

ON A CONSTRAINED VARIATIONAL PROBLEM AND THE SPACES OF HORIZONTAL PATHS

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In this paper we study geometry associated to an isoholonomy variational problem on a fat bundle. We prove that the energy function satisfies the Palais-Smale condition on the space of horizontal paths. We blow up the singularity of the horizontal loop space. Then we study the closed geodesics. We relate the number of connected components of the space of loops with trivial holonomy to the topology of the fat bundle.

1. Introduction. In this paper we will study geometry associated to the following constrained variational problem:

Problem 1. Consider a principal G -bundle (G compact) over a compact Riemannian manifold M

$$\pi: F \rightarrow M$$

with a given connection form ω on F . Fixing two points $x_0, x_1 \in F$, consider the space of H^1 -horizontal paths from x_0 to x_1 , denoted by $\Omega F(x_0, x_1)$. Then the problem is to find a horizontal path $\gamma_0 \in \Omega F(x_0, x_1)$ such that

$$E(\gamma_0) = \min_{\gamma \in \Omega F(x_0, x_1)} E(\gamma),$$

where $E(\gamma) = \int_0^1 \|\pi_* \dot{\gamma}(t)\|^2 dt$. Recall that a path γ in F is horizontal if

$$(1.1) \quad \omega(\dot{\gamma}) = 0.$$

The most important case in application (cf. [18]) is when the two end points lie on the same fiber, and the problem reduces to the following form:

Problem 2. Let $\Omega M(x_0, H_0)$ be the space of H^1 -loops on M based at $x_0 \in M$ with holonomy H_0 , then the problem is to find a loop γ_0 such that

$$E(\gamma_0) = \min_{\gamma \in \Omega M(x_0, H_0)} E(\gamma),$$

where $E(\gamma) = \int_0^1 \|\dot{\gamma}\|^2 dt$.

Define a metric on F by

$$d(x, y) = \min_{\gamma \in \Omega F(x, y)} (E(\gamma))^{1/2},$$

which is called the Carnot-Carathéodory metric on F (see the discussion in Gromov [10], Koranyi [15]). This metric has appeared in various contexts: the collapse of Riemannian metrics (cf. [10], [15], [17], [18], [19]), the growth of volumes in Riemannian manifold (cf. [19]), pseudo-hermitian structure on strongly pseudo-convex CR manifold (cf. [15], [24]), Vakanomic mechanics (cf. [1]), control theory (cf. [6]), microbiology and quantum physics (cf. [18]). This metric is also the underlying geometry of hypoelliptic operators (cf. [4], [22]).

We will study this problem from the point of view of sub-Riemannian geometry. It is well known that the path space plays a very important role in Riemannian geometry (cf. [10], [14]), and in the case of sub-Riemannian geometry, we expect that the horizontal path space will play a similar role.

In this paper we study the simplest case: F is a fat bundle in the sense of Weinstein, so that the Carnot-Carathéodory metric is strongly bracket generating (cf. Strichartz [20]). In this case $\Omega F(x, y)$ is smooth if $x \neq y$, and $\Omega F(x, x)$ has a singularity at the constant loop. One of our main results is the resolution of this singularity and its application to the study of closed geodesics.

We first develop some analytical tools. In the second section we study the holonomy map and show that the energy function E on $\Omega M(x_0, H_0)$ satisfies the P-S condition. In particular, we show that geodesics are smooth, which is a special case of regularity results for geodesics by Bar, Hamenstadt, Taylor, see [11], [22], [29] for details. In their treatment they only require that the horizontal bundle satisfies Hörmander's condition. In particular, their work implies the equivalence of the Lagrangian approach by Carathéodory, Hermann (cf. [13]) et al. and the Hamiltonian approach by Rayner [28] et al.

In the third section we define the concept of conjugate points and compute the Morse index of the energy E on $\Omega F(x_0, x_1)$ at a geodesic in terms of conjugate points as in Riemannian geometry. This amounts to the study of a 2nd-order differential-integral operator (the Jacobi field) subject to a constraint on the line. Although our result takes the same form as the Morse index theorem, its proof is more complicated than that of the Riemannian case. Then we show that there are arbitrary nearby conjugate points, thus sharpen a result of Rayner [28] and Strichartz [20].

In the fourth section we study the horizontal loop space, both free and based. We describe a blowing up of the singularity and show that there is a canonical correspondence between the blow-up and certain two-step nilpotent Lie algebras. For the free horizontal loop space, we show that there is a nature smooth S^1 action on its blow-up.

In the fifth section we apply the results obtained in §4 to the closed geodesics problem. Since the horizontal loop space is singular at the constant loops, we will consider either the punctured horizontal loop space, or the blow-up of the horizontal loop space. In particular, we show that the number of connected components on the based horizontal loop space is that of the based loop space on F . This somehow generalizes a result of Smale [27]. Further results concerning the topology of horizontal loop space will appear elsewhere.

In the appendix we derive the Jacobi field for the variational problem on M . However, we have not been able to define the curvature (cf. Strichartz [20] for a discussion).

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2. Horizontal paths spaces.

2.1. *Spaces of horizontal paths and Hörmander's condition.* Let H be the horizontal subbundle in TF . A classical result of Chow says that if the connection satisfies Hörmander's condition and F is connected, then for every $x_1, x_2 \in F$, there is a horizontal H^1 path connecting them (cf. [6], [20]).

The following result is due to Weinstein [26]. For the sake of completeness we include the proof.

PROPOSITION 2.1. *If F is connected and the connection satisfies Hörmander's condition, then the spaces $\Omega F(x_1, x_2)$, consisting of horizontal H^1 paths from x_1 to x_2 , are all homotopic to each other.*

Proof. We will only prove that $\Omega F(x_0, x_0)$ is homotopic to $\Omega F(x_0, x_1)$. Take a horizontal path γ_0 starting at x_0 and ending at x_1 . Define a map $A: \Omega F(x_0, x_0) \rightarrow \Omega F(x_0, x_1)$ by

$$\gamma \rightarrow \gamma_0\gamma,$$

and a map $B: \Omega F(x_0, x_1) \rightarrow \Omega F(x_0, x_0)$ by

$$\gamma \rightarrow \gamma_0^{-1}\gamma,$$

where $\gamma_0\gamma$ denotes the path

$$\gamma_0\gamma(t) = \begin{cases} \gamma(2t), & 0 \leq t \leq 1/2; \\ \gamma_0(2t-1), & 1/2 \leq t \leq 1 \end{cases}$$

and $\gamma^{-1}(t)$ denotes the path

$$\gamma^{-1}(t) = \gamma(1-t).$$

Now $AB: \Omega F(x_0, x_1) \rightarrow \Omega F(x_0, x_1)$, $\gamma \rightarrow \gamma_0\gamma_0^{-1}\gamma$ is homotopic to the identity. In fact, let γ_ε be the paths

$$\gamma_\varepsilon(t) = \gamma_0(\varepsilon(1-t));$$

then the map L_ε

$$\gamma \rightarrow \gamma_\varepsilon\gamma_\varepsilon^{-1}\gamma$$

is the identity map when $\varepsilon = 0$, and $L_1 = AB$. Similarly BA is also homotopic to the identity map.

2.2. Linear O.D.E.'s on Lie Groups. In this subsection we will study linear O.D.E.'s on a Lie group, which will be used in the study of the holonomy map.

Consider a (time-dependent) linear O.D.E. on Lie group G ,

$$(2.1) \quad \dot{q}(t) = TL_{q(t)}B(t)$$

where B is a g -valued function. If B is continuous with respect to t , then the solution $q(t)$ can be written as (cf. [8])

$$(2.2) \quad q(t) = \lim_{N \rightarrow \infty} \exp(tB(t_0)/N) \cdots \exp(tB(t_N)/N)$$

where $0 = t_0 < t_1 < \cdots < t_N = t$. If $B(t)$ is not continuous but only integrable, then (2.2) is no longer valid. Nevertheless, (2.2) is valid in the approximate sense, as we will see below. First we specify the space of B .

DEFINITION 2.1. $L([0, T], g)$ is the space of time-dependent vector fields $B(t)$ such that

$$\|B\| = \int_0^T \|B(t)\| dt < \infty.$$

We smooth the non-smooth $B(t)$ in the usual way. Take a smooth cutoff function $\rho(t)$ which vanishes outside $-1 < t < 1$, and has total integral 1 in $[-1, 1]$. We define

$$B_h(t) = h^{-1} \int_{-\infty}^{\infty} B(s)\rho((t-s)/h) ds.$$

PROPOSITION 2.2. *Let $q_h(t)$ be the solution of the time-dependent O.D.E. on G*

$$(2.3) \quad \dot{q}_h(t) = (L_{q_h(t)})_* B_h(t).$$

Then as $h \rightarrow 0$, $q_h(t)$ converges to $q(t)$, the solution to

$$(2.4) \quad \dot{q} = (L_q)_* B(t).$$

Proof. Let the Lie algebra \mathfrak{g} be a subalgebra in the Lie algebra of $k \times k$ matrices $\mathfrak{gl}(k)$ for suitable k ; then (2.4) ((2.3) respectively) can be extended to be a O.D.E. on matrices,

$$\dot{A}(t) = A(t)B(t) \quad (\dot{A}_h(t) = A_h(t)B_h(t) \text{ respectively}).$$

The solution can be written as

$$A(t) = 1 + \sum_{k=1}^{\infty} \int \cdots \int_{0 \leq t_1 \leq \dots \leq t_k \leq t} B(t_1) \cdots B(t_k) dt_1 \cdots dt_k / k!.$$

It is easy to prove that

$$\begin{aligned} & \left\| \int \cdots \int_{0 \leq t_1 \leq \dots \leq t_k \leq t} (B(t_1) \cdots B(t_k) - B_h(t_1) \cdots B_h(t_k)) dt_1 \cdots dt_k \right\| \\ & \leq k \left(\int_0^t (\|B(t)\| + \|B_h(t)\|) dt \right)^{k-1} \cdot \int_0^t \|B_h(t) - B(t)\| dt, \end{aligned}$$

so the proposition follows.

2.3. The holonomy map. We will study the holonomy map which assigns a loop to its holonomy

$$H: \Omega M(x_0) \rightarrow G$$

where $\Omega M(x_0)$ is the space of H^1 -loops on M based at x_0 . We first prove a bound on the holonomy of a loop in terms of its length (“isoperimetric inequality”). Let $D(q_1, q_2)$ denote the distance function of the bi-invariant metric $\| \cdot \|$ on G , and $\| \cdot \|_F$ the Kaluza-Klein metric on F (cf. [18]).

