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CLASSIFICATION OF ESSENTIAL COMMUTANTS OF ABELIAN VON NEUMANN ALGEBRAS

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The main purpose of this paper is to classify the C^*-algebras of the form $\mathfrak{A}' + \mathscr{K}$, where $\mathfrak{A}'$ denotes the commutant of an abelian von Neumann algebra $\mathfrak{A}$, and $\mathscr{K}$ is the set of compact operators. By the famous result of Johnson and Parrott, $\mathfrak{A}' + \mathscr{K}$ is the same as the essential commutant of $\mathfrak{A}$. These algebras were studied by Plastiras in the special case in which $\mathfrak{A}$ is generated by its minimal projections and in addition all of these projections are finite dimensional. Using a theorem of Andersen, we are able to generalize Plastiras' main results to general abelian von Neumann algebras. We also study the automorphism groups and derivations of these algebras.

If $\mathfrak{A}$ is an abelian von Neumann algebra, then its projection lattice $\mathscr{L}$ is a complemented commutative subspace lattice, and of course $\mathfrak{A}' = \mathscr{L}'$. Since our results are given in terms of the lattice, we simply start with such a lattice $\mathscr{L}$ and consider the algebra $\mathscr{L}' + \mathscr{K}$. In Corollaries 6 and 9, we give necessary and sufficient conditions for two such algebras $\mathscr{L}' + \mathscr{K}$ and $\mathscr{M}' + \mathscr{K}$ to be equal or isomorphic. We then turn to automorphisms, and first catagorize those algebras for which every unitary operator implementing an automorphism splits (Theorem 11), and then determine those algebras for which every automorphism is inner (Theorem 12). These four results generalize the most important results in **[P]**. We next calculate the outer automorphism group (Corollary 20), and finally show that every derivation of such an algebra is inner (Theorem 22). This latter portion of the paper was motivated primarily by similar studies with nests, namely **[W1]**, **[W2]**, and **[DW]**.

All Hilbert spaces in this paper will be separable and infinite dimensional, and will usually be denoted by $\mathscr{H}$. $\mathscr{B}(\mathscr{H})$ will be used to denote the set of bounded operators on $\mathscr{H}$, and the set of compact operators will be given by $\mathscr{K}(\mathscr{H})$, or just $\mathscr{K}$ if the Hilbert space is clear from the context. If $\mathscr{S} \subseteq \mathscr{B}(\mathscr{H})$, then $\mathscr{S}' = \{T \in \mathscr{B}(\mathscr{H}) : TS = ST \text{ for all } S \in \mathscr{S}\}$ is the *commutant* of $\mathscr{S}$, and $\mathscr{S}''$ denotes the *double commutant* $(\mathscr{S}')'$ of $\mathscr{S}$. The *essential commutant* of $\mathscr{S}$ is $\{T \in \mathscr{B}(\mathscr{H}) : TS - ST \in \mathscr{K} \text{ for all } S \in \mathscr{S}\}$. All projections on Hilbert space will be self-adjoint.

A *commutative subspace lattice* (CSL) is a commuting set of projections on $\mathscr{H}$ which is closed under the lattice operations $P \vee Q = P+Q-PQ$ and $P \wedge Q = PQ$, is closed in the strong operator topology, and contains 0 and I. A projection $E \in \mathscr{L}$ is an *atom* if $EP = 0$ or E for all $P \in \mathscr{L}$. Note that all atoms are mutually orthogonal. $\mathscr{L}$ is *nonatomic*, or *continuous*, if it has no atoms, and *purely atomic* if $\sum E_i = I$, where the sum is taken over all atoms E_i of $\mathscr{L}$ and convergence is in the strong operator topology. As is often done in the study of commutative subspace lattices, we will sometimes blur the distinction between a projection P and its range $P\mathscr{H}$. In this spirit, we will use $\dim P$ to denote the dimension of $P\mathscr{H}$. Also, a projection is *infinite* or *finite* if its range is infinite or finite dimensional, respectively. Define $\mathscr{A}(\mathscr{L}) = \{\text{atoms of } \mathscr{L}\}$, $\mathscr{I}(\mathscr{L}) = \{\text{infinite atoms of } \mathscr{L}\}$, and $\mathscr{F}(\mathscr{L}) = \{\text{finite atoms of } \mathscr{L}\}$, and let $i(\mathscr{L})$ denote the cardinality of $\mathscr{I}(\mathscr{L})$. Let $P_a = \sum_{A \in \mathscr{A}(\mathscr{L})} A$ and define $P_c = I - P_a$. Then $\mathscr{L}_a \subseteq \mathscr{B}(P_a\mathscr{H})$ is defined to be the purely atomic CSL $\mathscr{L}|_{P_a\mathscr{H}}$ and likewise $\mathscr{L}_c \subseteq \mathscr{B}(P_c\mathscr{H})$ is the nonatomic CSL $\mathscr{L}|_{P_c\mathscr{H}}$. Finally, we define the operator $\delta_{\mathscr{L}}$ by $\delta_{\mathscr{L}}(T) = \sum E_i T E_i$, where again the sum is taken over all atoms in $\mathscr{L}$ and convergence is in the strong operator topology. If $\mathscr{L}$ is purely atomic, then $T \in \mathscr{L}'$ if and only if $T = \delta_{\mathscr{L}}(T)$, and $T \in \mathscr{L}' + \mathscr{K}$ if and only if $T - \delta_{\mathscr{L}}(T) \in \mathscr{K}$ by [W3, Lemma 5] (the proof given in [W3] is for a special case, but it works for the general case as well).

If V is a unitary operator, then $\operatorname{Ad} V$ is the operator defined by $(\operatorname{Ad} V)(A) = VAV^*$. An isomorphism α of two C^*-algebras $\mathfrak{A}$ and $\mathfrak{B}$ is *spatial* if $\alpha = \operatorname{Ad} V$ for some V. Let $\operatorname{Aut}(\mathfrak{A}) = \{\text{automorphisms of } \mathfrak{A}\}$ and $\operatorname{Inn}(\mathfrak{A}) = \{\text{inner automorphisms of } \mathfrak{A}\} = \{\operatorname{Ad} V : V \in \mathfrak{A}\}$, and let the outer automorphism group $\operatorname{Aut}(\mathfrak{A})/\operatorname{Inn}(\mathfrak{A})$ be denoted by $\operatorname{Out}(\mathfrak{A})$. In addition, we will use s-$\operatorname{Aut}(\mathfrak{A})$ to represent the group of spatial automorphisms of $\mathfrak{A}$. The notation $\cong$ will be used for both unitary equivalence and group isomorphism as appropriate.

