked by a dot. Since $\pi$ is unitary, we can rewrite (2.13) as

\begin{equation}
||\pi(\exp(-Y))\{v_1(\cdot, Y)(D_\alpha \psi)\sigma(\cdot, Y)\}||
\end{equation}

which can be written

\begin{equation}
||v_2(\cdot, Y)(D_\alpha \psi)(\cdot)||
\end{equation}

with some polynomial $v_2$. This is because $\sigma(x, Y)$ (see (2.12)) comes from the multiplication of $\exp X$ (on the right) by $\exp Y$, and subsequent multiplication of $\exp \sigma(x, Y)$ by $\exp(-Y)$ makes it $x$ again. Suppose now that $Y$ depends on $\lambda$ via $Y = Y_\lambda = \sum \lambda^j Y_j$ (cf. (2.11)). It follows from (2.13b) that

\begin{equation}
|||\phi||| \leq \text{a polynomial in } \lambda \text{ of some degree } Q_1 \text{ with coefficients of the form } ||v_3(\cdot; Y_1, \ldots, Y_r)(D_\alpha \psi)(\cdot)||,
\end{equation}

($v_3$ being a polynomial in $x \in \mathbb{R}^P$ and $Y_1, \ldots, Y_r$) and with $Q_1$ independent of $Y$ or $x$. Thus $\tilde{u}(l)$ is well-defined and continuous for any $Q \geq Q_1 + 2$. As in [R-S] $P\tilde{u} = 0$, since the differentiation under the integral sign can
be justified as follows. (Recall that $P$ acts on the right and is left-invariant.)

\begin{equation}
(2.15) \quad |P(\pi_\lambda(l)u, \psi)| = |(\pi_\lambda(l)\pi(\lambda P)u, \psi)| = \lambda^d |(u, \pi(lP)(\lambda^{-1})\psi)| \leq \lambda^d C |\pi(lP)(\alpha(l^{-1}))\psi| \leq \lambda^d C_1 |\pi(lP)(\lambda^{-1})\psi||',
\end{equation}

where $d$ = degree of homogeneity of $P$ and $||| \cdot |||$' means that we've "absorbed" $\pi(P)$ into the Schwartz space seminorm $||| \cdot |||$. The last expression in (2.15) can now be estimated in exactly the same way as the one in (2.9), resulting in an estimate similar to (2.14), with a polynomial in $\lambda$ of degree $Q_2$, say. Thus $\int_{1}^{\infty} P(\pi_\lambda(l)u, \psi)\lambda^{-Q} d\lambda$ converges absolutely whenever $Q \geq Q_2 + 2$, and $P\check{u} = \int_{1}^{\infty} P(\pi_\lambda(l)u, \psi)\lambda^{-Q} d\lambda = \int_{1}^{\infty} (\pi_\lambda(l)\pi(lP)u, \psi)\lambda^{-Q} d\lambda = 0$.

The key thing now is that by the assumption of the lemma there is a $Z \in \mathcal{L} \cap \mathcal{L}$ such that $d\pi(Z) = ic \neq 0, c \in \mathbb{R}$. For this $Z$ we have $\pi_\lambda(\exp tZ) = e^{i\lambda' t}$ and

$\check{u}(\exp tZ) = \int_{1}^{\infty} (\pi(\exp tZ)u, \psi)\lambda^{-Q} d\lambda$

$= (u, \psi) \int_{1}^{\infty} e^{i\lambda t' \lambda'^{-Q} d\lambda}$

$= (u, \psi) r^{-1} \int_{1}^{\infty} e^{i\lambda' \lambda'^{-Q} d\lambda}$

where $Q_3 = Q/r + (r - 1)/r$ and $\lambda' = \lambda'$. We pick now $Q$ in (2.8) so that $Q \geq \max(Q_1, Q_2) + 2$, and that $Q_3$ is an integer $\geq 2$. With this choice of $Q$, as in [R-S] $\check{u}(l)$ defines a distribution on $L$, $P\check{u} \equiv 0$, yet $\check{u}$ restricted to $\exp(\mathbb{R} Z) \subset L$ is not smooth.

(2.16) Corollary. In particular, Lemma (2.7) holds true for $\mathcal{N} = \mathcal{F}$, the free nilpotent Lie algebra of step $r$ on $k + m$ generators $\tilde{X}_1, \ldots, \tilde{X}_k; \tilde{Y}_1, \ldots, \tilde{Y}_m$, and $\mathcal{L}$ the subalgebra of $\mathcal{F}$ generated by $\tilde{X}_1, \ldots, \tilde{X}_k$. Here $P = \tilde{X}_1^2 + \cdots + \tilde{X}_k^2$ is the Rockland operator (cf. [F-S], p. 130) and $p \in \widehat{\mathcal{F}}$ is such that $d\rho(\tilde{Z}) \neq 0$ for some $\tilde{Z} \in \tilde{L} \cap \mathcal{F}$.