PROPOSITION 2.3. *Let γ be a loop on M . Then*

$$(2.5) \quad D(H(\gamma), e) \leq \|\omega\| \text{ length}(\gamma)$$

where

$$\|\omega\| = \max_{x \in F} \max_{y \in T_x F} \frac{\|\omega(y)\|}{\|y\|_F}.$$

Proof. Take local coordinates (x, q) on F . Then ω can be written as

$$\omega(x, q) = \omega_0(q) + \bar{q}^{-1}(t)A(x(t))q(t)x(t).$$

We may confine ourself to the case where γ lies wholly in a single chart U_α on M (for which we have of course $\pi^{-1}(U_\alpha) = G \times U_\alpha$). Suppose that the loop is $t \rightarrow x(t)$ and the horizontal path is given by $t \rightarrow (x(t), q(t))$. Let $B(t) = \dot{x}^\mu(t)A_\mu(x(t))$; then $q(t)$ is the solution of the O.D.E.

$$\dot{q} - (L_{B(t)})_*q = 0.$$

As is shown in §2.2, we may only work with smooth $B(t)$. By the triangle inequality and the bi-invariance of the metric D , we see that

$$\begin{aligned} D(q_B(t), e) &= \lim_{N \rightarrow \infty} D(\exp(tB(t_0)/N) \cdots \exp(tB(t_N)/N), e) \\ &\leq \lim_{N \rightarrow \infty} \sum_{i=1}^N D(\exp(tB(t_i)/N), e) = \lim_{N \rightarrow \infty} \sum_{i=1}^N \|tB(t_i)\|/N \\ &\leq \int_0^t \|B(t)\| dt. \end{aligned}$$

So

$$\begin{aligned} D(q_B(t), e) &\leq \int_0^t \|B(t)\| dt \leq \int_0^t \|\dot{x}^\mu(t)A(x(t))\| dt \\ &\leq \|A\| \int_0^t \|\dot{x}(t)\| dt \end{aligned}$$

where

$$\|A\| = \max_x (\|A(x(t))\|/\|x\|) \leq \|\omega\|.$$

Now we prove the continuity of the holonomy map under some extra assumption on the energy function E .

PROPOSITION 2.4. *Suppose that $M \rightarrow R^k$ is an isometrical embedding. If $\gamma_n \in \Omega M(x_0)$ is a sequence on which E is bounded, $E(\gamma_n) < M_0 < \infty$, and*

$$\lim_{n \rightarrow \infty} \int_0^t (\|\dot{\gamma}_n(t) - \dot{\gamma}(t)\| + \|\gamma_1(t) - \gamma_2(t)\|) dt = 0,$$

then

$$\lim_{n \rightarrow \infty} H(\gamma_n) = H(\gamma).$$

Proof. Suppose the horizontal path covering γ_1 (γ_2 respectively) is $(x_1(t), q_{B_1}(t))$ ($(x_2(t), q_{B_2}(t))$ respectively). We need only to prove that

$$(2.6) \quad D(q_{B_1}(t), q_{B_2}(t)) \leq M \int_0^t \|B_1(t) - B_2(t)\| dt$$

where $B_i(t) = \dot{x}_i^\mu(t)A(x(t))$, $i = 1, 2$.

W.o.l.g. we need only to prove (2.6) for smooth B_1, B_2 .

$$\begin{aligned} D(q_{B_1}(t), q_{B_2}(t)) &= \lim_{N \rightarrow \infty} D(\exp(tB_1(t_1)/N) \cdots \exp(tB_1(t_N)/N), \\ &\quad \exp(tB_2(t_1)/N) \cdots \exp(tB_2(t_N)/N)) \\ &\leq \lim_{N \rightarrow \infty} \sum_{i=1}^N D(\exp(tB_1(t_1)/N) \cdots \exp(tB_1(t_{i-1})/N) \exp(tB_2(t_i)/N) \\ &\quad \cdots \exp(tB_2(t_1)/N), \exp(tB_1(t_1)/N) \\ &\quad \cdots \exp(tB_1(t_i)/N) \exp(tB_2(t_{i+1})/N) \\ &\quad \cdots \exp(tB_2(t_N)/N)) \\ &= \lim_{N \rightarrow \infty} \sum_{i=1}^N D(\exp(tB_1(t_i)/N), \exp(tB_2(t_i)/N)) \\ &\leq C \lim_{N \rightarrow \infty} \sum_{i=1}^N \|tB_1(t_i) - tB_2(t_i)\|/N \\ &\leq M \int_0^t \|B_1(t) - B_2(t)\| dt. \end{aligned}$$

So

$$\begin{aligned} D(q_{B_1}(t), q_{B_2}(t)) &\leq M \int_0^t \|B_1(t) - B_2(t)\| dt \\ &\leq M \left(\int_0^t \|(\dot{x}_1 - \dot{x}_2)A(x_1(t))\| dt + \int_0^t \|x_1(A(x_1)) - A(x_2)\| dt \right) \\ &\leq M \left(\max \|A(x(t))\| \int_0^t \|\dot{x}_1 - \dot{x}_2\| dt \right. \\ &\quad \left. + \max \|A_x(t)\| \int_0^t \|x_1 - x_2\| dt \max_i \max \|x_i(t)\| \right). \end{aligned}$$

Since $E(\gamma) < M_0 < \infty$, $\max \|x(t)\|$ is universally bounded. Hence

$$D(q_{B_1}(t), q_{B_2}(t)) \leq M' \int_0^t (\|\dot{x}_1(t) - \dot{x}_2(t)\| + \|x_1(t) - x_2(t)\|) dt.$$

In order to obtain the derivative of the holonomy map, we have to study the derivative of the solution to the following O.D.E.

$$\dot{q}(t) = q(t)B(t)$$

with respect to B . Here q and b are $k \times k$ matrices. We denote its solution by $q_{B(t)}$.

PROPOSITION 2.5.

$$(2.7) \quad q_{B+\delta B}(t) - q_B(t) - \int_0^t q_B(s)\delta B(s)q_B^{-1}(s) ds q_B(t) = o(\|\delta B\|).$$

Proof. As is shown in §2.2, we need only to prove (2.7) for smooth $B(t)$. Now

$$\begin{aligned} & q_{B+\delta B}(t) - q_B(t) - \int_0^t q_B(s)\delta B(s)q_B^{-1}(s) ds q_B(t) \\ &= \lim_{N \rightarrow \infty} (\exp(tB + \delta B)(t_0)/N) \cdots (\exp(t(B + \delta B)(t_N)/N) \\ & \quad - \exp(tB(t_0)/N) \cdots \exp(tB(t_N)/N)) \\ & \quad - \int_0^t q_B(s)\delta B(s)q_B^{-1}(s) ds q_B(t) \\ &= \lim_{N \rightarrow \infty} \sum_{i=1}^N \exp(t(B + \delta B)(t_0)/N) \\ & \quad \cdots \exp(t(B + \delta B)(t_{i-1})/N) \\ & \quad \cdot \{\exp(t(B + \delta B)(t_i)/N) - \exp(tB(t_i)/N)(1 + (\delta B(t_i)/N))\} \\ & \quad \cdots \exp(tB(t_{i+1})/N) \cdots \exp(tB(t_N)/N). \end{aligned}$$

Now we estimate

$$\begin{aligned} & \exp(t(B(t_i) + \delta B(t_i))/N) - \exp(tB(t_i)/N)(I + (\delta B(t_i)t/N)) \\ &= \exp(tB(t_i)/N)\alpha(\text{ad}(tB(t_i)/N))(tB(t_i)/N) + o(t|\delta B(t_i)|/N) \\ &= \exp(tB(t_i)/N)\alpha(\text{ad}(tB(t_i)/N))(t\delta B(t_i)/N) + o(t(|\delta B(t_i)|/N)) \\ & \quad \text{where } \alpha(t) = 1 + (x/2) + (x^2/3!) + \cdots = (\exp(x) - 1)/x. \end{aligned}$$

So

$$q_{B+\delta B}(t) - q_B(t) - \int_0^t q_B(s)\delta B(s)q_B^{-1}(s) ds q_B(t) = o(\|\delta B\|).$$

COROLLARY 2.1. *The map $H: \Omega M(x_0) \rightarrow G$ is C^1 and its derivative has the form*

$$dH(\gamma_\varepsilon)/d\varepsilon|_{\varepsilon=0} = - \oint U_\tau F_\omega(\dot{\gamma}, \delta\gamma) U_\tau^{-1} d\tau H(\gamma)$$

where U_τ denotes the operation of parallel translation along γ to the fiber over $\gamma(\tau)$, F_ω is the curvature of ω . Here we loosely use the notation $\text{Ad}_U F(\dot{\gamma}, \delta\gamma) = UF(\dot{\gamma}, \delta\gamma)U^{-1}$, $\delta\gamma = \partial\gamma_\varepsilon/\partial\varepsilon|_{\varepsilon=0}$.

Proof. We work in local coordinates. Suppose γ is given by $t \rightarrow x(t)$ and the horizontal path covering γ is given by $t \rightarrow (x(t), q(t))$. Then $q(t)$ is the solution of

$$\dot{q}(t) = q(t)B(t);$$

here $B(t) = \dot{x}(t)A(x(t))$. So

$$\begin{aligned} \delta B(t) &= (\dot{x} + \delta\dot{x})(t)A(x(t) + \delta x(t)) - \dot{x}(t)A(x(t)) \\ &= (\dot{x} + \delta\dot{x})(t)(A(x(t)) + A_x(x(t))\delta x(t) + o(|\delta x|)) - \dot{x}(t)A(x(t)) \\ &= \dot{x}(t)A_x(x(t))\delta x(t) + \delta\dot{x}(t)A(x(t)) + o(|\delta x|^2 + |\delta\dot{x}|^2). \end{aligned}$$

Use Proposition 2.5,

$$\begin{aligned} &\int_0^t q(s)\delta B(s)q^{-1}(s) ds \\ &= \int_0^t q(s)(\dot{x}(s)A_x(x(s))\delta x(s) + \delta\dot{x}(s)A(x(s)) \\ &\quad + o(|\delta x|^2 + |\delta\dot{x}|^2))q^{-1}(s) ds \\ &= \int_0^t \{q(s)\dot{x}(s)A_x(x(s))\delta x(s)q^{-1}(s) \\ &\quad + q(s)\delta\dot{x}(s)A(x(s))q^{-1}(s)\} ds + o(\|\delta x\|^2) \\ &= \int_0^t \{q(s)\dot{x}(s)A_x(x(s))\delta x(s)q^{-1}(s) \\ &\quad - q(s)\dot{x}(s)A(x(s))\delta x(s)A(x(s))q(s) \\ &\quad - q(s)\delta x(s)A_x(x(s))\dot{x}(s)q^{-1}(s) \\ &\quad + q(s)\delta x(s)A(x(s))\dot{x}(s)A(x(s))q^{-1}(s)\} ds \\ &\quad + q(t)\delta x(t)A(x(t))q^{-1}(t) + o(\|\delta x\|^2) \\ &= \int_0^t q(s)F_\omega(\dot{x}(s), \delta x(s))q^{-1}(s) ds \\ &\quad + q(t)\delta x(t)A(x(t))q^{-1}(t) + o(\|\delta x\|^2). \end{aligned}$$

Now $\delta x(0) = \delta x(1) = 0$; hence the corollary follows.