A CSL $\mathscr{L}$ is a *nest* if it is linearly ordered by the usual ordering of range inclusion, and it is *complemented* if $P^{\perp} \in \mathscr{L}$ for all $P \in \mathscr{L}$. A complemented CSL is the same as a complete Boolean algebra of projections, and the projection lattice of every abelian von Neumann algebra is a complemented CSL. On the other hand, if $\mathscr{L}$ is a CSL, then $\mathscr{L}''$ is an abelian von Neumann algebra whose projection lattice is the complemented subspace lattice $\mathscr{C}(\mathscr{L})$ generated by $\mathscr{L}$. Our focus in this paper is the C^*-algebra $(\mathscr{L}'')' + \mathscr{K} = (\mathscr{C}(\mathscr{L}))' + \mathscr{K} = \mathscr{L}' + \mathscr{K}$, and although this work was motivated by the study of nests, our results will be stated in terms of the projection lattice $\mathscr{C}(\mathscr{L})$ of

$\mathscr{L}''$. Thus, one can start with a nest $\mathscr{L}$ or any other type of CSL, and simply replace $\mathscr{L}$ by $\mathscr{C}(\mathscr{L})$. This will be a common practice in the sequel. $\mathscr{L}' + \mathscr{K}$ is the essential commutant of $\mathscr{L}''$ by [**JP**, Theorem 2.1], so one can also view this as the study of essential commutants of abelian von Neumann algebras.

We first mention two well-known results which we will use later.

LEMMA 1. *If $\mathscr{L}$ is nonatomic CSL, then $\mathscr{L}' \cap \mathscr{K} = \{0\}$.*

Proof. By [**Ar**, p. 482], there is a continuous nest $\mathscr{M} \subseteq \mathscr{L}$ such that $\mathscr{M}' = \mathscr{L}'$. The proof can then be completed by applying the argument given in [**W3**, Theorem 14] to $\mathscr{M}$. □

LEMMA 2. *If $\mathscr{L}$ and $\mathscr{M}$ are CSL's and α is a C^*-isomorphism of $\mathscr{L}' + \mathscr{K}$ onto $\mathscr{M}' + \mathscr{K}$, then α is spatial.*

Proof. The result follows by a slight modification of the argument given in [**JP**, Lemma 4.5]. □

DEFINITION 3. Suppose $\mathscr{L}$ is a complemented CSL and P is a finite projection in $\mathscr{L}'$. We define $\mathscr{L}^P$ to be the complemented CSL generated by P and $\{P^\perp L : L \in \mathscr{L}\}$, and we say that $\mathscr{L}^P$ is a *finite perturbation* of $\mathscr{L}$ (this usage differs slightly from that in [**D2**] and [**DW**], but the idea is the same). Note that $(\mathscr{L}^P)' + \mathscr{K} = \mathscr{L}' + \mathscr{K}$. Also, $P \leq P_a$ by Lemma 1, and therefore $\mathscr{L}_c^P = \mathscr{L}_c$.

Our main tool for analyzing isomorphisms and automorphisms of $\mathscr{L}' + \mathscr{K}$ is the following theorem of Andersen.

THEOREM 4 [**An**, *Proposition* 2.3.3]. *If $\mathscr{L}$ and $\mathscr{M}$ are complemented CSL's and $\mathscr{L}' + \mathscr{K} = \mathscr{M}' + \mathscr{K}$, then there are finite projections $P \in \mathscr{L}'$, $Q \in \mathscr{M}'$, and a unitary U with $U - I \in \mathscr{K}$ such that $U\mathscr{L}^P U^* = \mathscr{M}^Q$ and $UPU^* = Q$.*

It follows that $U(\mathscr{L}^P)''U^* = (\mathscr{M}^Q)''$ and $U(\mathscr{L}^P)'U^* = (\mathscr{M}^Q)'$.

LEMMA 5. *The projections P and Q in Theorem 4 can be chosen to be sums of finite atoms and subprojections of infinite atoms.*

Proof. This lemma is very similar to [**DW**, Lemma 3.3]. Let $P_0 \in \mathscr{L}'$, $Q_0 \in \mathscr{M}'$, and $U = I +$ compact be given by Theorem 4 so that $U(\mathscr{L}^{P_0})U^* = \mathscr{M}^{Q_0}$ and $UP_0U^* = Q_0$. We may write $P_0 = \sum_{i=1}^{m} A_i + A$ and $Q_0 = \sum_{j=1}^{n} B_j + B$, where each $A_i(B_j)$ is dominated by a finite atom $E_i(F_j)$ and A and B are each dominated by a sum of infinite atoms. Then $E_iA_i^{\perp}$ and $F_jB_j^{\perp}$ are finite atoms of $\mathscr{L}^{P_0}$ and $\mathscr{M}^{Q_0}$, respectively. Let $C_i = E_iA_i^{\perp}$, $i = 1, \dots, m$, and $C_{j+m} = U^*(F_jB_j^{\perp})U$, $j = 1, \dots, n$. Setting $P = P_0 + \sum_{i=1}^{m+n} C_i$ and $Q = Q_0 + \sum_{i=1}^{m+n} UC_iU^*$, it follows easily that P and Q have the desired form, $U(\mathscr{L}^{P})U^* = \mathscr{M}^{Q}$, and $UPU^* = Q$. □

For convenience, we will say that a finite projection which is a sum of finite atoms and subprojections of infinite atoms is *σ-finite*. We can now characterize when two such algebras $\mathscr{L}' + \mathscr{K}$ and $\mathscr{M}' + \mathscr{K}$ are equal or isomorphic.

COROLLARY 6. *Suppose $\mathscr{L}$ and $\mathscr{M}$ are complemented CSL's. Then $\mathscr{L}' + \mathscr{K} = \mathscr{M}' + \mathscr{K}$ iff there is a unitary U with $U - I \in \mathscr{K}$ and σ-finite projections $P \in \mathscr{L}'$ and $Q \in \mathscr{M}'$ such that*

(i) *$U\mathscr{L}_cU^* = \mathscr{M}_c$, and*
(ii) *$U\mathscr{L}_a^PU^* = \mathscr{M}_a^Q$ with $UPU^* = Q$.*

Proof. Necessity follows immediately from Theorem 4 and Lemma 5. On the other hand, given U, P, and Q satisfying (i) and (ii), we have $U(\mathscr{L}^P)U^* = \mathscr{M}^Q$ since $\mathscr{L}_c^P = \mathscr{L}_c$ and $\mathscr{M}_c^Q = \mathscr{M}_c$. Then $(\mathscr{M}^Q)' = U(\mathscr{L}^P)'U^* \subseteq (\mathscr{L}^P)' + \mathscr{K}$ and $(\mathscr{L}^P)' = U^*(\mathscr{M}^Q)'U \subseteq (\mathscr{M}^Q)' + \mathscr{K}$ since $U - I \in \mathscr{K}$. The result now follows because $(\mathscr{L}^P)' + \mathscr{K} = \mathscr{L}' + \mathscr{K}$ and $(\mathscr{M}^Q)' + \mathscr{K} = \mathscr{M}' + \mathscr{K}$. □

If $\mathscr{L}$ and $\mathscr{M}$ are purely atomic with no infinite atoms, then [**P**, Corollary 14(i)(ii)] follows immediately.