Proof of Lemma (2.6). Suppose $d\pi(X_1^2 + \cdots + X_k^2)u = 0$ for $u \in (H_{\infty}^{r, \mathbb{C}})'$ and let $Z \in \mathcal{L} \cap \mathcal{N}$ be such that $d\pi(Z) \neq 0$. By the above corollary, $u = 0$. This can be seen as follows. We construct a chain of subgroups from $L$ to $N$, each of codimension 1 in the next, so that $\mathcal{N} = \mathbb{R} Y_1 \times \cdots \times \mathbb{R} Y_m \ltimes \mathcal{L}$, for some $Y_1, \ldots, Y_m \in \mathcal{N}$. Let $\Phi$ be a homomorphism of $\mathcal{F}$ onto $\mathcal{N}$ given on the generators of $\mathcal{F}$ by
\[ \Phi(\tilde{X}_j) = X_j, \ j = 1, \ldots, k; \ \Phi(\tilde{Y}_j) = Y_j, \ j = 1, \ldots, m. \] Let \( \tilde{Z} \) be a preimage of \( Z \) in \( \widetilde{\mathcal{F}} \cap \mathcal{F}_r \). We apply the corollary to \( \rho = \pi \circ \Phi \in \tilde{F} \) noticing that \( H_\pi = H_\rho \) and \( d\rho(\tilde{X}_1^2 + \cdots + \tilde{X}_k^2) = d\pi(X_1^2 + \cdots + X_k^2) \).

Lemma (2.7) also implies the following version of Theorem (2.1) in case \( \mathcal{L} \), the Lie algebra generated by \( X_1, \ldots, X_k \), is graded.

(2.1') Theorem. Let \( M = \Gamma \backslash N \) be a compact nilmanifold and let \( \mathcal{L} \) be a graded subalgebra of \( \mathcal{N} \). Suppose that \( \mathcal{L} \) has the property (2.2). Let \( P \in \mathcal{U}(\mathcal{L}) \) be a Rockland operator on \( L \) acting on \( \Gamma \backslash N \). If \( u \in \mathcal{D}'(\Gamma \backslash N) \) is in the kernel of \( P \) then \( u = \text{const.} \)

Remark 1. Theorem (2.1') states that if \( \mathcal{L} \) is a large enough graded subalgebra of \( \mathcal{N} \) (i.e. \( \mathcal{L} \) satisfies (2.2)) and \( P \in \mathcal{U}(\mathcal{L}) \) is homogeneous, then injectivity of \( \rho^\circ \) on \( H_\rho^\infty \) for all non-trivial \( \rho \) in \( \tilde{L} \) implies injectivity of \( \pi^\circ \) on \( (H_\pi^\infty)' \) for all non-trivial \( \pi \) in \( (\Gamma \backslash N)^{\backslash} \).

Remark 2. The existence of \( Z \) in \( \mathcal{N}_r \cap \mathcal{L} \) at the end of the proof of Lemma (2.7) and the choice of \( \tilde{Z} \) in \( \widetilde{\mathcal{F}} \cap \mathcal{F}_r \) in the proof of Lemma (2.6) use condition (2.2). We don't know whether the assumption (2.2) in Theorem (2.1) can be replaced by the weaker condition (2.0) of Theorem (4.1).

3. A solvmanifold (counter)example. Here we produce an example of a (GH) system of two vector fields on a class of 3-dimensional ("hyperbolic") solvmanifolds. We show that the kernel of the sum of squares of these vector fields contains a distribution which is not a \( C^\infty \)-function. Also, Lemma (3.4) might be of independent interest.

Consider the following class of three-dimensional compact solvmanifolds, \( M = \Gamma \backslash S \) (see [A-G-H] and [Br1, 2] for details). \( S \) is the semidirect product of \( \mathbb{R} \) and \( \mathbb{R}^2 \) (with \( \mathbb{R}^2 \) normal in \( S \)), in which the group operation is

\[
(x, t)(x', t') = (x + A^t x', t + t'), \quad x, x' \in \mathbb{R}^2, \ t, t' \in \mathbb{R}.
\]