REMARK. Since we have used the left-handed fiber bundle, the above formula is different from that of Montgomery [18].

Now we recall the definition of fat bundles (cf. Weinstein [25]): we say a bundle $F \rightarrow M$ with connection ω is fat if for every $a \in g^*$,

$a \neq 0$, and horizontal tangent vector $x \in T_z F$, $x \neq 0$, there is a $y \in T_z F$, such that

$$\langle a, F_\omega(x, y) \rangle \neq 0$$

where F_ω is the curvature of ω .

This definition is equivalent to that (cf. [4]): the set

$$\Phi = \{(x, p) \in T^*F, \langle p, H \rangle = 0\}$$

forms a symplectic submanifold in T^*F , where H is the horizontal subbundle of TF .

See Bergery [3] for a classification of homogeneous fat bundles with reductive structure group.

COROLLARY 2.2. *If the fiber bundle is fat (so that the horizontal bundle is strongly bracket generating), then the space of loops having the same holonomy $H_0 \neq e$, $\Omega M(x_0, H_0)$, is a smooth Hilbert-submanifold of $\Omega M(x_0)$. Also, if $H_0 = e$, then $\Omega M(x_0, e)$ is a smooth submanifold except at the constant loop.*

Proof. If the bundle is fat, then the map $H: \Omega M(x_0) \rightarrow g$ is submersive except at the constant loop, so the corollary follows.

2.3. Existence and regularity. In this subsection we will prove the existence of minimizers and smoothness of extremals.

PROPOSITION 2.6. *If $\Omega M(x_0, H_0)$ is not empty, then the minimizing elements of the energy functional E in $\Omega M(x_0, H_0)$ exist. If the fiber function is fat, then the extremals (including the minimizing elements) are smooth.*

Proof. Let γ_n be a minimizing sequence,

$$\lim_{n \rightarrow \infty} E(\gamma_n) = \inf E(\gamma).$$

By the weak compactness of the unit ball in a Hilbert space, there is a $\gamma_0 \in \Omega M(x_0)$ and a subsequence of $\{\gamma_n\}$ (which we still denote by $\{\gamma_n\}$) such that

$$\begin{aligned} \lim_{n \rightarrow \infty} \max d(\gamma_n(t), \gamma_0(t)) &= 0, \\ \lim_{n \rightarrow \infty} E(\gamma_n) &\geq E(\gamma_0), \end{aligned}$$

where d is the Carnot-Carathéodory metric on F , see the introduction. We only need to prove that $H(\gamma_0) = H_0$. Without loss of generality we work in local coordinates. Let γ_n be $t \rightarrow (x_n(t), q_n(t))$,

γ be $t \rightarrow (x_0(t), q_0(t))$; there is a subsequence (which we still denote by γ_n) such that

$$\int_0^t (\|\dot{x}_n(t) - \dot{x}_0(t)\| + \|x_n(t) - x_0(t)\|) dt \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

By Proposition 2.4, $q(t) = \lim_{n \rightarrow \infty} q_n(t)$. Hence $H(\gamma_0) = H_0$. So γ_0 is a minimizing element.

For the proof of regularity, we just remark that it is a standard result in O.D.E. (cf. [14]).

2.4. The Palais-Smale Condition. Let E_0 denote the restriction of E to $\Omega M(x_0, H_0)$, and $\langle\langle \cdot, \cdot \rangle\rangle_0$ the metric on $\Omega M(x_0, H_0)$ induced from the metric $\langle\langle \cdot, \cdot \rangle\rangle$ of $\Omega M(x_0)$. using the metric $\langle\langle \cdot, \cdot \rangle\rangle_0$ on $\Omega M(x_0, H_0)$ we define the corresponding vector field $\text{grad } E_0$. To establish Palais-Smale condition we must show that whenever $\{\gamma_n\}$ is a sequence on which E is bounded and for which $\|\text{grad}(e_0)_{\gamma_n}\|_0 \rightarrow 0$ when $n \rightarrow \infty$, $\{\gamma_n\}$ has a convergent subsequence. In this subsection we always assume that $H_0 \neq e$.

We will also use another topology on $\Omega M(x_0)$: the d_∞ -topology on $\Omega M(x_0)$ is defined by the metric

$$\rho(\gamma_1, \gamma_2) = \max_{t \in [0,1]} D(\gamma_1(t), \gamma_2(t))$$

where D is the distance on M .

Let us first examine what the above assumption on γ_n implies.

PROPOSITION 2.7. *Let $\{\gamma_n\} \subset \Omega M(x_0)$ be a sequence in $\Omega M(x_0)$ on which E is bounded. Then $\{\gamma_n\}$ has a subsequence converging in d_∞ to a continuous path γ on M .*

For a proof see [14].

The following three propositions will reduce the condition $\|\text{grad } E_0(\gamma_n)\| \rightarrow 0$ to $\|\text{grad } E(\gamma_n) + (a, \text{grad } H(\gamma_n))\| \rightarrow 0$ for some $a \in g^*$ (a refined method of Lagrangian multipliers).

PROPOSITION 2.8. *If $\{\gamma_n\}$ is a sequence for which*

$$\|\text{grad } E_0(\gamma)\|_0 \rightarrow 0 \quad \text{as } n \rightarrow \infty,$$

then there exist $a_n \in g^$ such that*

$$\|\text{grad } E(\gamma_n) + (\text{grad } H(\gamma_n), a_n)\| \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Proof. Let Σ_γ be the space of H^1 -vector fields X along γ such that

$$(dH(\gamma), X) = 0.$$

Then we have

$$(2.8) \quad \lim_{n \rightarrow \infty} (dE(\gamma_n), X) = 0,$$

for $X \in \Sigma_{\gamma_n}$, $\|X\| \leq 1$. Let $P_\gamma: T_\gamma \Omega M(x_0) \rightarrow \Sigma_\gamma$ be the orthogonal projection; then from (2.8) we have

$$\lim_{n \rightarrow \infty} (P_{\gamma_n} dE(\gamma_n), X) = \lim_{n \rightarrow \infty} (dE(\gamma_n), P_{\gamma_n} X) = 0$$

for $\|X\| \leq 1$. But $P_\gamma dE(\gamma) = dE(\gamma) - (a_\gamma, dH(\gamma))$ for some $a_\gamma \in g^*$, so the proposition is proved.

PROPOSITION 2.9. *Suppose that $F \rightarrow M$ is a fat bundle. If $\{\gamma\}$ is a sequence for which $\|\text{grad } E(\gamma_n)\|_0 \rightarrow 0$ as $n \rightarrow \infty$, then $\{\gamma_n\}$ has a subsequence (which we still denote by $\{\gamma_n\}$) and some $a \in g^*$ such that*

$$\|\text{grad } E(\gamma_n) + (\text{grad } H(\gamma_n), a)\| \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Proof. Let $\{a_n\}$ be as in Proposition 2.8. We first want to prove that $\{a_n\}$ is a bounded set in g^* . Assuming the contrary: $a_n \rightarrow \infty$, as $n \rightarrow \infty$, then

$$\|(\text{grad } E(\gamma_n)/\|a_n\|) + (\text{grad } H(\gamma_n), (a_n/\|a_n\|))\| \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Note that since M is compact, $\|\text{grad } E(\gamma)\|$ is bounded by $E(\gamma)$,

$$\|\text{grad } E(\gamma)\| \leq ME(\gamma),$$

from which we have

$$\|(\text{grad } H(\gamma_n), a_n/\|a_n\|)\| \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Now by Proposition 2.7 we can assume that $a_n/\|a_n\| \rightarrow a \neq 0$, and that $\gamma_n \rightarrow \gamma$, as $n \rightarrow \infty$, in the d_∞ -topology, to obtain

$$(\text{grad } H(\gamma), a) = 0,$$

which is in contradiction with the assumption that the bundle is fat. So $\{a_n\}$ is bounded. W.l.o.g. we assume that $a_n \rightarrow a$, as $n \rightarrow \infty$. Then from $\|\text{grad } H(\gamma)\| \leq ME(\gamma)$ we get

$$\begin{aligned} & \|\text{grad } E(\gamma_n) + (\text{grad } H(\gamma_n), a_n) - (\text{grad } E(\gamma_n) + (\text{grad } H_a(\gamma_n), a))\| \\ &= \|(\text{grad } H(\gamma_n), a_n - a)\| \rightarrow 0 \quad \text{as } n \rightarrow \infty. \end{aligned}$$

So the proof is completed.

By a nature chart around $\gamma \in \Omega M(x_0)$ we understand a chart defined as follows

$$\exp_\gamma: T_\gamma \Omega M(x_0) \rightarrow \Omega M(x_0)$$

is given by

$$\exp_\gamma(X)(t) = \exp_t(X(t))$$

for all $X \in T_\gamma \Omega M(x_0)$ and for all $t \in R$. Here $\exp_t: T_{\gamma(t)} M \rightarrow M$ is the Riemannian exponential map. \exp_γ is a diffeomorphism of a neighborhood around the zero-section 0 in $T_\gamma \Omega M(x_0)$ onto a neighborhood around $\gamma \in \Omega M(x_0)$, i.e. it is a chart on $\Omega M(x_0)$.

THEOREM 2.1. *If $F \rightarrow M$ is a fat bundle and $H_0 \neq e$, then the energy $E: \Omega M(x_0, H_0) \rightarrow R$ satisfies the Palais-Smale condition.*

Proof. Let $\{\gamma_n\}$ be a sequence in $\Omega M(x_0, H_0)$ on which E is bounded (say $E(\gamma_n) < k < \infty$, $n \in Z$) and for which $\|\text{grad } E_0(\gamma_n)\|_0 \rightarrow 0$ as $n \rightarrow \infty$. We want to show that $\{\gamma_n\}$ has a convergent subsequence.

By Proposition 2.9, we can assume w.l.o.g. that

$$\|\text{grad } E(\gamma_n) + (\text{grad } H_a(\gamma_n), a)\| \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Now by Proposition 2.7 we can assume that γ_n converges to a continuous loop γ in the d_∞ -topology.