COROLLARY 7 [**P**, *Corollary* 14]. *Suppose $\mathscr{L}$ and $\mathscr{M}$ are purely atomic CSL's with atoms $\{E_i\colon 1 \le i < \infty\}$ and $\{F_i\colon 1 \le i < \infty\}$, respectively, such that* $\dim E_i$, $\dim F_i < \infty$ *for all i. Then $\mathscr{L}' + \mathscr{K} = \mathscr{M}' + \mathscr{K}$ iff there is a unitary U with $U - I \in \mathscr{K}$, finite subsets $N_0, N_1 \subseteq \mathbb{N}$, and a bijection $\pi\colon \mathbb{N} \setminus N_0 \to \mathbb{N} \setminus N_1$ such that $UE_iU^* = E_{\pi(i)}$ for all $i \in \mathbb{N} \setminus N_0$ and $U(\sum_{i \in N_0} E_i)U^* = \sum_{j \in N_1} F_j$.*

Proof. Simply apply Corollary 6 to the complemented lattices generated by $\mathscr{L}$ and $\mathscr{M}$, letting $P = \sum_{i \in N_0} E_i$ and $Q = \sum_{j \in N_1} F_j$. □

LEMMA 8. *Suppose $\mathscr{L}$ and $\mathscr{M}$ are complemented CSL's and V is a unitary operator such that $V(\mathscr{L}'+\mathscr{K})V^* = \mathscr{M}'+\mathscr{K}$. Then there is a unitary U with $U - I \in \mathscr{K}$ and σ-finite projections $P \in \mathscr{L}'$, $Q \in \mathscr{M}'$ such that*

(i) $(\operatorname{Ad} UV)(\mathscr{L}_c) = \mathscr{M}_c$, *and*

(ii) $(\operatorname{Ad} UV)(\mathscr{L}_a^P) = \mathscr{M}_a^Q$ *with* $(\operatorname{Ad} UV)(P) = Q$.

Proof. $\mathscr{M}'+\mathscr{K} = V(\mathscr{L}'+\mathscr{K})V^* = V\mathscr{L}'V^*+\mathscr{K} = (V\mathscr{L}V^*)'+\mathscr{K}$, so Corollary 6 can be applied to $\mathscr{M}$ and $V\mathscr{L}V^*$. Thus, there is a unitary U with $U - I \in \mathscr{K}$ and σ-finite projections $R \in (V\mathscr{L}V^*)'$ and $Q \in \mathscr{M}'$ such that $U(V\mathscr{L}V^*)_c U^* = \mathscr{M}_c$ and $U((V\mathscr{L}V^*)_a^R)U^* = \mathscr{M}_a^Q$ with $URU^* = Q$. Now $(V\mathscr{L}V^*)_c = V\mathscr{L}_cV^*$, so $(\operatorname{Ad} UV)(\mathscr{L}_c) = \mathscr{M}_c$. Also, $(V\mathscr{L}V^*)_a^R = V\mathscr{L}_a^{V^*RV}V^*$, so $(\operatorname{Ad} UV)(\mathscr{L}_a^{V^*RV}) = \mathscr{M}_a^Q$ with $(UV)V^*RV(UV)^* = Q$. Since V^*RV is a σ-finite projection in $\mathscr{L}'$, the result follows by letting $P = VRV^*$. □

By Lemma 2, C^*-isomorphism is the same as unitary equivalence for essential commutants of abelian von Neumann algebras, so the next corollary specifies when two such algebras are isomorphic.

COROLLARY 9. *Suppose $\mathscr{L}$ and $\mathscr{M}$ are complemented CSL's. Then $\mathscr{L}' + \mathscr{K} \cong \mathscr{M}' + \mathscr{K}$ iff*

(i) $\mathscr{L}_c \cong \mathscr{M}_c$,

(ii) $i(\mathscr{L}) = i(\mathscr{M})$, *and*

(iii) *there are finite subsets $\mathscr{P} \subseteq \mathscr{F}(\mathscr{L})$ and $\mathscr{Q} \subseteq \mathscr{F}(\mathscr{M})$ and a bijection $\tau\colon \mathscr{F}(\mathscr{L}) \setminus \mathscr{P} \to \mathscr{F}(\mathscr{M}) \setminus \mathscr{Q}$ such that* $\dim A = \dim \tau(A)$ *for all $A \in \mathscr{F}(\mathscr{L}) \setminus \mathscr{P}$. In addition, if $i(\mathscr{L}) = i(\mathscr{M}) = 0$, then* $\dim(\sum_{A\in\mathscr{P}} A) = \dim(\sum_{B\in\mathscr{Q}} B)$.

Proof. Suppose V is a unitary operator such that $V(\mathscr{L}'+\mathscr{K})V^* = \mathscr{M}' + \mathscr{K}$. By Lemma 8, there is a unitary U with $U - I \in \mathscr{K}$ and σ-finite projections $P \in \mathscr{L}'$, $Q \in \mathscr{M}'$ such that $(\operatorname{Ad} UV)(\mathscr{L}_c) = \mathscr{M}_c$, $(\operatorname{Ad} UV)(\mathscr{L}_a^P) = \mathscr{M}_a^Q$, and $(\operatorname{Ad} UV)(P) = Q$. Then $\mathscr{L}_c \cong \mathscr{M}_c$ via UV, proving (i), and $\operatorname{Ad} UV$ is a bijection of $\mathscr{A}(\mathscr{L}_a^P)$ onto $\mathscr{A}(\mathscr{M}_a^Q)$ such that $(\operatorname{Ad} UV)(P) = Q$. (ii) now follows since $i(\mathscr{L}) = i(\mathscr{L}_a^P)$ and $i(\mathscr{M}) = i(\mathscr{M}_a^Q)$. Finally, let $\mathscr{P} = \{A \in \mathscr{A}(\mathscr{L})\colon A \leq P\}$ and $\mathscr{Q} = \{A \in \mathscr{A}(\mathscr{M})\colon A \leq Q\}$. Then $\mathscr{F}(\mathscr{L}^P) = \{P\} \cup (\mathscr{F}(\mathscr{L}) \setminus \mathscr{P})$ and $\mathscr{F}(\mathscr{M}^Q) = \{Q\} \cup (\mathscr{F}(\mathscr{M}) \setminus \mathscr{Q})$. Now just let $\tau = (\operatorname{Ad} UV)|_{\mathscr{F}(\mathscr{L})\setminus\mathscr{P}}$, and (iii) follows (note that if $i(\mathscr{L}) = 0 = i(\mathscr{M})$, then $P = \sum_{A\in\mathscr{P}} A$ and $Q = \sum_{B\in\mathscr{Q}} B$).