Here \( A^t, t \in \mathbb{R} \), is a 1-parameter subgroup of \( \text{SL}(2, \mathbb{R}) \) through a fixed matrix \( A \in \text{SL}(2, \mathbb{Z}) \). The discrete subgroup \( \Gamma \) can be taken to be the set of points in \( S \) with integer coordinates. (The fact that \( A \in \text{SL}(2, \mathbb{Z}) \) is equivalent to \( A \) mapping the integer lattice \( \mathbb{Z}^2 \) into itself.) We'll consider the case in which \( A \) has unequal positive eigenvalues \( \lambda \) and \( \lambda^{-1} \). Choosing the eigenvectors of \( A \) as a basis of \( \mathbb{R}^2 \) we can
write the group operation in \( S \) in the new \( u, v \) coordinates

\[
(u, v, t)(u', v', t') = (u + \lambda'u', v + \lambda^{-1}v', t + t'),
\]

\[ u', u, v', v, t, t' \in \mathbb{R}. \]

In these new coordinates \( \mathbb{R}^2 \cap \Gamma \) is no longer \( \mathbb{Z}^2 \). (For each \( \lambda > 1 \) such that \( \lambda, \lambda^{-1} \) are eigenvalues of a matrix \( A \in \text{SL}(2, \mathbb{Z}) \) we get a distinct solv-manifold \( \Gamma_\lambda \backslash S \), although \( S \) is not changed up to isomorphism by altering \( \lambda \).) Letting \( T, U \), and \( V \) be the infinitesimal generators of the one-parameter subgroups of \( S \) corresponding to \( t, u, \) and \( v \) we have \([T, U] = \ln \lambda U, [T, V] = -\ln \lambda V\). We consider the operator \( P = T^2 + U^2 \) and the system \( \{T, U\} \) of vector fields induced on \( \Gamma \backslash S \) by \( T \) and \( U \).

(3.2) **PROPOSITION.** Let \( T, U \) and \( \Gamma \backslash S \) be as described above. Then

(a) The system of vector fields \( \{T, U\} \) on \( M = \Gamma \backslash S \) is \((GH)\);

(b) The operator \( P = T^2 + U^2 \) is not \((GH)\) on \( \Gamma \backslash S \). In fact, there is a distribution \( u \in \mathcal{D}'(\Gamma \backslash S) \sim L^2(\Gamma \backslash S) \) such that \( Pu = 0 \).

Proof of (a). Let \( u \in \mathcal{D}'(\Gamma \backslash S) \) be such that \( Tu = f, \ Uu = g, \) and \( Vu = h, \) with \( f, g, h \in C^\infty(\Gamma \backslash S) \). Let \( u = u_0 + \sum_{\pi, j} u_{\pi, j} \) be an irreducible Fourier series of \( u \), the summation being over infinite dimensional \( \pi \in (\Gamma \backslash S)^\sim \) with \( j \) counting the multiplicities and with \( u_0 \in \mathcal{D}'(\Gamma[S, S]\backslash S) \), so \( u_0 \) lives on the associated torus. On that 1-dim torus, \( T \) acts as \( d/dt \). Thus \( Tu_0 = f_0 \in C^\infty(\Gamma[S, S]\backslash S) \) and \( u_0 \) is smooth. As for the \( u_{\pi, j} \)'s, each \( \infty \)-dimensional \( \pi \in (\Gamma \backslash S)^\sim \) acts on \( L^2(\mathbb{R}, dt) \) and \( d\pi(U)u_{\pi, j} = 2\pi i \omega \lambda^j \lambda^j \) for some \( 0 \neq \alpha \in \mathbb{R} \). Since \( d\pi(V) \) acts by multiplication by \( 2\pi \beta \lambda^{-i} \) with \( 0 \neq \beta \in \mathbb{R}, u_{\pi, j} = (-4\pi^2 \alpha \beta)^{-1} d\pi(V) d\pi(U)u_{\pi, j} = (-4\pi^2 \alpha \beta)^{-1}(Vg)_{\pi, j} \). In fact, for any \( R \in \mathcal{D}'(\mathcal{S}) \) we have \( (Ru)_{\pi, j} = (-4\pi^2 \alpha \beta)^{-1}(RVg)_{\pi, j} \), and

\[
\|Ru\|^2_{L^2(\Gamma \backslash S)} = (4\pi^2)^{-2} \sum_{\pi, j} \|RVg\|^2(\alpha \beta)^{-2} + \|(Ru)_0\|^2
\]

\[ \leq C \sum_{\pi, j} \|RVg\|^2 + \|(Ru)_0\|^2 \]

\[ \leq C\|RVg\|^2_{L^2(\Gamma \backslash S)} < \infty. \]

The last expression is finite since \( g \in C^\infty(\Gamma \backslash S) \). The first inequality in (3.3) is a consequence of the following.