From now on we work locally in a natural chart around γ . Let N_γ be an open neighborhood of the zero-section such that $\exp_\gamma: L^2(N_\gamma) \rightarrow \Omega M(x_0)$ is a natural chart around γ in $\Omega M(x_0)$, where $L^2(N_\gamma)$ is the L^2 -sections of $\gamma^* TM$ belonging to N_γ and vanishing at the end points, i.e., an open neighborhood of the zero section 0 in $T_\gamma \Omega M(x_0) = L^2(\gamma^* TM)_0$.

Let $X_n = \exp(\gamma_n)$ so that there is X_∞ such that $\|X_n - X_\infty\|_\infty \rightarrow 0$, as $n \rightarrow \infty$. Using the local expression for the energy (see [14]) it follows that $\|X_n\|$ is bounded. Furthermore, E is locally coercive (see [14])

$$\begin{aligned} & (dE(x) + (dH(X), a) - (dE(Y) + (dH(Y), a)))(X - Y) \\ & \geq \lambda \|X - Y\| - C \|X - Y\|_\infty \end{aligned}$$

for sufficiently small $\|X\|_\infty$ and $\|Y\|_\infty$. We have used E (respectively H) for $E \circ \exp_\gamma$ (respectively $H \circ \exp_\gamma$) which should cause no

confusion. Now

$$\lambda \|X_i - X_j\| \leq (dE(X_i) + (dH(X_i), a) - (dE(X_j) + (dH(X_j), a)))(X_i - X_j) + c \|X_i - X_j\|_\infty;$$

thus X_n is a Cauchy sequence and convergent. So $\{\gamma_n\}$ is convergent.

REMARK. The above proof also works for the punctured horizontal loop space $\Omega M(x_0, e) \cap E_\varepsilon^+$, where E_ε^+ is the space of loops γ with $E(\gamma) \geq \varepsilon$ (for any positive number ε).

3. Index theorem.

3.1. Index theorem on M . In this subsection we will prove an index theorem for a quadratic functional defined via a 2nd order differential-integral operator subject to a constraint. In particular, the second variation of E on $\Omega M(x_0, H_0)$ has the form considered here (cf. Appendix).

The quadratic form in question is $Q^\tau(x, x)$, where

$$Q^\tau(x_1, x_2) = \int_0^\tau \{(\dot{x}_1, \dot{x}_2) + (B(t)\dot{x}_1, x_2) + (Cx_1, x_2)\} dt + \left\langle b, \int_0^\tau dt \int_0^t [F(t)x_1(t), F(s)x_2(s)] ds \right\rangle$$

which is defined on the space $H^\tau = \{x: [0, \tau] \rightarrow R^m, x \text{ continuous and piecewise smooth, } x(0) = x(\tau) = 0, \int_0^\tau F(t)x(t) dt = 0\}$ where B, C are $m \times m$ matrices: $B^t = -B, C^t = C$; $(,)$ is the inner product on R^m . For $t \in [0, \tau]$, $F(t)$ is a linear map $R^m \rightarrow g$, $[,]$ is the Lie bracket of g , and \langle, \rangle is the dual bracket between g and g^* , and $b \in g^*$ is constant.

We always assume that $F(t): R^m \rightarrow g$ is a submersion for $t \in [0, \tau]$, which is essentially the fatness assumption (cf. Appendix).

We will also write $Q^\tau(x, x)$ as $Q^\tau(x)$ for simplicity.

Note we have the formula

$$(3.1) \quad Q(x_1 + x_2) = Q(x_1) + Q(x_2) + \int_0^\tau (Lx_1, x_2) dt - \sum x_2(t_i) \Delta \dot{x}_1(t_i)$$

where L is a 2nd order linear differential-integral operator

$$(Lx)(t) = -2(\partial^2 x(t)/\partial t^2) + (B(t)\partial x(t)/\partial t) + \partial(B(t)x(t))/\partial t + 2C(t)x(t) + 2(F(t))^* \text{ad}^* \int_0^t F(s)x(s) ds^b$$

and $\Delta\dot{x}(t)$ is the jump of \dot{x} at t

$$\Delta\dot{x}(t) = \dot{x}(t+0) - \dot{x}(t-0)$$

and the summation in (3.1) is taken over the t where $\dot{x}(t)$ is discontinuous.

Recall that the index of Q^τ is the maximal dimension of subspaces in H^τ on which Q^τ is negative definite.

To compute the index we take a finite dimensional approximation A^τ of H^τ in the following way:

First take a subdivision $0 = t_0 < t_1 < \dots < t_N = \tau$. Let A^τ be the vector space of such x satisfying

- (1) x is continuous on $[0, \tau]$;
- (2) x is smooth on (x_i, x_{i+1}) , $i = 0, 1, \dots, N-1$;
- (3) $x(0) = x(\tau) = 0$;
- (4) $Lx = F^*c$ ($c \in g^*$ is constant);
- (5) $\int_0^\tau F(t)x(t) dt = 0$.

The construction of A^τ is as follows: Let e_1, \dots, e_n be an orthonormal basis in g^* for an inner product $\langle \cdot, \cdot \rangle_{g^*}$ on g^* . Every x in A^τ can be written as

$$(3.2) \quad x = u_1 + b_1 u^1 + \dots + b_n u^n$$

where u_1 satisfies (1), (2) above, and

$$Lu_1 = 0, \quad u_1(t_i) = x(t_i), \quad i = 1, \dots, N-1;$$

and u^k satisfies (1), (2) above, and

$$Lu^k = F^*(e_k), \quad u^k(t_i) = 0, \quad i = 0, 1, \dots, N.$$

Also b_1, b_2, \dots, b_n are constants satisfying

$$(3.3) \quad \int_0^\tau (u_1 + b_1 u^1 + \dots + b_n u^n) dt = 0.$$

PROPOSITION 3.1. *Suppose that $t_{i+1} - t_i = \tau/N$, $i = 1, \dots, N-1$. Then if N is sufficiently big, u_1 is uniquely determined by $x(t_i)$, $i = 1, \dots, N-1$; and u^1, \dots, u^n are unique. Moreover, the $n \times n$ matrix*

$$\left(\left\langle \int_0^\tau F(s)u^i(s) ds, e_j \right\rangle_{g^*} \right)$$

is non-singular.

Proof. The first two statements are trivial; we need only to prove the last statement. By a rescaling of each interval $[t_i, t_{i+1}]$, we see that the function $v(t) = u^k(t\tau/N + t_i)$, $t \in [0, 1]$, satisfies $v(0) = v(1) = 0$ and

$$\frac{d^2v(t)}{dt^2} = -(\tau/N)^2 F^*(e_k)(t_i) + o(\tau/N)^2;$$

hence

$$u^k(t) = F^*(e_k)(t_i)(t - t_i)(t - t_{i+1}) + o(\tau/N)^2, \quad t_i \leq t \leq t_{i+1},$$

so

$$\begin{aligned} & \left(\left\langle \int_0^\tau F(s)u^i(s)ds, e_j \right\rangle_{g^*} \right) \\ &= -\frac{\tau^2}{6N^2} \left(\int_0^\tau \langle F^*(s)e_i(s), F^*(s)e_j(s) \rangle_{g^*} ds \right) + o(\tau/N)^2. \end{aligned}$$

Since $F^*: g^* \rightarrow R^m$ is an immersion, the first term above is a non-singular matrix; hence the proposition follows.

By the above result, we see that the map $\Gamma^\tau: A^\tau \rightarrow R^{(N-1)m}$

$$x \rightarrow (x(t_1), x(t_2), \dots, x(t_{N-1}))$$

is an isomorphism.

PROPOSITION 3.2. *Let the subdivision*

$$0 = t_0 < t_1 < \dots < t_N = \tau + \varepsilon$$

be sufficiently small. Then the isomorphism $\Gamma^{\tau+\varepsilon}$ depends smoothly on ε if ε is small enough with t_1, \dots, t_{N-1} fixed.

Proof. We need to prove that the solution to

$$Lu = F^*c, \quad u(0) = 0, \quad u(b) = a$$

is a continuous function of b for $b \neq 0$ sufficiently small (keeping F, a fixed), and this is a well-known fact in the theory of O.D.E.

We will see that A^τ is a “faithful” approximation of H^τ .

PROPOSITION 3.3. *H^τ splits as a direct sum:*

$$H^\tau = A^\tau + B^\tau$$

such that

$$(3.4) \quad Q^\tau(x_1 + x_2) = Q^\tau(x_1) + Q^\tau(x_2),$$

for $x_1 \in A^\tau$, $x_2 \in B^\tau$, and if the subdivision is small enough, then

$$Q^\tau(x) > 0 \quad \text{for } x \in B^\tau, \quad x \neq 0.$$

(Hence the index of Q^τ on H^τ is that of Q^τ on A^τ .)

Proof. B^τ is the space of functions $x \in H^\tau$ such that $x(t_i) = 0$, $i = 1, \dots, N$. Given any $x \in H^\tau$, there is a unique $u \in A^\tau$ such that

$$u(t_i) = x(t_i), \quad i = 0, 1, \dots, N.$$

Clearly $x - u \in B^\tau$. Thus the two subspaces A^τ and B^τ generate H^τ , and only have the zero function in common.

Then (3.3) follows from (3.1), and we need only to prove the positivity of Q^τ on B^τ . Now

$$Q^\tau(x) = \int_0^\tau \left\{ (\dot{x}, \dot{x}) + (B\dot{x}, x) + (Cx, x) + \left\langle b, \int_0^\tau dt \int_0^t [F(t)x(t), F(x)x(s)] ds \right\rangle \right\}.$$

The last term in the above expression is

$$\begin{aligned} & \left\langle b, \int_0^\tau dt \int_0^t [F(t)x(t), F(s)x(s)] ds \right\rangle \\ & \geq -M_0 \int_0^\tau dt |x(t)| \int_0^t |x(s)| ds \geq -M_0 \left(\int_0^\tau |x(t)| dt \right)^2 \\ & \geq -M_1 \int_0^\tau |x(t)|^2 dt. \end{aligned}$$

So

$$Q^\tau(x) \geq \int_0^\tau \{ (\dot{x}, \dot{x}) + (Bx, \dot{x}) + (Cx, x) - M|x|^2 \} dt = P^\tau(x).$$

Now we need only to prove that if τ is sufficiently small, $P^\tau(x) \geq 0$, for $x \in B^\tau$, and the equality holds only if $x = 0$. Using

$$\int_0^\tau \|x(t)\|^2 dt \leq \tau^2 \left(\int_0^\tau \|\dot{x}(t)\|^2 dt \right) / 2,$$

$$\begin{aligned}
P^\tau(x) &= \int_0^\tau \{(\dot{x}, \dot{x}) + (B\dot{x}, x) + (Cx, x) - M|x|^2\} dt \\
&\geq \int_0^\tau \{(\dot{x}, \dot{x}) - 2((B\dot{x}, B\dot{x})/k) + k(x, x) \\
&\quad + (Cx, x) - M|x|^2\} dt \\
&\geq \int_0^\tau \{(1 - (2\|B\|/k))(\dot{x}, \dot{x}) - (2k + \|C\| + M)(x, x)\} dt \\
&\geq \int_0^\tau \{(1 - (2\|B\|/k))/\tau^2 - (2k + \|C\| + M)\}(x, x) dt
\end{aligned}$$

where $\|B\| = \max \|B(t)\|$, $\|C\| = \max \|c(t)\|$, and k is taken to be $k = 4\|B\|$. Then when τ is sufficiently small,

$$d = 1/(2\tau^2) - 4\|B\| - \|C\| - M > 0,$$

$$P^\tau(x) \geq d \int_0^\tau (x, x) dt \geq 0,$$

and the equality only holds for $x = 0$.