Conversely, suppose (i)–(iii) hold. If $\mathscr{L} \subseteq \mathscr{B}(\mathscr{H}_1)$ and $\mathscr{M} \subseteq \mathscr{B}(\mathscr{H}_2)$, let $W: \mathscr{H}_1 \to \mathscr{H}_2$ be a partial isometry such that $W\mathscr{L}_c W^* = \mathscr{M}_c$. Let $\{I_i: 1 \le i \le i(\mathscr{L})\}$ and $\{J_i: 1 \le i \le i(\mathscr{M}) = i(\mathscr{L})\}$ be enumerations of $\mathscr{I}(\mathscr{L})$ and $\mathscr{I}(\mathscr{M})$, respectively. Let $X: \mathscr{H}_1 \to \mathscr{H}_2$ be a partial isometry such that $XAX^* = \tau(A)$ for all $A \in \mathscr{F}(\mathscr{L}) \setminus \mathscr{P}$. Now if $i(\mathscr{L}) = i(\mathscr{M}) = 0$, let $Y: \mathscr{H}_1 \to \mathscr{H}_2$ be a partial isometry such that $Y(\sum_{A\in\mathscr{P}} A)Y^* = \sum_{B\in\mathscr{Q}} B$, and define $V = W + X + Y$. Otherwise, let $m = \dim(\sum_{A\in\mathscr{P}} A) - \dim(\sum_{B\in\mathscr{Q}} B)$. If $m < 0$, let $C \le I_1$ be a projection of dimension $|m|$, and let $D = 0$. If $m > 0$, let $D \le J_1$ be a projection of dimension m, and let $C = 0$. If $m = 0$, let $C = D = 0$. Define $P = C + \sum_{A\in\mathscr{P}} A$ and $Q = D + \sum_{B\in\mathscr{Q}} B$. Now define $Y: \mathscr{H}_1 \to \mathscr{H}_2$ to be a partial isometry such that $YPY^* = Q$, $YI_iY^* = J_i$ for all $i \ge 2$, and $Y(I_1 - C)Y^* = J_1 - D$, and define $V = W + X + Y$. Then in either case $V\mathscr{L}^P V^* = \mathscr{M}^Q$, and it follows that $V(\mathscr{L}^P)'V^* = (\mathscr{M}^Q)'$ and therefore $V(\mathscr{L}' + \mathscr{K})V^* = \mathscr{M}' + \mathscr{K}$. □

If $\mathscr{L}$ or $\mathscr{M}$ is not complemented, then the same result holds with item (i) changed to $\mathscr{C}(\mathscr{L}_c) \cong \mathscr{C}(\mathscr{M}_c)$, simply by applying the corollary to $\mathscr{C}(\mathscr{L})$ and $\mathscr{C}(\mathscr{M})$. Also, the result for the special case considered in **[P]** again follows immediately.

COROLLARY 10 [**P**, *Theorem* 1]. *Suppose* $\mathscr{L}$ *and* $\mathscr{M}$ *are purely atomic CSL's with atoms* $\{E_i: 1 \le i < \infty\}$ *and* $\{F_i: 1 \le i < \infty\}$, *respectively, such that* $\dim E_i, \dim F_i < \infty$ *for all* i. *Then* $\mathscr{L}' + \mathscr{K} \cong \mathscr{M}' + \mathscr{K}$ *iff there are finite subsets* $N_0, N_1 \subseteq \mathbb{N}$ *and a bijection* $\pi: \mathbb{N} \setminus N_0 \to \mathbb{N} \setminus N_1$ *such that* $\dim E_i = \dim E_{\pi(i)}$ *for all* $i \in \mathbb{N} \setminus N_0$ *and* $\dim(\sum_{i\in N_0} E_i) = \dim(\sum_{j\in N_1} F_j)$.

Corollary 9 shows that isomorphism is determined by simple conditions on the atoms and unitary equivalence of the nonatomic parts of $\mathscr{L}$ and $\mathscr{M}$. But the latter is equivalent to unitary equivalence of the nonatomic von Neumann algebras $\mathscr{L}_c''$ and $\mathscr{M}_c''$. Let $\mathscr{H}_c = L^2([0,1], \mathscr{B}, \mu)$, where $\mathscr{B}$ is the σ-algebra of Borel sets and μ is Lebesgue measure. If $f \in L^\infty([0,1], \mathscr{B}, \mu)$, let M_f be the multiplication operator acting on $\mathscr{H}_c$. For $1 \le m \le \infty$, define $\mathscr{Z}^{(m)} = \{M_f \oplus \cdots \oplus M_f: f \in L^\infty([0,1], \mathscr{B}, \mu)\}$ acting on $\mathscr{H}_c^{(m)}$, the direct sum of m copies of $\mathscr{H}_c$. Then each nonatomic abelian von Neumann algebra $\mathfrak{A}$ is unitarily equivalent to $\sum_{m\in\mathscr{J}}^{\oplus} \mathscr{Z}^{(m)}$, where $\mathscr{J}$ is a uniquely determined subset of $\{1, 2, \dots, \infty\}$ [**KR**, §9.4]. We will call $\mathscr{J}$ the *multiplicity sequence of* $\mathfrak{A}$. Thus, $\mathscr{L}_c \cong \mathscr{M}_c$ if and only if $\mathscr{L}_c''$ and $\mathscr{M}_c''$ have the same multiplicity sequences.

Another result in **[P]** characterizes certain lattices $\mathscr{L}$ for which every unitary V implementing an automorphism of $\mathscr{L}'+\mathscr{K}$ *splits*. This means that $V = W + K$ where W is a unitary satisfying $W\mathscr{L}'W^* = \mathscr{L}'$ and $K \in \mathscr{K}$. Equivalently, $V = W_1W_2$ for some unitaries W_i satisfying $W_1\mathscr{L}'W_1^* = \mathscr{L}'$ and $W_2 - I \in \mathscr{K}$ (just let $W_1 = W$ and $W_2 = W^*V$). If $\mathscr{L}$ is a multiplicity-free CSL ($\mathscr{L}' = \mathscr{L}''$ is maximal abelian), then every such unitary splits **[JP**, Theorem 4.10]. However, as shown below, there are also other cases for which this is true. In the following proof, $\operatorname{ind}(T)$ denotes the Fredholm index of an operator T.

THEOREM 11. *Let $\mathscr{L}$ be a CSL. Define*

$$c(k) = \operatorname{card}\{A \in \mathscr{A}(\mathscr{L})\colon \dim A = k\}.$$

Then every unitary which implements an automorphism of $\mathscr{L}' + \mathscr{K}$ splits if and only if either

(i) $c(\infty) = 0$ *and* $c(k) = \infty$ *for at most one* k, *or*
(ii) $c(\infty) = 1$ *and* $c(k) < \infty$ *for all* k.