(3.4) **LEMMA.** Let \( S = \mathbb{R}^2 \times \mathbb{R} \) be a solvable Lie group with the group law (3.1). Let \( \Gamma \) be a cocompact discrete subgroup of \( S \). Then
\[ \Gamma \cap \mathbb{R}^2 \times \{0\} \text{ is an abelian lattice of points } (a, b) \in \mathbb{R}^2 \text{ having the property that the product } ab \text{ is bounded away from zero, except of course for the group identity.} \]

\[ (3.5) \text{ Corollary. In the setting of the lemma above, the dual lattice } \Gamma^* = \{ \chi_{\alpha, \beta} : \Gamma \to 1 \} \text{ is also a lattice of points } (\alpha, \beta) \text{ such that the product } \alpha \beta \text{ is bounded away from 0, except for } (\alpha, \beta) = (0, 0). \]

\[ \text{Proof of Lemma (3.4). Let } (0, 0, m) \text{ and } (a, b, 0) \in \Gamma \subset S. \text{ Then } (0, 0, m)(a, b, 0)(0, 0, m)^{-1} = (\lambda^m a, \lambda^{-m} b, 0). \text{ Suppose } (a_n, b_n, 0), \ n = 1, 2, \ldots \text{ were a sequence of points in } \Gamma \cap \mathbb{R}^2 \times \{0\} \text{ such that } a_n b_n \to 0 \text{ as } n \to \infty. \text{ Wlog we may suppose } a_n \geq b_n > 0. \text{ Then for every } n \text{ there would be an integer } k_n \text{ such that } \]

\[ (3.6) \lambda^{2(k_n - 1)} < b_n / a_n \leq \lambda^{2k_n}. \]

Define a new sequence of points of \( \Gamma \) by

\[ (a'_n, b'_n, 0) := (\lambda^{k_n} a_n, \lambda^{-k_n} b_n, 0), \quad n = 1, 2, \ldots. \]

We have \( a'_n b'_n = a_n b_n \to 0 \) and \( b'_n / a'_n = \lambda^{-2k_n} b_n / a_n \). The inequalities (3.6) imply now

\[ \lambda^{-2} < b'_n / a'_n \leq 1, \quad n = 1, 2, \ldots. \]

Thus \( \Gamma \ni (a'_n, b'_n, 0) \to (0, 0, 0) \) as \( n \to \infty \), which violates the discreteness of \( \Gamma \).

\[ \text{Proof of (b). For a fixed infinite dimensional } \pi \in (\Gamma \setminus S)^\sim \text{ acting on } L^2(\mathbb{R}, dt) \text{ we'll construct a non-zero function } u(t) \text{ on } \mathbb{R} \text{ such that } d\pi(P)u = 0 \text{ and } d\pi(U)u \in L^2(\mathbb{R}). \text{ Such a } u \text{ defines a distribution } \hat{u} \text{ on } \phi \in C^\infty(\Gamma \setminus S): \]

\[ |(\hat{u}, \phi)| := |(u, Q\phi_{\pi})| \]

\[ = |(u, (\pi U\pi V Q\phi_{\pi})^{-1})| \]

\[ = |(\pi(U)u, (VQ\phi_{\pi})^{-1})| \]

\[ \leq C \|\pi(U)|u||\|V\phi_{\pi}\| \leq C_1 \|V\phi_{\pi}\|_{L^2(\Gamma \setminus S)} \]

\[ \leq C_2 \|V\phi_{\pi}\|_{L^\infty(\Gamma \setminus S)}, \]

where \( \phi_{\pi} \) denotes a projection onto a fixed irreducible subspace \( H_{\pi} \) of \( L^2(\Gamma \setminus S) \) corresponding to \( \pi \), and \( Q : H_{\pi} \to L^2(\mathbb{R}) \) is an intertwining operator onto a Schrödinger model for \( \pi \).

To find such \( u \) we write

\[ (3.7) \quad d\pi(P)u = (d^2 / dt^2 - 4\pi^2 \alpha^2 \lambda^2)u \]

\[ = \{(\ln \lambda)r^2(d/dr^2 + r^{-1}d/dr) - (2\pi \alpha)^2\}u. \]
where we have put \( r = \lambda t \), and we have defined \( u_1(r) = u(t), \ r > 0 \), and we take advantage of the fact that \( d^2/dr^2 + r^{-1}d/dr, \ r > 0 \), is the radial part of the Laplacian \( \Delta \) on the plane. Thus (3.7) becomes equivalent to

\[
(\Delta - a^2)u_2 = 0, \quad a = 2\pi\alpha/\ln \lambda
\]