Now the computation of the index of Q^τ on H^τ reduces to that of Q^τ on the finite-dimensional space A^τ . Next we define the notion of conjugate points as in Riemannian geometry.

DEFINITION 3.1. We say that τ_1 is conjugate to 0 if there is a smooth function $x: [0, \tau_1] \rightarrow R^m$ satisfying

- (1) $Lx = F^*c$ ($c \in g^*$ is constant);
- (2) $x(0) = x(\tau_1) = 0$;
- (3) $\int_0^{\tau_1} F(t)x(t) dt = 0$.

If τ_1 is conjugate to 0, then its multiplicity is the maximal number of linearly independent x satisfying (1), (2), (3).

Now we state the index theorem.

THEOREM 3.1. *The index of Q^τ on H^τ is equal to the number of conjugate points to 0 in $[0, \tau)$, counted together with their multiplicity.*

Proof. We will prove the theorem in four steps.

Step 1. Let $\mu(\tau)$ be the index of Q^τ on H^τ . Then for τ sufficiently small, $\mu(\tau) = 0$.

The proof of this is similar to that of Proposition 3.3 and we will not repeat.

Step 2. Let $\tau < \tau'$; then $\mu(\tau) \leq \mu(\tau')$.

In fact, by Definition 3.1, there is a subspace of H^τ , V of dimension $\mu(\tau)$ such that Q^τ is negative definite on V . Extend $x \in V$ by zero; then we can consider V as a subspace of $H^{\tau'}$. So $\mu(\tau') \geq \mu(\tau)$.

Step 3. If τ is not conjugate to 0, then for ε sufficiently small, $\mu(\tau + \varepsilon) = \mu(\tau)$.

If τ is conjugate to 0 with multiplicity k , then

$$\mu(\tau + \varepsilon) \leq \mu(\tau) + k.$$

We take subdivision $0 = t_0 < t_1 < \dots < t_N = \tau$. According to Proposition 3.3, $\mu(\tau)$ is the index of Q^τ on A^τ , which is isomorphic to $R^{(N-1)m}$. So we have a one-parameter family of quadratic forms (which we still denote by Q^τ) on $R^{(N-1)m}$ depending smoothly on τ . Then Q^τ is positive definite on a subspace of codimension $\mu(\tau) + k$ in $R^{(N-1)m}$, and so is $Q^{\tau+\varepsilon}$ if ε is small enough. So $\mu(\tau + \varepsilon) \leq \mu(\tau) + \varepsilon$.

Step 4. We want to prove that if τ is conjugate to 0 with multiplicity k , then $\mu(\tau + \varepsilon) \geq \mu(\tau) + k$.

Now there are linearly independent $w_1, \dots, w_{\mu(\tau)} \in H^\tau$ such that Q^τ is negative on them.

Assume w.l.o.g. that $u_1, \dots, u_c \in H^\tau$ (respectively $v_1, \dots, v_d \in H^\tau$, $c + d = k$) such that

$$Lu_i = F^*c_i, \quad i = 1, \dots, c;$$

and $\dot{u}_1(\tau), \dots, \dot{u}_c(\tau)$ are linearly independent (respectively $Lv_i = F^*e_i$, $\dot{v}_i(\tau) = 0$, $i = 1, \dots, d$). Then there are $x_1, \dots, x_c \in H^{\tau+\varepsilon}$, such that the $c \times c$ matrix

$$((\dot{u}_i(\tau), x_j(\tau)))$$

is the identity matrix. Also there are $y_1, \dots, y_d \in H^{\tau+\varepsilon}$ such that the $d \times d$ matrix

$$\left(\int_0^\tau (F^*e_i, y) dt \right)$$

is the identity matrix, and $y_i(\tau) = 0$, $i = 1, \dots, d$.

From this and (3.2) we immediately get

$$\begin{aligned} Q^{\tau+\varepsilon}(u_i, x_j) &= -\delta_{ij}, & i = 1, \dots, c; j = 1, \dots, c; \\ Q^{\tau+\varepsilon}(v_i, y_j) &= -\delta_{ij}, & i = 1, \dots, d, j = 1, \dots, d; \\ Q^{\tau+\varepsilon}(u_i, y_j) &= Q^{\tau+\varepsilon}(v_j, x_i) = 0, & i = 1, \dots, c, j = 1, \dots, d. \end{aligned}$$

Now $Q^{\tau+\varepsilon}$ is negative on $w_1, \dots, w_{\mu(\tau)}$, $\frac{1}{\nu}u_1 + \nu x_1, \dots, \frac{1}{\nu}u_c + \nu x_c$, $\frac{1}{\nu}v_1 + \nu y_1, \dots, \frac{1}{\nu}v_d + \nu y_d$ for sufficiently small ν . In fact, the

compound matrix of $Q^{\tau+\varepsilon}$ with respect to these vector fields is

$$\begin{vmatrix} (Q^\tau(w_i, w_j)) & \nu C \\ \nu D & -I + \nu^2 B \end{vmatrix}$$

where B, C, D are matrix independent of ν . Obviously, if ν is sufficiently small, the above matrix is negative definite. Hence $\mu(\tau + \varepsilon) \geq \mu(\tau) + c + d = \mu(\tau) + k$.

3.2. Index Theorem on F . We now come to the index theorem for Problem 1.

Suppose that $\gamma_0 \in \Omega F(x_1, x_2)$ is a critical point of E on $\Omega F(x_1, x_2)$. The Morse index of E at γ_0 is the maximal dimension of subspaces of $T_{\gamma_0} \Omega F(x_1, x_2)$ on which the quadratic differential E_{**} is negative definite. We want to express the Morse index in terms of the conjugate points as in Riemannian geometry.

DEFINITION 3.2. We say that x_2 is conjugate to x_1 along γ_0 if there is a smooth one-parameter family of horizontal paths $\gamma_\varepsilon \in \Omega F(x_1^\varepsilon, x_2^\varepsilon)$ such that for each ε , γ_ε is a horizontal extremal of E on $\Omega F(x_1^\varepsilon, x_2^\varepsilon)$, and the vector field on γ_0

$$\partial(\gamma_\varepsilon)/\partial\varepsilon|_{\varepsilon=0}$$

is not identically zero and vanishes at the end points x_1, x_2 . The index of x_2 along γ_0 is the maximal number of one-parameter families of horizontal extremals $\gamma_\varepsilon^1, \dots, \gamma_\varepsilon^m$, such that the vector fields along γ_0

$$\partial(\gamma_\varepsilon^1)/\partial\varepsilon|_{\varepsilon=0}, \dots, \partial(\gamma_\varepsilon^m)/\partial\varepsilon|_{\varepsilon=0}$$

are linear independent and vanish at the end points x_1, x_2 .

THEOREM 3.2. *Suppose that the bundle is fat and γ_0 is an extremal of E on $\Omega F(x_1, x_2)$. Then the Morse index of E on $\Omega F(x_1, x_2)$ at γ_0 is the number of conjugate points to x_1 along γ_0 , counted together with their multiplicity (x_2 should not be counted).*

Proof. Project γ_0 down to a path γ_1 on M . Then the Morse index of E at γ_0 is just that of the quadratic form defined via the 2nd order differential-integral operator (cf. Appendix)

$$\begin{aligned} LX &= D^2 X/dt^2 + R(V, X)V - Q(t)(\nabla_J F_\omega)(X,) \\ (*) \quad &- Q(t)F_\omega(DX/dt,) \\ &- U(t)\text{ad}^*_{\int_0^t U F_\omega(V, X)U^{-1} ds} Q(0)F_\omega(V,)U^{-1}(t) \end{aligned}$$

(where $V(t) = \dot{\gamma}(t)$) defined on the subspace of vector fields along γ_0 :

$$H = \left\{ X, \int_0^1 UF_\omega(V, X)U^{-1} dt = 0 \right\}.$$

By Theorem 3.1, the index of the quadratic form is just the number of conjugate points (in the sense of Definition 3.1) to $\gamma_1(0)$ along γ_1 , counted together with their multiplicity. So we only need to prove that $\gamma_0(\tau)$ is conjugate to $\gamma_0(0)$ with multiplicity m along γ_0 (in the sense of Definition 3.2) if and only if $\gamma_1(\tau) = \pi\gamma_0(\tau)$ is conjugate to $\gamma_1(0)$ with multiplicity m along $\pi\gamma_0$ (in the sense of Definition 3.1).

Obviously, if a family of horizontal extremals γ_ε on F is such that $X_0 = \partial\gamma/\partial\varepsilon|_{\varepsilon=0} \neq 0$, then $X = \pi_*\partial\gamma/\partial\varepsilon|_{\varepsilon=0} \neq 0$ satisfies $LX = QUF_\omega(V,)U^{-1}$, and if X_0 vanishes at the end points, then X satisfies

$$\int_0^1 UF_\omega(V, X)U^{-1} dt = 0.$$

We need only to show that every X along γ_1 satisfying $LX - QUF_\omega(V,)U^{-1}$, and if X_0 vanishes at the end points, then X satisfies

$$\int_0^1 UF_\omega(V, X)U^{-1} dt = 0.$$

We need only to show that every X along γ_1 satisfying

$$LX = QUF_\omega(V,)U^{-1}$$

can be given in this way: there is a family of horizontal extremals γ_ε such that $X = \pi_*\partial\gamma/\partial\varepsilon|_{\varepsilon=0}$. To see this, define map $e_x: T_xF \rightarrow F$ by

$$(3.5) \quad e_x(v) = \exp(-\omega(y)) \exp_x(v)$$

where $v = y + z \in T_xF$, y is tangent to the fiber, z is horizontal, $\exp_x: T_xF \rightarrow F$ is the exponential map of the Kaluza-Klein metric on F , and \exp is the exponential map of G . We will write e for e_x for simplicity. Then γ_0 can be written as $t \rightarrow e(tv_0)$ (Montgomery [18]) for some $v \in T - xF$. Moreover, for t_0 sufficiently small, e_ε is regular at v_0t_0 (this will be proved in §3.3). So there is a family of v_ε depending smoothly on ε , such that the differential of e at t_0v_0 ,

$$e_*(v_\varepsilon) = \int_0^{t_0} UF_\omega(V,)U^{-1} ds e(t_0v_0) + X_1$$

where X_1 is horizontal satisfying $\pi_*X_1 = X(t_0)$. Now define a family

of horizontal extremals γ_ε by

$$\gamma_\varepsilon(t) = e(tv_\varepsilon).$$

Then $\pi_*\partial\gamma/\partial\varepsilon|_{\varepsilon=0} = X$. Hence the theorem is proved.

