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OBSTRUCTION TO PRESCRIBED POSITIVE RICCI CURVATURE

PH. DELANOË

Obstruction to positive curvature is a phenomenon currently explored in global Riemannian geometry; the strongest results bear of course on the scalar curvature. Hereafter we consider the Ricci curvature and we adapt DeTurck and Koiso's device to non-compact manifolds. We also record a simple non-existence result on Kähler manifolds.

1. Statement of results. Let X be a connected non-compact C^3 n-manifold, $n > 2$, and $\mathbf{h}$ be a fixed C^2 Riemannian metric on X. We are interested in finding conditions on $\mathbf{h}$ which prevent it from being the Ricci tensor of any Riemannian metric on X. Following [**5**] we consider the largest eigenvalue $\lambda(\mathbf{h})$ of the curvature operator acting on covariant symmetric 2-tensors (see [**1**]). Given any C^2 metric $\mathbf{g}$ on X, we let $e(\mathbf{g})$ denote the energy density of the identity map from $(X, \mathbf{g})$ to $(X, \mathbf{h})$.

THEOREM 1. *Assume $\lambda(\mathbf{h}) \leq 1 - \varepsilon$ on X, for some positive real ε. Then there is no complete C^2 metric $\mathbf{g}$ on X which admits $\mathbf{h}$ as Ricci curvature.*

THEOREM 2. *Assume $\lambda(\mathbf{h}) \leq 1$ on X and $\mathbf{h}$ complete. Then there is no C^2 metric $\mathbf{g}$ on X, with $e(\mathbf{g})$ assuming a local maximum, which admits $\mathbf{h}$ as Ricci curvature.*

THEOREM 3. *Assume $\lambda(\mathbf{h}) \leq 1$ on X. Then there is no C^2 metric $\mathbf{g}$ on X, with $e(\mathbf{g})$ vanishing at infinity, which admits $\mathbf{h}$ as Ricci curvature.*

2. Remarks and examples. Our results and methods of proof extend [**5**] from compact to non-compact manifolds. Related, though weaker, results, obtained by different techniques, are those of [**0**] (a reference kindly pointed out to us by the referee).

Theorem 1 may be viewed as the "true" extension of [**5**, Theorem 3.2-b]. Interestingly, Theorem 2 looks somewhat stronger than

[5, Theorem 3.2-b] due to the non-compactness of X; an example here for $(X, \mathbf{h})$ is the Poincaré disk, since constant curvature -1 implies at once $\lambda(\mathbf{h}) \equiv 1$ by [1, Proposition 4.3]. Theorem 3 typically applies when $(X, \mathbf{g})$ is asymptotically flat; as such, it generalizes [8].

It is not possible to drop the completeness of both metrics and just assume $\lambda(\mathbf{h}) \leq 1$, as the following example shows: X is the euclidean n-space, $\mathbf{h}$ the conformal metric $4(n-1)\sigma^{-4}E$, E denoting the standard euclidean metric and $\sigma := \sqrt{1+|\mathbf{x}|^2}$. $\mathbf{h}$ satisfies $\lambda(\mathbf{h}) \equiv 1$ and $\mathrm{Ricci}(\mathbf{h}) = \mathbf{h}$ because it is constructed in the following way: start with the round n-sphere $(S^n, \mathbf{g}_0)$ of radius $r = \sqrt{n-1}$ so that $\mathrm{Ricci}(\mathbf{g}_0) = \mathbf{g}_0$. By [1, Proposition 4.3] we see at once that $\lambda(\mathbf{g}_0) \equiv 1$. Now $\mathbf{h}$ is obtained as the pull-forward of $\mathbf{g}_0$ by a stereographic projection composed with the dilation of ratio $1/r$.

From the identity $\lambda(c\mathbf{h}) = \frac{1}{c}\lambda(\mathbf{h})$ valid for any positive constant c, one would like to infer that, given *any* C^2 metric $\mathbf{h}$ on X, the preceding theorems hold with $c\mathbf{h}$ for suitable $c \gg 1$. This is what DeTurck and Koiso do on compact X. However, this cannot be done on non-compact X without *assuming* that $\lambda(\mathbf{h})$ is uniformly bounded from above (a mistake to be corrected in [8]). Keeping this in mind, one can formulate in an obvious way corollaries of our three theorems analogous to those of [5].

3. Proofs. For each theorem we argue by contradiction and assume the existence of a metric $\mathbf{g}$ with the asserted properties. As observed in [5], the Bianchi identity thus satisfied by $\mathbf{h}$ with respect to the metric $\mathbf{g}$ means that the identity map from $(X, \mathbf{g})$ to $(X, \mathbf{h})$ is harmonic. Hence the energy density $e(\mathbf{g})$ satisfies on X the elliptic differential inequality

$$\Delta[e(\mathbf{g})] \leq -2\|T\|^2 - [1-\lambda(\mathbf{h})]|\mathbf{h}|^2 \tag{1}$$

deduced in [5] from an identity discovered by R. Hamilton [6]. Here Δ stands for the Laplacian (with *negative* symbol) of $\mathbf{g}$, T for the $\binom{1}{2}$-tensor difference between the Christoffel symbols of $\mathbf{g}$ and $\mathbf{h}$, $|\cdot|$ for the norm in the metric $\mathbf{g}$, $\|\cdot\|$ for another norm (see [5]). Under the assumption $\lambda(\mathbf{h}) \leq 1$, made in all three theorems, $e(\mathbf{g})$ is thus C^2 positive subharmonic on $(X, \mathbf{g})$.