for a radial function \( u_2 \) on \( \mathbb{R}^2 \sim 0 \). A solution \( u_2 \) of

\[
(\Delta - a^2)u_2 = \delta_0 \quad \text{on} \ \mathbb{R}^2,
\]

where \( \delta_0 \) is the Dirac function supported at the origin 0 of \( \mathbb{R}^2 \), satisfies (3.8), and if it is radial, also (3.7). Applying the Fourier transform on \( \mathbb{R}^2 \) to (3.9) we obtain (cf. e.g. [S-W], page 6)

\[
\hat{u}_2(\xi, \eta) = -(4\pi^2)^{-1}(\xi^2 + \eta^2 + a^2)^{-1} \cdot \exp[-(\xi^2 + \eta^2 + a^2)s] ds
\]

\[
= \int_{\mathbb{R}^2} \exp[-2\pi i(\xi x + \eta y)]u_2(x, y) \, dx \, dy
\]

where

\[
u_2(x, y) = u_1(r) = -(16\pi^2)^{-1}\int_0^{\infty} s^{-1}e^{-bs} \exp(-\pi^2 r^2/s) \, ds
\]

with \( r^2 = x^2 + y^2 \) and \( b = (\alpha/\ln \lambda)^2 \). Thus letting \( r = \lambda^t, \ -\infty < t < \infty, \ u_1(r) \) given by (3.10) is a non-zero solution to (3.7) we were after. It remains to show that \( d\pi(U)u \in L^2(\mathbb{R}) \). For this we write

\[
\|d\pi(U)u\|^2 = \int_{-\infty}^{\infty} |2\pi\alpha u_1(r)|^2 \, dt
\]

\[
= c \int_0^\infty r \left( \int_0^\infty \ldots ds \int_0^\infty \ldots d\sigma \right) \, dr,
\]

where \( c_1 \int_0^{\infty} \ldots ds = u_1(r) = c_1 \int_0^{\infty} \ldots d\sigma \) are given by (3.10). Changing the order of integration in (3.11) to \( dr \, ds \, d\sigma \) and grouping the terms containing \( r \) only, the \( dr \) integral becomes

\[
\int_0^\infty \exp[-\pi^2 r^2(s^{-1} + \sigma^{-1})]r \, dr = (s^{-1} + \sigma^{-1})^{-1}\int_0^\infty \exp(-\pi^2 r^2) r \, dr
\]

\[
= 2\pi^2(s^{-1} + \sigma^{-1})^{-1}.
\]
Substituting this back into (3.10) we obtain

\[ \frac{(3.11)}{109} = \frac{C'}{cxp[-b(s + \sigma)](s\sigma)^{-1}} \int_0^\infty \int_0^\infty \exp(-bs) \exp(-b\sigma)(s + \sigma)^{-1} ds \, d\sigma \]

\[ \leq \left( \frac{C'/2}{109} \right) \left( \int_0^\infty \exp(-bs)s^{-1/2} ds \right)^2 < \infty. \]

**Remark.** Similarly one can show that \( \infty = \|u\|_{L^2(\mathbb{R})} \). We claim \( \tilde{u} \in \mathcal{D}'(\Gamma \setminus S) \) is not given by any \( L^2 \)-function on \( \Gamma \setminus S \). Suppose the negation, i.e. \( \tilde{u} \) is given by some \( w \in L^2(\Gamma \setminus S) \). Since \( \tilde{u} : H^\infty_\pi \to 0 \) for all \( \pi' \neq \pi, \pi' \in (\Gamma \setminus S)^\infty \), we have \( w \in H_\pi \). Then \( Qw \in L^2(\mathbb{R}) \) and \( Qw = u \) a.e. because it gives the same distribution—a contradiction. In particular, \( \tilde{u} \) cannot be continuous or smooth on \( \Gamma \setminus S \).

4. **Necessary and sufficient conditions for (GH) of systems.** Theorem 1 on page 366 of [C-R2] states a necessary and sufficient condition for (GH) of a system \( L \) on \( \Gamma \setminus N \). The proof of necessity, however, has a gap in the last paragraph of page 367. Namely, it is not clear whether or not there is a \( \Lambda' \in \mathcal{O}_N(\Lambda) \) such that \( \Lambda'(\mathcal{L}) = 0 \). On the other hand, the proof of sufficiency establishes the (at least formally) stronger sufficiency theorem below.