3.3. The existence of non-conjugate points. Fixing $x_0 \in F$, we will prove that almost every point on F is not conjugate to $x_0 \in F$ along any horizontal extremals.

Let $e_x: T_xF \rightarrow F$ be the map defined by (3.5). It is known that for $v \in T_xF$, $\gamma(t) = ex(tv)$ is an extremal of the energy E on $\Omega F(x, x_1)$, where $x_1 = e_x(v)$. Conversely, if the bundle is fat, then every extremal γ of E on $\Omega F(x, x_1)$ can be written in the form $\gamma(t) = e_x(tv)$ for some $v \in T - xF$ (cf. [18], [20]).

PROPOSITION 3.4. *Suppose that $F \rightarrow M$ is a fat bundle. Then $x_1 = e_x(v)$ is conjugate to x (with $x \neq x_1$) along $\gamma(t) = e_x(tv)$ if and only if the map e_x is singular at v .*

Proof. Suppose that e_x is singular at $v_0 \in T_xF$. Then $e_{x^*}(v_1) = 0$ for some non-zero $v_1 \in T_xF$.

Let $s \rightarrow v(s)$ be a path in T_xF such that $v(0) = v_0$, and $dv(0)/ds|_{s=0} = v_1$. Then the one-parameter family of paths defined by

$$\gamma_s(t) = e_x(tv(s))$$

is a variation through geodesics of the geodesic γ_0 . Therefore the vector field W given by $t \rightarrow \partial(e_x(tv(s)))/\partial s|_{s=0}$ is an extremal field along γ_0 . Obviously $W(0) = 0$. We also have

$$W(1) = e_x(v(s))/\partial s|_{s=0} = e_{x^*}(v_1) = 0.$$

Now, we need only to show that W is not identically zero. If $W = 0$, then (we write $v_1 = y + z$, y is tangent to the fiber, z is horizontal)

$$DW/dt|_{t=0} = z$$

is zero, and also

$$D^2(\pi_*W)/dt^2|_{t=0} = (y, F_\omega(v_0, \cdot)) = 0.$$

Since the bundle is fat, either $z = 0$, $y = 0$, $v_1 = y + z = 0$, or $v_0 = 0$ in which case $x_1 = x$. Either case is a contradiction.

Using Sard's theorem, we get the following result, which has been used in the proof of Theorem 3.2.

COROLLARY 3.1. *Fixing $x_0 \in F$, then for almost every point x_1 of F , x_1 is not conjugate to x_0 along any extremals.*

3.4. The existence of arbitrary nearby conjugate points. In [20] it is proved that for $x \in F$, there are cut points arbitrarily near x . In this subsection we prove a slightly stronger result:

PROPOSITION 3.5. *There are conjugate points arbitrarily near x .*

Proof. Assume the contrary. Consider the sphere $B_\varepsilon = \{y: d(y, x) = \varepsilon\}$ and the set $e_x^{-1}(B_\varepsilon)$ in $T_x F$, where d is the Carnot-Carathéodory distance on F . Let $H_\varepsilon = e_x^{-1}(B_\varepsilon) \cap \{v \in T_x F, |\pi_* v| = \varepsilon\}$. Then the map

$$e_x: H_\varepsilon \rightarrow B_\varepsilon$$

is an onto map. And it is everywhere regular, hence is a covering map. But H_ε is a set in the cylinder $S^m \times R^n$, and it is impossible for a subset in the cylinder to cover the sphere S^{m+n} , hence a contradiction.

4. Horizontal loop spaces.

4.1. An abstract model. Let l^2 be the Hilbert space

$$l^2 = \left\{ x = (a_1, a_2, \dots), |x| = \left(\sum |a_i|^2 \right)^{1/2} \right\}$$

and Pl^2 is projection. Suppose that we have a smooth map $f: l^2 \rightarrow R^n$, where $f = f_0 + f_1$ satisfies

(1) $f_0 = (f_1^0, \dots, f_n^0)$ is a quadratic map and a submersion everywhere except at $x = 0 \in l^2$, i.e. for $(b_1, \dots, b_n) \in R^n - \{0\}$, $x \neq 0$

$$(4.1) \quad b_1 df_1^0(x) + \dots + b_n df_n^0(x) \neq 0;$$

(2) f_1 consists of higher terms

$$(4.2) \quad |f_1(x)| \leq M|x|^3;$$

(3) f is a submersion except at $x = 0 \in l^2$.

The set $f^{-1}(0)$ is smooth everywhere except at 0, and 0 is the singularity. In this subsection we will resolve this singularity.

As in the finite dimensional case, we define the local charts for Pl^2 to be

$$\phi_i: l^2 \rightarrow Pl^2,$$

$$\phi_i(\alpha_1, \dots, \alpha_{i-1}, \alpha_{i+1}, \dots) = \{\alpha_1, \dots, \alpha_{i-1}, 1, \alpha_{i+1}, \dots\},$$

and transition maps are

$$\begin{aligned} \phi_j^{-1} \circ \phi_i(\alpha_1, \dots, \alpha_{i-1}, \alpha_{i+1}, \dots) \\ = (\alpha_1/\alpha_j, \dots, \alpha_{i-1}/\alpha_j, 1/\alpha_j, \alpha_{i+1}/\alpha_j, \dots) \quad (\alpha_j \neq 0). \end{aligned}$$

Note that $\Phi_j^{-1} \circ \Phi_i$ can be written as

$$\Phi_j^{-1} \circ \Phi_i = ab$$

where b is a map $b: l^2 \rightarrow l^2$

$$b(\alpha_1, \alpha_2, \dots) = (\alpha_1, \dots, \alpha_{i-1}, 1, \alpha_{i+1}, \dots)$$

and $a: l^2 \cap \{\alpha_j \neq 0\} \rightarrow R$

$$a(\alpha_1, \alpha_2, \dots) = 1/\alpha_j.$$

Since a and b are smooth, so is $\Phi_j^{-1} \circ \Phi_i$.

Let $B: Bl^2 \rightarrow l^2$ be the real blow-up of l^2 at 0. Recall that Bl^2 is the set in $Pl^2 \times l^2$ defined by the quadratic relations:

$$\begin{aligned} Bl^2 = \{(\{\alpha_1, \alpha_2, \dots\}, (\beta_1, \beta_2, \dots)) \in Pl^2 \times l^2, \\ \alpha_i \beta_j = \alpha_j \beta_i, \quad i, j = 1, 2, \dots\} \end{aligned}$$

and B is the restriction of the projection $\text{pr}: Pl^2 \times l^2 \rightarrow l^2$ to Bl^2 .

DEFINITION 4.1. Define the blow-up of $f^{-1}(0)$, $B_{f=0}$, to be the closure of the set $B^{-1}(f^{-1}(0) - \{0\})$ in Bl^2 , and B^f to be the restriction of B to $B_{f=0}$

$$B^f: B_{f=0} \rightarrow f^{-1}(0).$$

THEOREM 4.1. (1) $B_{f=0}$ is a smooth manifold;

(2) $B^f: B^{-1}(f^{-1}(0) - \{0\}) \rightarrow f^{-1}(0) - \{0\}$ is a diffeomorphism;

(3) $(B^f)^{-1}(0)$ is an embedding of a projective variety into $B_{f=0}$ as a submanifold of codimension one. This projective variety is defined by $f_0(x) = 0$.

Proof. (1) Let $(\{\alpha_1, \alpha_2, \dots\}, (\beta_1, \beta_2, \dots))$ be the coordinates on $Pl^2 \times l^2$. Define a map A on $(Pl^2 \cap \{\alpha_1 \neq 0\}) \times l^2$ by means of the local chart Φ_1 in (4.3):

$$\begin{aligned} A: (Pl^2 \cap \{\alpha_1 \neq 0\}) \times l^2 &\rightarrow l_\infty \times R^n, \\ A(\{1, \alpha_2, \alpha_3, \dots\}, (\beta_1, \beta_2, \dots)) \\ &= ((\beta_2 - \alpha_2 \beta_1, \beta_3 - \alpha_3 \beta_1, \dots), \\ &\quad (f_0(1, \alpha_2, \alpha_3, \dots) + f_1(\beta_1, \alpha_2 \beta_1, \alpha_3 \beta_1, \dots)/\beta_1^2)) \end{aligned}$$

where l_∞ is the Banach space

$$l_\infty = \{x = (c_1, c_2, \dots), |x| = \max |c_i| < \infty\}.$$

Obviously A is smooth and submersive, so $A^{-1}(0)$ is a smooth manifold. Note that

$$A^{-1}(0) \cap \{(\beta_1, \beta_2, \dots) \neq 0\} = B^{-1}(f^{-1}(0) - \{0\}) \cap \{\alpha_1 \neq 0\},$$

and $A^{-1}(0) \cap \{(\beta_1, \beta_2, \dots) = 0\}$ is a submanifold of codimension 1 in $A^{-1}(0)$. Hence after taking the closure of $B^{-1}(f^{-1}(0) - \{0\})$, we have

$$A^{-1}(0) = B_{f=0} \cap \{\alpha_1 \neq 0\}.$$

So $B_{f=0}$ is smooth.

(2) This follows from the fact that $B: Bl^2 - B^{-1}\{0\} \rightarrow l^2 - \{0\}$ is a diffeomorphism.