Proof of Theorem 1. By Schwarz inequality $e(\mathbf{g}) \leq \sqrt{n}|\mathbf{h}|$; so (1) implies that $e(\mathbf{g})$ solves on X the inequality

$$\Delta u \leq -f(u) \tag{2}$$

where

$$f(t) := (2\varepsilon/n)t^2.$$

The function f is positive strictly increasing on $(0, \infty)$ and it readily satisfies the following condition: for all $a < b$ in $(0, \infty)$,

$$\int_b^\infty \left(\int_a^s f(t)\,dt\right)^{-1/2} ds < \infty. \tag{3}$$

Assume provisionally that $(X, \mathbf{g})$ is of class C^3. Since $\mathbf{h} = \mathrm{Ricci}(\mathbf{g})$ is non-negative, $(X, \mathbf{g})$ and f fulfill all the conditions required for the proof of Calabi's extension of Hopf's maximum principle [2] (Theorem 4). Fixing $a \in (0, \min_X[e(\mathbf{g})])$ in (3) and arguing as in [2] yields an impossibility for $e(\mathbf{g})$ to satisfy (2) on X. So we get the desired contradiction.

We are left with the C^3 regularity of $(X, \mathbf{g})$. It follows basically from local elliptic regularity, as a repeated use of [4] now shows. Fix α in $(0, 1)$. Since $\mathbf{g}$ is $C^{1,\alpha}$, X admits a $C^{2,\alpha}$ atlas of coordinates harmonic for $\mathbf{g}$ [4] (Lemma 1.2). Being $C^{1,\alpha}$ in the original atlas, $\mathbf{h}$ remains so in the harmonic atlas [4] (Corollary 1.4). Since $\mathrm{Ricci}(\mathbf{g}) = \mathbf{h}$, $\mathbf{g}$ is $C^{3,\alpha}$ in the harmonic atlas [4] (Theorem 4.5-b) and the atlas itself actually is $C^{4,\alpha}$ [4] (Lemma 1.2). □

Proof of Theorem 2. By Hopf's maximum principle [7], $e(\mathbf{g})$ is necessarily constant on X. It follows from (1) that $T \equiv 0$ hence $\mathrm{Ricci}(\mathbf{h}) = \mathbf{h}$ on X. Moreover, the regularity argument above, now applied to $\mathbf{h}$, combined with a bootstrap argument, provides a harmonic atlas in which $(X, \mathbf{h})$ is a C^∞ Riemannian manifold. So Myers' theorem [10] holds for $(X, \mathbf{h})$, contradicting the noncompactness of X. □

Proof of Theorem 3. Since $e(\mathbf{g})$ vanishes at infinity, it assumes a positive global maximum M. Fix μ in $(0, M)$ and let K be a compact subdomain of X outside which $e(\mathbf{g}) \leq \mu$. Hopf's maximum principle [7] applied to $e(\mathbf{g})$ inside K implies that either $e(\mathbf{g})$ is constant on K, or $e(\mathbf{g}) \leq \mu$ on K. In both cases it contradicts $\mu < M$. □

4. A non-existence result on Kähler manifolds. Let X be a connected complex manifold, of complex dimension $n \geq 1$, admitting a C^2 Kähler metric $\mathbf{h}$. Denote by $|\mathbf{h}|$ the Riemannian density of $\mathbf{h}$.

THEOREM 4. *Assume that the scalar curvature of* $\mathbf{h}$ *is bounded above by* n*, but not identical to* n*. Then there exists no* C^2 *Kähler metric* $\mathbf{g}$

on X, with relative density $|\mathbf{g}|/|\mathbf{h}|$ assuming a local minimum, which admits $\mathbf{h}$ as Ricci curvature.

Proof. Again by contradiction; let $\mathbf{g}$ be such a metric. Then the C^2 function $f := \text{Log}(|\mathbf{g}|/|\mathbf{h}|)$ satisfies on X the equation $\Delta f = n - S$, S standing for the scalar curvature of $\mathbf{h}$, Δ for its (complex) Laplacian. From the assumption, f is superharmonic on $(X, \mathbf{h})$; moreover, it assumes a local minimum, so it must be *constant* according to Hopf's maximum principle [7]. It implies that $S \equiv n$, contradicting the assumption. □

For non-compact X, Theorem 4 typically applies when $(X, \mathbf{g})$ is Kähler asymptotically $\mathbb{C}^n$ [3]. For compact X, recalling that $S(c\mathbf{h}) = S(\mathbf{h})/c$ for any positive constant c, we obtain a simple proof of the following

COROLLARY. *Let $(X, \mathbf{h})$ be a C^2 compact Kähler manifold. Then there exists a positive real $c(\mathbf{h})$ such that, for any real $c > c(\mathbf{h})$, no C^2 Kähler metric on X admits $c\mathbf{h}$ as Ricci curvature.*

Of course, as emphasized by J.-P. Bourguignon (in a letter to us), the classical cohomological constraint bearing on Ricci tensors of compact Kähler manifolds makes Theorem 4 rather relevant for *non-compact* simply connected X.

Acknowledgment. This work originated from a question posed to me by Albert Jeune, about the contradiction between **[3]** and Jeune's Corollary 1 in **[8]**; as pointed out in §2, the latter turns out to be incorrect without a boundedness assumption on $\lambda(\mathbf{h})$.

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CHARGÉ DE RECHERCHES AU C.N.R.S.
I.M.S.P., PARC VALROSE
F-06034 NICE CEDEX, FRANCE