(4.1) **Theorem.** If (1°) \( L + [\mathcal{N}, \mathcal{N}] \) is (GH) on \( \Gamma[N, N]\setminus N \), and (2°) for each \( \pi \in (\Gamma \setminus N)^\infty \), \( (\mathcal{L} + W_\pi) \cap \mathcal{L}(\mathcal{N}/W_\pi) \neq \{0\} \), then \( L \) is (GH) on \( \Gamma[N] \). (Here, \( W_\pi \) is an ideal in \( \ker(d\pi) \) such that \( \dim\mathcal{L}(\mathcal{N}/W_\pi)_\pi = 1 \) and \( \pi \mid Z(N/W_\pi) \neq I \). Also, \( \pi \in (\Gamma \setminus N)^\infty \) means \( \pi \) is infinite dimensional, and \( \mathcal{L} \) is a Lie subalgebra of \( \mathcal{N} \) generated by \( L \).

(4.2) **Conjecture.** Conditions (1°) and (2°) of Theorem (4.1) are necessary for (GH) of \( \mathcal{L} \) on \( \Gamma \setminus N \).

Although we do not have any counter-example to this conjecture, we have been able to prove it only under special conditions.

To prove the conjecture, we assume that \( \mathcal{L} \) is (GH) on \( \Gamma \setminus N \), so that (1°) is automatically satisfied. Then we suppose that \( \mathcal{L} \cap \mathcal{L}(\mathcal{N}/W_\pi)_\pi = 0 \), and we try to prove there exists \( \Lambda' \in \mathcal{O}_N(\Lambda) \) such that \( \Lambda'(\mathcal{L}) = 0 \). By the lemma on page 368 of [C-R2], this would contradict (GH) of \( \mathcal{L} \) on \( \Gamma \setminus N \).
(4.3) Proposition. Suppose that $\pi \in (\Gamma \setminus N)_{\infty}$ implies either that
the corresponding co-adjoint orbit $\mathcal{O}(\pi)$ in $N^*$ is flat, or else that
$\pi$ is induceable from a polarization of codimension one in $N$. If
$L = \{X_1, \ldots, X_k\}$ is a (GH) system of vector fields on $\Gamma \setminus N$, then (1°)
$L + [N, N]$ is (GH) on $\Gamma[N, N] \setminus N$, and (2°) for each $\pi \in (\Gamma \setminus N)_{\infty}$,
$(\mathcal{L} + \mathcal{H}_n) \cap \mathcal{Z}(N / \mathcal{H}_n) \neq \{0\}$.

Before proving this theorem and giving examples, we state the following immediate consequences. If $N$ is of step 2, then Proposition
(4.3) shows that Conjecture (4.2) is true for $N_9$ since all orbits will
be flat. Also, for each natural number $n \geq 2$, there exist nontrivial
rational nilpotent Lie algebras of step $n$ such that all orbits (i.e., of
all dimensions) will be flat [R3]. Thus the conjecture will have been
proved for a large class of nilmanifolds. Also, the conjecture will have
been proved for the important class of nilpotent semi-direct products
$\mathbb{R} \ltimes \mathbb{R}^n$, with arbitrarily long lower central series.

Proof. The case of flat orbits is easiest. Fix a $\pi \in (\Gamma \setminus N)_{\infty}$. By
repeatedly factoring out the part of the center in the kernel of $d\pi$,
we may assume wlog that $\mathcal{Z}(N) = \mathbb{R}Z$, that $\mathcal{L} \cap \mathcal{Z} = \{0\}$, and that
$\pi = \pi_\Lambda$ where $\Lambda = Z^*$. Then $\mathcal{O}_N(\Lambda) = Z^* + (Z)^\perp$. $\mathcal{L}$ is spanned by
a basis $L_1, \ldots, L_k$, not in $\mathcal{Z}$. Pick $\Lambda_1, \ldots, \Lambda_k$ in $Z^\perp$ such that
$$\Lambda_j(L_i) = \begin{cases} 0, & \text{if } i \neq j; \\ -\Lambda(L_j), & \text{if } i = j. \end{cases}$$

Then let $\Lambda' = Z^* + \sum_1^k \Lambda_j \in \mathcal{O}_N(\Lambda)$, and $\Lambda'(\mathcal{L}) = 0$. This proves the
conjecture in the flat orbit case.

Now, suppose $\pi$ is induced from a (rational) polarization $\mathcal{M}$ of
codimension 1. (Hence $\mathcal{M}$ is an ideal.)

(4.4) Lemma. If $\mathcal{Z}(N) = \mathbb{R}Z$ and $\mathcal{M}$ is a polarizing ideal for any
$\Lambda \in N^*$ with $\Lambda(Z) \neq 0$, then $\mathcal{M}$ is abelian.