(3) Obviously

$$\begin{aligned} (B^f)^{-1}(0) \cap \{\alpha_1 \neq 0\} \\ = \{(\{1, \alpha_2, \dots\}, (0, 0, \dots)) \in Pl^2 \times l^2 \mid f_0(1, \alpha_2, \dots) = 0\}; \end{aligned}$$

hence $(B^f)^{-1}(0)$ is diffeomorphic to the projective variety:

$$f_0(\alpha_1, \alpha_2, \dots) = 0.$$

Intersecting $f^{-1}(0)$ with the ε -sphere $S_\varepsilon = \{x \in l^2, |x| = \varepsilon\}$, we get

$$M_\varepsilon = f^{-1}(0) \cap S_\varepsilon.$$

Then $T_\varepsilon = (B^f)^{-1}(\bigcup_{\rho \leq \varepsilon} M_\rho)$ is a tubular neighborhood of $(B^f)^{-1}(0)$ with boundary $(B^f)^{-1}(M_\varepsilon)$. Note that for sufficiently small ε , the function

$$h: B_{f=0} \rightarrow R, \quad h(y) = |B^f y|^2$$

has no critical points in T_ε other than $(B^f)^{-1}(0)$, takes the minimal value on $(B^f)^{-1}(0)$, and is constant on $(B^f)^{-1}(M_\varepsilon)$, so we get

PROPOSITION 4.1. *For sufficiently small ε , m_ε is a double covering of $(B^f)^{-1}(0)$.*

REMARK. The above arguments also show that $\bigcup_{\rho \leq \varepsilon} M_\rho$ is contractible.

Next we consider $g^{-1}(0)$, where g is a smooth function

$$g: [0, 1]^m \times l^2 \rightarrow R^n, \quad g = g_0 + g_1.$$

(1) $g(t, 0) = 0$ and g is submersive except at $(t, 0) \in [0, 1]^m \times l^2$.

(2) For fixed $t \in [0, 1]^m$, $g_0(t, x)$, $g_1(t, x)$ satisfy (4.1), (4.2) respectively.

$g^{-1}(0)$ is smooth except at $[0, 1]^m \times \{0\}$, and we are going to blow up $g^{-1}(0)$ along $[0, 1]^m \times \{0\}$.

Let $\overline{B}: [0, 1]^m \times Bl^2 \rightarrow [0, 1]^m \times l^2$ be the blow-up along $[0, 1]^m \times \{0\}$.

DEFINITION 4.2. Define the blow-up of $g^{-1}(0)$ along $[0, 1]^m \times \{0\}$, $B_{g=0}$, to be the closure of $(\overline{B})^{-1}(g^{-1}(0) - [0, 1]^m \times \{0\})$ in $[0, 1]^m \times Pl^2$, and B^g be the restriction of \overline{B} to $B_{g=0}$

$$B^g: B_{g=0} \rightarrow g^{-1}(0).$$

PROPOSITION 4.2. (1) $B_{g=0}$ is a smooth manifold;

(2) $B^g: B_{g=0} - (B^g)^{-1}([0, 1]^m \times \{0\}) \rightarrow g^{-1}(0) - ([0, 1]^m \times \{0\})$ is a diffeomorphism;

(3) $(B^g)^{-1}([0, 1]^m \times \{0\})$ is an embedding of $\{(t, x) \in [0, 1]^m \times Pl^2, g_0(t, x) = 0\}$ into $B_{g=0}$ as a submanifold of codimension 1.

The proof is the same as that of Theorem 4.1.

4.2. Horizontal loop spaces. We first study the singularity of the based horizontal loop space $\Omega F(x_0, x_0)$ using the results of §4.1. Then we study the free horizontal loop space. From now on we assume that F is a fat bundle.

Since the singularity is of local character, we choose local coordinates on F and may identify $\overline{F}(x_0, x_0)$ with the subset $\overline{R}^m(0, e)$ in $\overline{R}^m(0)$, satisfying $H(\gamma) = e$; here $\overline{R}^m(0)$ is the Hilbert space

$$\left\{ \begin{array}{l} \gamma: [0, 1] \rightarrow R^m, \quad \gamma(0) = \gamma(1) = 0, \\ \|\gamma\| = \left(\int_0^1 (|\gamma|^2 + |\dot{\gamma}|^2) dt \right)^{1/2} < \infty \end{array} \right\}.$$

In terms of the local coordinates, we can compute the Hessian of the holonomy map at the constant loop 0.

PROPOSITION 4.3. $(d^2H(0), (u, v)) = \int_0^1 F_{\omega 0}(u, \dot{v}) dt$ where $F_{\omega 0}$ is the curvature at $0 \in M$.

There is a two-step nilpotent Lie algebra which is independent of the local coordinates. This construction is well known to those working in the subelliptic operators (cf. [20], [22]).

DEFINITION 4.3. $n_0 = V_1 + V_2$ is the graded two step nilpotent Lie algebra defined by

$$\begin{aligned}(x, y) \in V_1 \times V_1 &\rightarrow F_{\omega_0}[x, y] \in V_2, \\(x, y) \in V_1 \times V_2 &\rightarrow 0, \\(x, y) \in V_2 \times V_2 &\rightarrow 0.\end{aligned}$$

Now we see that H can be written as

$$H(\gamma) = \int_0^1 F_{\omega_0}(\gamma, \dot{\gamma}) dt + b(\gamma)$$

where b is a smooth map $b: \Omega R^m(0) \rightarrow g$ satisfying

$$|b(\gamma)| < M\|\gamma\|^3.$$

Since the bundle F is fat, the quadratic function $\int_0^1 F_{\omega_0}(\gamma, \dot{\gamma}) dt$ satisfies (4.1). Note that l^2 is isomorphic to $\Omega R^m(0)$, applying Theorem 4.1 and Proposition 4.1, we get

THEOREM 4.2. *There is a smooth map B^H and a smooth manifold $B_{H=e}$, called the blow-up of $\Omega M(0, e)$,*

$$B^H: B_{H=e} \rightarrow \Omega M(0, e)$$

such that

- (1) $B^H: B_{H=e} - (B^H)^{-1}\{0\} \rightarrow \Omega M(0, e) - \{0\}$ is a diffeomorphism;
- (2) $(B^H)^{-1}(0)$ is an embedding of the projective variety $P_{F,0}$ defined by

$$\int_0^1 F_{\omega_0}(\gamma, \dot{\gamma}) dt = 0,$$

into $B_{H=e}$ as a submanifold of codimension 1.

- (3) Let $m_\varepsilon = \Omega M(0, e) \cap \{\gamma: E(\gamma) = \varepsilon\}$. Then for ε sufficiently small, M_ε is a double covering of $(B^H)^{-1}(0)$.

REMARK. If N_{x_0} is Lie algebra isomorphic to n_{x_1} , then P_{F, x_0} differs from P_{F, x_1} by an affine transformation. In particular, this shows that the blow-up is independent of the local coordinates.

Now we turn to the free horizontal loop space on F , denoted by $\Omega_H F$. It is smooth except at constant loops. We want to resolve the singularities.

Let $F \subset \Omega_H F$ be embedded as constant loops. We blow up $\Omega_H F$ along F as in Proposition 4.2. Then we get

PROPOSITION 4.4. *There is a blow-up of $\Omega_H F$ along F , denoted by $B\Omega_H F$,*

$$B_H: B\Omega_H F \rightarrow \Omega_H F$$

such that

- (1) $B_H, B\Omega_H F$ are smooth;
- (2) $B_H: B\Omega_H F - B_H^{-1}(F) \rightarrow \Omega_H F - F$ is a diffeomorphism;
- (3) $B_H^{-1}(F)$ is an embedded submanifold of codimension 1 in $B\Omega_H F$.

To describe $B_H^{-1}(F)$, we look at the free loop space ΩF , and the normal bundle of F in ΩF , denoted by n_F . The fibers of N_F are isomorphic to the vector space $\Omega R^{m+n}(0)$. Let the multiplicative group $R - \{0\}$ act on $\Omega R^{m+n}(0) = \Omega R^m(0) \times \Omega R^n(0)$ by

$$(t, (\gamma_1, \gamma_2)) \rightarrow (t\gamma_1, t^2\gamma_2),$$

and $P_w \Omega R^{m+n}(0)$ be the quotient space. Let $P_w N_F$ be the corresponding quotient bundle of N_F . Then $B_H^{-1}(F)$ can be written as

$$\left\{ (z, \{\gamma_1, \gamma_2\}) \in P_w N_F, \gamma_2(t) = \int_0^t F_{\omega z}(\gamma_1, \dot{\gamma}_1) ds, \int_0^1 F_{\omega z}(\gamma_1, \dot{\gamma}_1) ds = 0 \right\}.$$

REMARK. Take local coordinates on F . Then locally $B_H^{-1}(F)$ can be written as

$$\left\{ (z, H, \{\gamma\}) \in U \times G \times P\Omega R^m(0), \int_0^1 F_{\omega z}(\gamma, \dot{\gamma}) ds = 0 \right\},$$

which is isomorphic to

$$\left\{ (z, H, \{\gamma\}) \in U \times G \times PH^1(S^1, R^m), \int_0^1 \gamma ds = 0, \int_0^1 F(\gamma, \dot{\gamma}) ds = 0 \right\}.$$

Taking the Fourier series, then the above space is isomorphic to

$$\left\{ (z, H, \{\alpha_1, \alpha_2, \dots\}) \in U \times G \times Pl^2, \sum_{k \text{ odd}}^{\infty} F_{\omega z}((\alpha_{kn+1}, \dots, \alpha_{kn+n}), (\alpha_{kn+n+1}, \dots, \alpha_{kn+2n})) = 0 \right\}.$$

Then $B_H^{-1}(F)$ is the total space of a fiber bundle over F if $\dim(G) = 1$. $B_H^{-1}(F)$ might still be the total space of a fiber bundle over F in general, but we are not able to prove this.

Let $\widetilde{M}_\varepsilon$ be the set of those free horizontal loops γ with $E(\gamma) = \varepsilon$; then using the same method as in Proposition 4.1, we arrive at

PROPOSITION 4.5. *For ε sufficiently small, $\widetilde{M}_\varepsilon$ is a double covering of $B_H^{-1}(F)$.*

The S^1 action on $\Omega_H F$ (rotate the loops) lifts to $B\Omega_H F - B_H^{-1}(F)$ then extends smoothly to $B\Omega_H F$.

PROPOSITION 4.6. *There is a smooth S^1 -action on $B\Omega_H F$ with respect to which the blow-up map B_H is equivariant.*

Proof. It suffices to show that the action of S^1 on $B\Omega_H F - B_H^{-1}(F)$ extends smoothly to $B_H^{-1}(F)$.