Proof. By replacing $\mathscr{L}$ with the complemented lattice generated by $\mathscr{L}$, we can assume without loss of generality that $\mathscr{L}$ is complemented. Suppose (i) holds, and suppose that V is a unitary satisfying $V(\mathscr{L}' + \mathscr{K})V^* = \mathscr{L}' + \mathscr{K}$. Then by Lemma 8, there is a unitary U with $U - I \in \mathscr{K}$ such that $(UV)\mathscr{L}_c(UV)^* = \mathscr{L}_c$, and finite subsets $\mathscr{P}, \mathscr{Q} \subseteq \mathscr{F}(\mathscr{L}) = \mathscr{A}(\mathscr{L})$ such that $\operatorname{Ad} UV$ is a bijection of $\mathscr{F}(\mathscr{L}) \setminus \mathscr{P}$ onto $\mathscr{F}(\mathscr{L}) \setminus \mathscr{Q}$ and $(UV)(\sum_{A\in\mathscr{P}} A)(UV)^* = \sum_{B\in\mathscr{Q}} B$. Let $e(k) = \operatorname{card}\{A \in \mathscr{A}(\mathscr{L})\colon \dim A = k \text{ and } A \in \mathscr{P}\}$ and $f(k) = \operatorname{card}\{A \in \mathscr{A}(\mathscr{L})\colon \dim A = k \text{ and } A \in \mathscr{Q}\}$. Then if $c(k) < \infty$, it follows that $e(k) = f(k)$, and (i) then implies that $e(k) = f(k)$ for all k. Thus, there is a partial isometry Y such that $\operatorname{Ad} Y$ is a bijection of $\mathscr{F}(\mathscr{L}) \cap \mathscr{P}$ onto $\mathscr{F}(\mathscr{L}) \cap \mathscr{Q}$. Let $W = Y + UV(\sum_{A\in\mathscr{P}} A)^\perp$. Then $\operatorname{Ad} W$ implements an automorphism of $\mathscr{L}''$ since $W\mathscr{L}_cW^* = \mathscr{L}_c$ and $W\mathscr{L}_aW^* = \mathscr{L}_a$, and therefore implements an automorphism of $\mathscr{L}'$ as well. Also, W is a compact perturbation of UV since Y and $\sum_{A\in\mathscr{P}} A$ are compact, and UV is in turn a compact perturbation of V since $U - I \in \mathscr{K}$.

Suppose instead that (ii) holds, and let A_0 be the one infinite atom. If V is a unitary such that $V(\mathscr{L}'+\mathscr{K})V^* = \mathscr{L}'+\mathscr{K}$, then we again obtain a unitary U with $U - I \in \mathscr{K}$ such that $(UV)\mathscr{L}_c(UV)^*$, and also there are σ-finite projections $P, Q \in \mathscr{L}'$ such that $(UV)P(UV)^* = Q$

and $(UV)\mathcal{L}_a^P(UV)^* = \mathcal{L}_a^Q$. Then $\operatorname{Ad} UV$ is a bijection of $\mathcal{F}(\mathcal{L})\backslash\mathcal{P}$ onto $\mathcal{F}(\mathcal{L}) \setminus \mathcal{Q}$, where $\mathcal{P} = \{A \in \mathcal{F}(\mathcal{L}): A \le P\}$ and $\mathcal{Q} = \{A \in \mathcal{F}(\mathcal{L}): A \le Q\}$. Let $P_0 = PA_0$ and $Q_0 = QA_0$. Since $c(k) < \infty$ for all k, it follows that $e(k) = f(k)$ for all k, and therefore $\dim P_0 = \dim Q_0$. Now let Y be a partial isometry such that $\operatorname{Ad} Y$ is a bijection of $\mathcal{F}(\mathcal{L}) \cap \mathcal{P}$ onto $\mathcal{F}(\mathcal{L}) \cap \mathcal{Q}$ and also $YP_0Y^* = Q_0$. Finally, define $W = Y + UVP^\perp$. As in the argument for case (i), $V - W \in \mathcal{K}$ and $W\mathcal{L}'W^* = \mathcal{L}'$.

To prove that either (i) or (ii) is necessary, first suppose that $\mathcal{L} \subseteq \mathcal{B}(\mathcal{H})$ has at least two infinite atoms A_0 and A_1. Let $P_0 \le A_0$ be a rank-one projection and let $F = I - A_0 - A_1$. Define V to be a unitary such that $V|_{F\mathcal{H}} = I|_{F\mathcal{H}}$, $V(A_0 - P_0)V^* = A_0$, and $V(A_1 + P_0)V^* = A_1$. Then certainly $V(\mathcal{L}' + \mathcal{K})V^* = \mathcal{L}' + \mathcal{K}$. However, there is no unitary W with $V - W \in \mathcal{K}$ such that $W\mathcal{L}'W^* = \mathcal{L}'$. For if there were, it would follow that $W\mathcal{L}''W^* = \mathcal{L}''$, and therefore WA_0W^* is an infinite atom of $\mathcal{L}$. But $WA_0W^* = A_0 + \text{compact}$, so in fact $WA_0W^* = A_0$. Similarly, $WA_1W^* = A_1$. Let $R = A_1 + P_0$ and $S = A_1 + WP_0W^*$. Then, viewing $SW|_{R\mathcal{H}}$ and $SV|_{R\mathcal{H}}$ as operators from $R\mathcal{H}$ to $S\mathcal{H}$, $\operatorname{ind}(SW|_{R\mathcal{H}}) = 0$ and $\operatorname{ind}(SV|_{R\mathcal{H}}) = -1$, contradicting the fact that the Fredholm index is invariant under compact perturbations.

Next, suppose $c(\infty) = 1$ and $c(k_0) = \infty$. Let A_0 be the infinite atom and let $E_1, E_2, \ldots$ be the atoms of dimension k_0. Let $G = I - A_0 - \sum_{i=1}^\infty E_i$ and define V to be a unitary such that $V|_{G\mathcal{H}} = I|_{G\mathcal{H}}$, $V(A_0 + E_1)V^* = A_0$, and $VE_iV^* = E_{i-1}$ for all $i \ge 2$. Then $V(\mathcal{L}' + \mathcal{K})V^* = \mathcal{L}' + \mathcal{K}$, but again there is no unitary W with $V - W \in \mathcal{K}$ and $W\mathcal{L}'W^* = \mathcal{L}'$. For W would have to satisfy $WA_0W^* = A_0$, so if $R = A_0 + E_1$ and $S = A_0 + WE_1W^*$, then $\operatorname{ind}(SW|_{R\mathcal{H}}) = 0$ and $\operatorname{ind}(SV|_{R\mathcal{H}}) = -k_0$, a contradiction.

Finally, if $c(\infty) = 0$, $c(k_0) = \infty$, and $c(k_1) = \infty$ for $k_0 \ne k_1$, then a minor variation of the argument given in [**P**, Corollary 12] yields a unitary which implements an automorphism of $\mathcal{L}' + \mathcal{K}$ but does not split. □

We can also extend [**P**, Corollary 13] to the general situation.