Proof of Lemma. Since $\mathcal{M}$ is subordinate, $Z \not\in [\mathcal{M}, \mathcal{M}]$, and $\mathcal{M} \triangleleft N$
implies $[\mathcal{M}, \mathcal{M}]$ is an ideal too, since $(\text{ad } X)[M_1, M_2] = [(\text{ad } X) M_1,$
$(\text{ad } X) M_2]$. If there exists $0 \neq [M_1, M_2] \in [\mathcal{M}, \mathcal{M}]$, then $[M_1, M_2] \not\in \mathcal{Z}$,
so there exists $U_1 \in N$ such that $(\text{ad } U_1)[M_1, M_2] = [(\text{ad } U_1) M_1,$
$(\text{ad } U_1) M_2] \in [\mathcal{M}, \mathcal{M}] \sim \{0\}$ too. Hence $(\text{ad } U_1)[M_1, M_2] \in [\mathcal{M}, \mathcal{M}] \sim \mathcal{Z}$.
So there is a $U_2$ such that $[(\text{ad } U_2)(\text{ad } U_1) M_1, (\text{ad } U_2)(\text{ad } U_1) M_2]$
$\neq 0$, and so on. Since there is no end to this process, the nilpotence
of $N$ has been violated. Thus $[\mathcal{M}, \mathcal{M}] = 0$. 

This proves the lemma.

Now we have $\mathcal{N} = \mathbb{R} \times \mathcal{M}$, where $\mathcal{M} \cong \mathbb{R}^n$ is a rational abelian ideal of codimension one in $\mathcal{N}$. Since $\mathcal{L}$ is $(\text{GH})$ on $\Gamma\setminus N$, and since $\mathcal{M}$ is rational, $\mathcal{L} \nsubseteq \mathcal{M}$. Thus there exists $X \in \mathcal{L} \sim \mathcal{M}$. We are supposing $\mathcal{L} \cap \mathcal{I} = \{0\}$, and it will suffice to prove that there exists $\Lambda' \in \mathcal{O}_N(\Lambda)$ such that $\Lambda'(\mathcal{L}) = 0$.

If $\mathcal{L}$ were not abelian, there would exist a central element $C \neq 0$, $C \in [\mathcal{N}, \mathcal{N}] \subset \mathcal{M}$, so $[C, \mathcal{M}] = 0$. But $[C, X] = 0$, and $\mathcal{N} = \mathbb{R}X \oplus \mathcal{M}$. Thus $[C, \mathcal{N}] = 0$, so that $C \in \mathcal{L}(\mathcal{N}) \sim \{0\}$.

Now, suppose there existed $L \in \mathcal{L} \cap \mathcal{I} \sim \{0\}$. Then $[X, L] = 0 = [\mathcal{M}, L]$, so $L \in \mathcal{I} \sim \{0\}$. This is a contradiction. So if $X \neq L \in \mathcal{L} \sim \{0\}$, then $L = \alpha X + M$, for some $\alpha \neq 0$ and $M \in \mathcal{M}$. Also, since $\mathcal{L}$ is abelian, $[X, M] = 0 = [\mathcal{M}, M]$. Hence $M \in \mathcal{I} \sim \{0\}$, so that $\mathcal{L}$ contains the $\mathbb{R}$-span of $X$ and $\alpha X + M$. But then $\mathcal{I} \subset \mathcal{L}$. This is a contradiction.

Hence $\mathcal{L} = \mathbb{R}X$. Now, pick $Y \in \mathcal{M} \sim \mathcal{I}$ such that $[X, Y] = Z$, and $(\text{Ad}^* \exp \mathbb{R}Y)\Lambda = Z^* + \mathbb{R}X^*$. Hence there exists $\Lambda' \in \mathcal{O}_N(\Lambda)$ such that $\Lambda'(X) = 0 = \Lambda'(\mathcal{L})$.

This proves the proposition.

(4.5) Example. Let $\mathcal{N}$ be spanned by $X_1$, $X_2$, $Y_1$, $Y_2$, and $Z$, where all non-zero brackets are generated by $[X_1, Y_1] = Y_2$, $[X_1, Y_2] = Z$, and $[X_2, Y_1] = Z$. Thus $\mathcal{N}$ is 3-step, with $\mathcal{N}_2 = [\mathcal{N}, \mathcal{N}]$ spanned by $Y_2$ and $Z$, and $\mathcal{I} = \mathbb{R}Z$.