We take local coordinates on F . Then locally $B\Omega_H F$ is a smooth submanifold of $G \times (\Omega R^m / \sim) \times \Omega R^m$. Here $\Omega R^m / \sim$ is the quotient space of ΩR^m (the space of free H^1 loops on R^m) by the equivalence relation “ \sim ”: we say that $\gamma_1 \sim \gamma_2$ if and only if

$$\gamma_1(s) - \gamma_1(0) = k(\gamma_2(s) - \gamma_2(0))$$

for a non-zero constant k . The S^1 -action on ΩR^m obviously respects this equivalence relation and projects down to a smooth action on $\Omega R^m / \sim$ and so for $G \times (\Omega R^m / \sim) \times \Omega R^m$. Restrict the S^1 -action on $G \times (\Omega R^m / \sim) \times \Omega R^m$ to $B\Omega_H F$, which is what we want.

5. Application to closed geodesics. In this section we will apply the results obtained in previous sections to the study of closed geodesics.

By a closed geodesic we mean a critical point of E on $\Omega M(x_0, e)$ for some $x_0 \in M$. Now we fix an $x_0 \in F$. Since the space $\Omega M(x_0, e)$ has a singularity at the constant loop, we cannot apply Morse theory directly to $\Omega M(x_0, e)$. To overcome this difficulty, we will consider either the “punctured” space $\Omega M_\varepsilon(x_0, e) = \Omega M(x_0, e) \cap E_\varepsilon^+$, where E_ε^+ is the space of loops γ with $E(\gamma) \geq \varepsilon$, or the blow-up of the horizontal loop spaces.

PROPOSITION. *For sufficiently small $\varepsilon > 0$, there is no non-trivial closed geodesic γ with $E(\gamma) < \varepsilon$.*

Proof. We will use the asymptotics of the exponential map $e_x(tv)$ as $t \rightarrow 0$ (cf. [20]). Let (x, z) denote local coordinates in F , $v = (v_1, v_2) \in T_{(x_0, z_0)}F = \mathbb{R}^m \times \mathbb{R}^n$, $(y, z) = e_x(v_1, v_2)$. Then

$$\begin{vmatrix} y \\ z \end{vmatrix} = \begin{vmatrix} x_0 \\ z_0 \end{vmatrix} + \begin{vmatrix} I & 0 \\ 0 & A(v) \end{vmatrix} \begin{vmatrix} v_1 \\ v_2 \end{vmatrix} + \begin{vmatrix} o(|v|^2) \\ o(|v|^4) \end{vmatrix}$$

where A is a (matrix valued) homogeneous polynomial of degree 2. It is proved in [20] that

$$\det A(v) \neq 0.$$

If there is a sequence of $v_n \neq 0$ such that $e_x(v_n) = x = (y_0, z_0)$, and $v_n \rightarrow 0$ as $n \rightarrow \infty$, then there is a $v_0 = (v_1, v_2) \neq 0$ such that $v_1 = 0$, $A(v_0)v_2 = 0$, which is in contradiction with $\det A(v_0) \neq 0$.

From now on we assume that ε is chosen so that there is no non-trivial closed geodesic with energy less than ε .

PROPOSITION. *Suppose that E is a non-degenerate Morse function on $\Omega M(x_0, e)$, and the Morse indices of the closed geodesics are $m_1 \leq m_2 \leq m_3 \leq \dots$. Then the Morse polynomial*

$$M(t) = \sum t^{m_k}$$

satisfies $M(t) \geq P(t)$ where

$$P(t) = \sum \dim H^k(\Omega M(x_0, e)) t^k.$$

Proof. We first apply Morse theory to $\Omega M_\varepsilon(x_0, e)$. Then we have

$$M(t) \geq P_\varepsilon(t)$$

where

$$P_\varepsilon(t) = \sum (\dim H^k(\Omega M(x_0, e), \Omega M(x_0, e) - \Omega M_\varepsilon(x_0, e))) t^k.$$

On the other hand, $\Omega M(x_0, e) - \Omega M_\varepsilon(x_0, e)$ is contractible (cf. the remark after Proposition 4.1), so $P(t) = P_\varepsilon(t)$.

REMARK. One can also use the blow-up of $\Omega_H F$ and its S^1 -equivariant homology to study the closed geodesics, as in [5].

The following lemma gives the number of connected components of $\Omega M(x_0, e)$.

PROPOSITION. *The number of connected components of $\Omega M(x_0, e)$ is that of the elements of $\pi_1(F)$.*

Proof. Let γ_1 be a loop on F , $\gamma_1(0) = x_0$. We first want to deform γ_1 to a horizontal loop γ on F also satisfying $\gamma(0) = x_0$.

Choose a subdivision $0 = t_1 < \dots < t_N = 1$ such that the distance (with respect to a Riemannian metric on F) of each point $\gamma(t)$ from $\gamma(t_i)$ is less than ε for $t \in [t_i, t_{i+1}]$. Connect each pair $\gamma(t_i), \gamma(t_{i+1})$ by one of the horizontal geodesics, say, $\gamma_i: [t_i, t_{i+1}] \rightarrow F$. Then by the estimates in [20], each $\gamma_i(t)$ is within a distance $M\varepsilon^{1/2}$ from $\gamma(t_i)$. If ε is small enough, $\gamma|_{[t_i, t_{i+1}]}$ and γ_i are in the same connected component of the space of horizontal paths connecting $x(t_i)$ to $x(t_{i+1})$. Now define a horizontal loop γ by

$$\gamma(t) = \gamma_i(t), \quad t \in [t_i, t_{i+1}];$$

then γ is homotopic to γ_1 .

Now we prove that if $\gamma_1, \gamma_2 \in \Omega F(x_0, x_0)$ are homotopic, then they are in the same connected component of $\Omega F(x_0, x_0)$. Let γ_ε be a family of loops connecting γ_1, γ_2 , and then choose a subdivision $0 = s_0 < \dots < s_m = 1$ such that

$$d(\gamma_{s_i}, \gamma_{s_{i+1}}) < \varepsilon,$$

where ε is a small positive number. Then for each i , $0 \leq i \leq m$, we can choose a broken horizontal loop γ'_i homotopic to γ_i such that

$$d(\gamma_i(t), \gamma'_i(t)) < \varepsilon.$$

We want to prove that γ'_i and γ'_{i+1} are in the same connected component of $\Omega F(x_0, x_0)$. In fact, if ε is sufficiently small, $D(\gamma'_i, \gamma'_{i+1}) < 2\varepsilon$, since $\Omega F(x_0, x_0)$ is locally path connected by Theorem 4.2, γ'_i, γ'_{i+1} are in the same connected component of $\Omega F(x_0, x_0)$. Hence $\gamma'_0 = \gamma_0, \gamma'_m = \gamma_1$ are in the same connected component of $\Omega F(x_0, x_0)$.

REMARK. Using the regularity result of Bar [29] for geodesics, we can show that the above result remains true even if the horizontal bundle H only satisfies Hörmander's condition.

As an application, consider the three-dimensional Heisenberg group H_3 and let $\gamma \subset H_3$ be a discrete subgroup with k free generators such that H_3/Γ is compact. It is well known that there is a left invariant contact structure on H_3 (cf. [15], [19]). Take a Γ -invariant

Riemannian metric on H^3 and we can get the Γ -invariant Carnot-Carathéodory metric on H_3/Γ . By Proposition 4.9 then there are at least k distinct closed geodesics on H_3/Γ .

Appendix. The extremal fields. In this appendix we will derive the second variation of E on $\Omega M(x_0, H_0)$. Since the extremals are given by Wong's equation ([18]), we need only to study the linearization of Wong's equation. We will continue to use the left-handed fiber bundle.

According to [18], Wong's equation is

$$\nabla_{\dot{\gamma}} \dot{\gamma} = Q(t)F_{\omega}(\dot{\gamma}, \cdot)$$

where $Q(t) = U(t)Q(0)U^{-1}(t)$, $U(t)$ is the parallel translation along the path $\gamma(t)$.

Now suppose we have a one-parameter family of solutions γ_{ε} ,

$$(1) \quad \nabla_{\dot{\gamma}_{\varepsilon}} \dot{\gamma}_{\varepsilon} = Q^{\varepsilon}(t)F_{\omega}(\dot{\gamma}_{\varepsilon}, \cdot)$$

where $Q^{\varepsilon}(t) = U_{\varepsilon}(t)Q^{\varepsilon}(0)U_{\varepsilon}^{-1}(t)$, U_{ε} is the parallel translation along the path γ_{ε} . We want to find the equation that the vector field along γ , $J(t) = \partial \gamma_{\varepsilon}(t)/\partial \varepsilon|_{\varepsilon=0}$, has to satisfy. We have

$$(2) \quad \left. \frac{D}{d\varepsilon} \right|_{\varepsilon=0} \nabla_{\dot{\gamma}_{\varepsilon}} \dot{\gamma}_{\varepsilon} = \left. \frac{D}{d\varepsilon} \right|_{\varepsilon=0} Q^{\varepsilon}(t)F_{\omega}(\dot{\gamma}_{\varepsilon}, \cdot).$$

As is well known, the left-hand side of the above is

$$(3) \quad (\partial^2 J/dt^2) + R(V, J)V$$

where $V(t) = \dot{\gamma}(t)$ and R is the Riemannian curvature on the base manifold M . Now we compute the right-hand side of (2) (we will omit $\varepsilon = 0$ for simplicity).

$$\begin{aligned} \partial(Q^{\varepsilon}(t)F_{\omega}(V, \cdot))/\partial \varepsilon &= (DQ^{\varepsilon}(t)/\partial \varepsilon)F_{\omega}(V, \cdot) \\ &\quad + Q(t)(D_J F_{\omega})(V, \cdot) + Q(t)F_{\omega}(J, \cdot) \end{aligned}$$

and

$$\begin{aligned} \partial Q^{\varepsilon}(t)/\partial \varepsilon &= U^{-1}(\text{ad}^*_{\int_0^t U(s)F_{\omega}(V, J)U^{-1}(s)ds} Q(0))U \\ &\quad + U(t)(dQ(0))/d\varepsilon U^{-1}(t) \end{aligned}$$

here we have used the formula in §2.3 that

$$\partial U^{\varepsilon}(t)/\partial \varepsilon = \int_0^t U(t)F_{\omega}(V(t), J(t))U^{-1}(t) dt$$

modulo the term which is independent of J . Then we see that the linearization of Wong's equation (and the Jacobi field for the variational problem) is the 2nd order differential-integral operator

$$(D^2 J/dt^2) + R(V, J)V - Q(t)(\nabla_J F_\omega)(V, \cdot) - Q(t)F_\omega((DJ/dt), \cdot) - U(t)\text{ad}^* \int_0^t U F_\omega(V, J)U^{-1} ds Q(0)F_\omega(V, \cdot)U^{-1}(t) = 0.$$

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