THEOREM 12. *Let $\mathcal{L}$ be a CSL. Then all automorphisms of $\mathcal{L}' + \mathcal{K}$ are inner if and only if*

(i) *$\mathcal{L}$ is purely atomic,*
(ii) *$i(\mathcal{L}) \le 1$, and*

(iii) *there is a finite set* $\mathcal{S} \subseteq \mathcal{A}(\mathcal{L})$ *such that* $\dim A \neq \dim B$ *for all distinct atoms* $A, B \in \mathcal{A}(\mathcal{L}) \setminus \mathcal{S}$.

Proof. Again, by replacing $\mathcal{L}$ with the complemented lattice generated by $\mathcal{L}$, we can assume without loss of generality that $\mathcal{L}$ is complemented. Suppose (i)–(iii) hold, and suppose $\operatorname{Ad} V$ is an automorphism of $\mathcal{L}' + \mathcal{K}$. Then Theorem 11 implies that V splits, since $c(k) < \infty$ for all k by (iii). Thus, $V = W + K$ with K compact and W a unitary satisfying $W\mathcal{L}'W^* = \mathcal{L}'$. Denote the atoms of $\mathcal{L}$ by $\{A_i\}_{i=1}^M$, $M \leq \infty$. Then it follows from (iii) that there is some N such that $(\operatorname{Ad} W)(A_i) = A_i$ for all $i > N$ and, if an infinite atom A_{inf} exists, then $(\operatorname{Ad} W)(A_{\mathrm{inf}}) = A_{\mathrm{inf}}$. By increasing N if necessary, we can assume $A_{\mathrm{inf}} = A_k$ for some $k \leq N$. Now let $P = \sum_{i=1}^N A_i$ if $i(\mathcal{L}) = 0$ and $P = \sum_{i=1}^N A_i - A_k$ if $i(\mathcal{L}) = 1$. Then $WP^\perp \in \mathcal{L}'$ and $W - WP^\perp \in \mathcal{K}$, so $V \in \mathcal{L}' + \mathcal{K}$.

To show that (i)–(iii) are necessary, first suppose $\mathcal{L}$ is not purely atomic. Let $P_a = \sum_{A \in \mathcal{A}(\mathcal{L})} A$ and $P_c = I - P_a$. Then $\mathcal{L}_c$ is a nonatomic complemented CSL on $P_c\mathcal{H}$. From the multiplicity theory of abelian von Neumann algebras, there are orthogonal projections $\{P_m : 1 \leq m \leq \infty\}$ such that for $P_m \neq 0$, $(\mathcal{L}_c|_{P_m\mathcal{H}})''$ is unitarily equivalent via a unitary X_m to the algebra $\mathcal{Z}^{(m)}$ defined after Corollary 10. Choose n so that $P_n \neq 0$, and recall that μ is being used to denote Lebesgue measure. Now if we let $f(x) = x^2$, then there is a unitary operator Y such that $YL^2([0, f(t)], \mu)^{(n)} = L^2([0, t], \mu)^{(n)}$ for all $t \in [0, 1]$ [**D1**, Corollary 7.16]. Let P_t be the projection onto $L^2([0, t], \mu)^{(n)}$, so $YP_{f(t)}Y^* = P_t$ for all t, and Y thus implements an automorphism of $\mathcal{Z}^{(n)}$. But then $P_{1/4}^\perp Y P_{1/4} Y^* = P_{1/4}^\perp P_{1/2}$, and it follows that $Y \notin \{P_t\}' + \mathcal{K}$, for otherwise the left side would be compact but the right side would not. Thus, $X_n^* Y X_n \notin (\mathcal{L}_c|_{P_n\mathcal{H}})' + \mathcal{K}$, and therefore $V = P_a + \sum_{m \neq n} P_m + X_n^* Y X_n$ is a unitary which implements an automorphism of $\mathcal{L}' + \mathcal{K}$ that is not inner.

Now suppose $\mathcal{L}$ is purely atomic but that $\mathcal{L}$ has at least two infinite atoms. Again, denote the atoms of $\mathcal{L}$ by $\{A_i\}_{i=1}^M$, $M \leq \infty$. By relabeling if necessary, we can assume A_1 and A_2 are infinite. Let $R = \sum_{i=3}^M A_i$ and define a unitary $V = I|_{R\mathcal{H}} + W$, where W is a partial isometry satisfying $WA_1W^* = A_2$ and $WA_2W^* = A_1$. Then $V\mathcal{L}'V^* = \mathcal{L}'$, so $\operatorname{Ad} V$ certainly implements an isomorphism of $\mathcal{L}' + \mathcal{K}$. However, $V - \delta_{\mathcal{L}}(V) = VR^\perp = W \notin \mathcal{K}$, and therefore $V \notin \mathcal{L}' + \mathcal{K}$ by [**W3**, Lemma 5].

Finally, suppose (i) and (ii) hold, but (iii) is false. Then there is an infinite sequence of atoms $\{B_i\}$ such that $\dim B_{2k-1} = \dim B_{2k}$ for $k = 1, 2, 3, \ldots$. Let $S = \sum_{i=1}^{\infty} B_i$ and define a unitary $V = I|_{S^{\perp}\mathscr{H}} + W$, where W is a partial isometry satisfying $WB_{2k-1}W^* = B_{2k}$ and $WB_{2k}W^* = B_{2k-1}$ for all k. Then $\operatorname{Ad} V$ implements an isomorphism of $\mathscr{L}'$, and therefore $\mathscr{L}' + \mathscr{K}$ also, but again $V \notin \mathscr{L}' + \mathscr{K}$ since $V - \delta_{\mathscr{L}}(V) = VS = W \notin \mathscr{K}$. □