Let $X = X_1 + \alpha Y_1$ and $Y = X_2 + \alpha Y_2 + \beta Z$, where $\alpha$ is an irrational, non-Liouville number. Then $\mathcal{I} = \mathbb{R}X \oplus \mathbb{R}Y$ is an abelian Lie subalgebra of $\mathcal{N}$, $(\text{GH})$ on $\Gamma[N, N]\setminus N$. Since $\mathcal{L} \cap \mathcal{I} = \{0\}$, condition (2°) of Proposition (4.3) is not satisfied. However, every Kirillov orbit $\mathcal{O}_N \subset \mathcal{N}^*$ is flat (of all possible dimensions) [R3]. By Proposition (4.3), $\mathcal{L}$ is not $(\text{GH})$ on $\Gamma\setminus N$, regardless of the choice of $\beta$, and regardless of the choice of $\alpha$. This example illustrates the necessity of conditions (1°) and (2°) of Proposition (4.3).

(4.6) Example. Let $\mathcal{N} = \mathbb{R} \times \mathbb{R}^{n+1}$ be the $n$-step nilpotent “chain” Lie algebra spanned by $X$, $Y_1$, $\ldots$, $Y_n$, $Y_{n+1} = Z$, with all non-zero brackets generated by $[X, Y_i] = Y_{i+1}$, $i = 1, \ldots, n$. Then let $\mathcal{L} = \mathbb{R}L$, where $L = X + \alpha_1 Y_1 + \cdots + \alpha_n Y_n + \alpha_{n+1} Z$. Then $\mathcal{L}$ is $(\text{GH})$ on $\Gamma[N, N]\setminus N$, but $\mathcal{L}$ is not $(\text{GH})$ on $\Gamma\setminus N$, since condition (2°) of Proposition (4.3) is not satisfied. That is $\mathcal{L} \cap \mathcal{I} = \{0\}$.
There are of course many variations and combinations of these two examples.

The following example supports Conjecture (4.2) by showing how \( \mathcal{L} \cap \mathcal{I} = \{0\} \) can lead to \( \mathcal{L} \) not being (GH) even under circumstances not covered by Proposition (4.3). In particular, it will be a 3-step non-flat orbit example in which \( \mathcal{L} \) is (GH) on \( \Gamma[N, N]\backslash N \) and also on \( \Gamma[Z\backslash N \) and on \( \Gamma[M\backslash N \), and yet \( \mathcal{L} \) is not (GH) on \( \Gamma \backslash N \), apparently because \( \mathcal{L} \cap \mathcal{I} = \{0\} \). Here, \( \mathcal{N}/\mathcal{M} \) will be of dimension two.

\[(4.7) \text{ Example.} \text{ Let } \mathcal{N} \text{ be the } \mathbb{R}-\text{span of } W_1, W_2, X_1, X_2, Y_1, Y_2, \text{ and } Z. \text{ Let all non-zero brackets be generated by } [W_i, X_i] = Y_i, \text{ and } [W_i, Y_i] = Z, \text{ } i = 1, 2. \text{ Let } \mathcal{L} \text{ be the } \mathbb{R}-\text{span of } \{W_i + \alpha W_2 + \beta X_1 + \gamma X_2 + \xi Z, Y_1 - \alpha Y_2 + \eta Z\} \text{ where the real numbers, } 1, \alpha, \beta, \gamma \text{ are linearly independent over } \mathbb{Q} \text{ and satisfy the growth condition (1.2), and } \xi, \eta \in \mathbb{R} \text{ are arbitrary but fixed. The abelian Lie algebra } \mathcal{L} \text{ is (GH) on } \Gamma[N, N]\backslash N \text{ and on } \Gamma[Z\backslash N, \text{ but } \mathcal{L} \cap \mathcal{I} = \{0\}. \text{ Let } \Lambda = Z^*, \text{ so the polarizing subalgebra is the ideal } \mathcal{M} \text{ spanned by } \{X_1, X_2, Y_1, Y_2, Z\}. \text{ However, we can act on } \Lambda \text{ by } \exp t(W_1 - W_2) \text{ to get } \Lambda': Y_1 - \alpha Y_2 + \eta Z_2 \mapsto 0. \text{ And we can act on } \Lambda' \text{ by } \exp s(Y_1 + Y_2) \text{ to get } \Lambda'': W_1 + \alpha W_2 + \beta X_1 + \gamma X_2 + \xi Z \mapsto 0. \text{ Thus } \Lambda'' \in \mathcal{B}_N(\Lambda), \text{ and } \Lambda''(\mathcal{L}) = \{0\}. \text{ By Lemma on page 368 of } [\text{C-R2}] \text{ } \mathcal{L} \text{ is not (GH) on } \Gamma\backslash N.\]

**References**


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LOUISIANA STATE UNIVERSITY
BATON ROUGE, LA 70803