The last proof used several techniques for constructing automorphisms which are not inner. These techniques give an indication of how to calculate the outer automorphism group of $\mathscr{L}' + \mathscr{K}$. First, we will use $\operatorname{Sym}(S)$ to denote the group of all permutations of a set S. Now let $\mathscr{L}$ be a CSL and let $P_{\mathscr{F}} = \sum_{A \in \mathscr{F}(\mathscr{L})} A$. If E is a finite sum of atoms in $\mathscr{F}(\mathscr{L})$, define $\mathscr{L}_E$ to be the lattice $\mathscr{L} P_{\mathscr{F}} E^{\perp}$ on $P_{\mathscr{F}} E^{\perp}\mathscr{H}$. Given $\mathscr{L}_E$ and $\mathscr{L}_F$, let $\operatorname{ISO}(\mathscr{L}_E, \mathscr{L}_F)$ denote the set of bijections $\theta\colon \mathscr{A}(\mathscr{L}_E) \to \mathscr{A}(\mathscr{L}_F)$ which preserve dimension. Note that θ extends to a map which takes sums of atoms to sums of atoms. Let $\operatorname{ISO}_{\mathscr{F}}(\mathscr{L}) = \bigcup \operatorname{ISO}(\mathscr{L}_E, \mathscr{L}_F)$, where the union is taken over all pairs (E, F) of finite sums of atoms in $\mathscr{F}(\mathscr{L})$ with the property that if $i(\mathscr{L}) = 0$, then $\dim E = \dim F$. Now if $\theta \in \operatorname{ISO}(\mathscr{L}_E, \mathscr{L}_F)$ and G is a finite sum of atoms in $\mathscr{L}_E$, then define $\theta_G \in \operatorname{ISO}\left((\mathscr{L}_E)_G, (\mathscr{L}_F)_{\theta(G)}\right)$ by restriction. We can now define an equivalence relation on $\operatorname{ISO}_{\mathscr{F}}(\mathscr{L})$ by $\theta \sim \tau$ if there are finite sums G and H of atoms such that $\theta_G = \tau_H$. The class of θ will be denoted by $[\theta]$.

We next define a multiplication on $\operatorname{ISO}_{\mathscr{F}}(\mathscr{L})/\sim$. Suppose $\theta \in \operatorname{ISO}(\mathscr{L}_E, \mathscr{L}_F)$ and $\tau \in \operatorname{ISO}(\mathscr{L}_G, \mathscr{L}_H)$, and define $[\tau][\theta] = [\rho]$, where $\rho = \tau_{FG^{\perp}} \circ \theta_{\theta^{-1}(F^{\perp}G)}$. It is not hard to see that this multiplication is well-defined and associative, [id] is an identity, and $[\theta]^{-1} = [\theta^{-1}]$. Thus, $\operatorname{ISO}_{\mathscr{F}}(\mathscr{L})/\sim$ is a group, denoted a-$\operatorname{ISO}_{\mathscr{F}}(\mathscr{L})$. We note that this group is very similar to, and was motivated by, the group a-$\operatorname{Aut}\mathscr{M}$ in [**DW**, p. 615-616].

We are now in a position to calculate $\operatorname{Out}(\mathscr{L}' + \mathscr{K})$. For simplicity, we will compute this group separately for the purely atomic and nonatomic cases, and then combine these results to prove the general case in Corollary 20. In the following theorem, we note that $\operatorname{Sym}(\mathscr{I}(\mathscr{L})) = \{\mathrm{id}\}$ if $i(\mathscr{L}) = 0$ or 1, and a-$\operatorname{ISO}_{\mathscr{F}}(\mathscr{L}) = \{[\mathrm{id}]\}$ if $\mathscr{F}(\mathscr{L})$ is finite.

THEOREM 13. *Suppose $\mathscr{L}$ is a purely atomic CSL. Then*

$$\operatorname{Out}(\mathscr{L}' + \mathscr{K}) \cong \operatorname{Sym}(\mathscr{I}(\mathscr{L})) \times a\text{-}ISO_{\mathscr{F}}(\mathscr{L}).$$

The proof will be completed by a succession of lemmas. First, note that each of the groups in the statement of the theorem is the same for $\mathscr{L}$ as for the complemented lattice generated by $\mathscr{L}$, so we can assume $\mathscr{L}$ is complemented. After the next technical lemma, we will define a map Γ: $\operatorname{Aut}(\mathscr{L}' + \mathscr{K}) \to \operatorname{Sym}(\mathscr{I}(\mathscr{L})) \times a\text{-ISO}_{\mathscr{F}}(\mathscr{L})$, and then show that Γ is a surjective group homomorphism whose kernel is $\operatorname{Inn}(\mathscr{L}' + \mathscr{K})$, completing the proof.

LEMMA 14. *Suppose $\mathscr{L}$ is a complemented CSL and W is a unitary in $\mathscr{L}'+\mathscr{K}$ such that $\operatorname{Ad} W$ is a bijection of $\mathscr{F}(\mathscr{L})\backslash\mathscr{R}$ onto $\mathscr{F}(\mathscr{L})\backslash\mathscr{S}$ for some finite sets $\mathscr{R}$, $\mathscr{S} \subseteq \mathscr{F}(\mathscr{L})$. Then there is a finite set $\widetilde{\mathscr{R}} \supseteq \mathscr{R}$ such that $(\operatorname{Ad} W)(A) = A$ for all $A \in \mathscr{F}(\mathscr{L}) \backslash \widetilde{\mathscr{R}}$.*

Proof. Write $W = S + K$ for some $S \in \mathscr{L}'$ and $K \in \mathscr{K}$, and suppose $(\operatorname{Ad} W)(E_i) \neq E_i$ for an infinite set $\{E_i\colon 1 \leq i < \infty\} \subseteq \mathscr{F}(\mathscr{L})\backslash\mathscr{R}$. Let $e_i \in E_i\mathscr{H}$ with $\|e_i\| = 1$, and let $f_i = We_i$. $\|Ke_i\| \to 0$ since $e_i \to 0$ weakly and K is compact. But then

$$\begin{aligned} 1 = \langle f_i , f_i\rangle &= \langle E_i^{\perp} W E_i e_i , f_i\rangle = \langle E_i^{\perp}(S+K)E_i e_i , f_i\rangle \\ &= \langle E_i^{\perp} K E_i e_i , f_i\rangle = \langle K e_i , E_i^{\perp} f_i\rangle , \end{aligned}$$

a contradiction. □

If $\alpha \in \operatorname{Aut}(\mathscr{L}'+\mathscr{K})$, then $\alpha = \operatorname{Ad} V$ for some unitary V by Lemma 2. If $\operatorname{Ad} V = \operatorname{Ad} V'$ on $\mathscr{L}'+\mathscr{K}$, then $V^*V' \in (\mathscr{L}'+\mathscr{K})' = \mathbb{C}I$, so V is unique up to multiplication by a scalar of modulus 1. By Lemma 8, there is a unitary U with $U - I \in \mathscr{K}$ and σ-finite projections P, $Q \in \mathscr{L}'$ such that $(\operatorname{Ad} UV)(P) = Q$ and $(\operatorname{Ad} UV)(\mathscr{L}^P) = \mathscr{L}^Q$. Thus, $\operatorname{Ad} UV$ is a bijection of $\mathscr{I}(\mathscr{L}^P)$ onto $\mathscr{I}(\mathscr{L}^Q)$. Because of the form of P and Q, this induces a unique bijection ι_α of $\mathscr{I}(\mathscr{L})$ satisfying $\iota_\alpha(A)Q^{\perp} = (\operatorname{Ad} UV)(AP^{\perp})$ for all $A \in \mathscr{I}(\mathscr{L})$. It follows that $\iota_\alpha(A) - \alpha(A) \in \mathscr{K}$ for all $A \in \mathscr{I}(\mathscr{L})$. From the above, a different choice of V has no effect on $\operatorname{Ad} UV$. Suppose U', P', and Q' satisfy the same properties as U, P, and Q, and ι'_α is the corresponding induced bijection of $\mathscr{I}(\mathscr{L})$. But then $\iota_\alpha(A) - \iota'_\alpha(A) \in \mathscr{K}$ for all $A \in \mathscr{I}(\mathscr{L})$, so $\iota_\alpha = \iota'_\alpha$. This shows that ι_α is well-defined, i.e., it is determined by any V implementing α and any U, P, and Q satisfying Lemma 8 for V and $\mathscr{L}$. An equivalent characterization of ι_α is that it is the unique bijection of $\mathscr{I}(\mathscr{L})$ such that $\iota_\alpha(A) - \alpha(A) \in \mathscr{K}$ for all $A \in \mathscr{I}(\mathscr{L})$.

Now $\operatorname{Ad} UV$ is also a bijection of $\mathscr{F}(\mathscr{L}) \backslash \mathscr{P}$ onto $\mathscr{F}(\mathscr{L}) \backslash \mathscr{Q}$, where $\mathscr{P} = \{A \in \mathscr{F}(\mathscr{L})\colon A \leq P\}$ and $\mathscr{Q} = \{A \in \mathscr{F}(\mathscr{L})\colon A \leq Q\}$.

Thus, $(\operatorname{Ad} UV)|_{\mathscr{F}(\mathscr{L})\setminus\mathscr{P}} \in \operatorname{ISO}(\mathscr{L}_E, \mathscr{L}_F)$, where $E = \sum_{A\in\mathscr{P}} A$ and $F = \sum_{A\in\mathscr{Q}} A$. Moreover, if $i(\mathscr{L}) = 0$, then $E = P$ and $F = Q$, so $\dim E = \dim F$. Let $\varphi_\alpha = [(\operatorname{Ad} UV)|_{\mathscr{F}(\mathscr{L})\setminus\mathscr{P}}]$. If U', P', and Q' satisfy the same properties as U, P, and Q, we will obtain analogous sets $\mathscr{P}'$ and $\mathscr{Q}'$ and projections E' and F'. Let

$$\mathscr{R} = \mathscr{P} \cup \{(\operatorname{Ad} V^*U^*)(A) \colon A \in \mathscr{Q}' \setminus \mathscr{Q}\}$$

and

$$\mathscr{S} = \mathscr{P}' \cup \{(\operatorname{Ad} V^*U'^*)(A) \colon A \in \mathscr{Q} \setminus \mathscr{Q}'\}.$$

Then $\operatorname{Ad} V^*U'^*UV$ is a bijection of $\mathscr{F}(\mathscr{L})\setminus\mathscr{R}$ onto $\mathscr{F}(\mathscr{L})\setminus\mathscr{S}$ and $V^*U'^*UV = I + \text{compact}$, so Lemma 14, with $W = V^*U'^*UV$, implies that there is a finite set $\widetilde{\mathscr{R}} \supseteq \mathscr{R}$ such that $(\operatorname{Ad} UV)(A) = (\operatorname{Ad} U'V)(A)$ for all $A \in \mathscr{F}(\mathscr{L}) \setminus \widetilde{\mathscr{R}}$. Therefore $(\operatorname{Ad} UV)|_{\mathscr{F}(\mathscr{L})\setminus\mathscr{P}} \sim (\operatorname{Ad} U'V)|_{\mathscr{F}(\mathscr{L})\setminus\mathscr{P}'}$, and it follows that φ_α is well-defined. Thus, we have proved

LEMMA 15. *The map* $\Gamma\colon \operatorname{Aut}(\mathscr{L}'+\mathscr{K}) \to \operatorname{Sym}(\mathscr{J}(\mathscr{L}))\times a\text{-}ISO_{\mathscr{F}}(\mathscr{L})$ *by* $\Gamma(\alpha) = (\imath_\alpha, \varphi_\alpha)$ *is well-defined.*

LEMMA 16. Γ *is surjective.*

Proof. Suppose $(\imath, [\varphi]) \in \operatorname{Sym}(\mathscr{J}(\mathscr{L}))\times a\text{-ISO}_{\mathscr{F}}(\mathscr{L})$. Let $\psi \in [\varphi]$, so $\psi \in \operatorname{ISO}(\mathscr{L}_E, \mathscr{L}_F)$ for some E, F. Let W be a partial isometry such that $WAW^* = \psi(A)$ for all $A \in \mathscr{A}(\mathscr{L}_E) = \mathscr{F}(\mathscr{L}_E)$ and $WAW^* = \imath(A)$ for all $A \in \mathscr{J}(\mathscr{L})$. If $\dim E = \dim F$ (in particular, if $i(\mathscr{L}) = 0$), let X be a partial isometry satisfying $XEX^* = F$, and define $V = X + W$, $P_0 = 0$, and $Q_0 = 0$. On the other hand, if $\dim E \neq \dim F$ (so $i(\mathscr{L}) > 0$), choose an infinite atom A_0 and finite subprojections $P_0 < A_0$ and $Q_0 < \imath(A_0)$ such that $\dim(E + P_0) = \dim(F + Q_0)$. Let Y be a partial isometry with $Y(A_0P_0^\perp)Y^* = \imath(A_0)Q_0^\perp$ and $Y(E+P_0)Y^* = F + Q_0$, and define $V = Y + W(E + A_0)^\perp$. Then in either case $(\operatorname{Ad} V)(\mathscr{L}^{E+P_0}) = \mathscr{L}^{F+Q_0}$, so $\alpha = \operatorname{Ad} V$ is an automorphism of $\mathscr{L}' + \mathscr{K}$, and it is readily apparent that $\Gamma(\alpha) = (\imath, [\varphi])$ (simply let $U = I$, $P = E + P_0$, and $Q = F + Q_0$). □

LEMMA 17. Γ *is a group homomorphism.*

Proof. Let $\alpha_1, \alpha_2 \in \operatorname{Aut}(\mathscr{L}' + \mathscr{K})$ with $\alpha_1 = \operatorname{Ad} V_1$ and $\alpha_2 = \operatorname{Ad} V_2$. Define $\alpha_3 = \alpha_2 \circ \alpha_1 = \operatorname{Ad} V_2V_1$. Use Lemma 8 to obtain unitaries U_1, U_2, and U_3 with $U_i - I \in \mathscr{K}$ and σ-finite projections

P_1, P_2, P_3, Q_1, Q_2, Q_3 in $\mathscr{L}'$ such that $(\mathrm{Ad}\, U_iV_i)(P_i) = Q_i$ and $(\mathrm{Ad}\, U_iV_i)(\mathscr{L}^{P_i}) = \mathscr{L}^{Q_i}$ for $i = 1, 2$, $(\mathrm{Ad}\, U_3V_2V_1)(P_3) = Q_3$, and $(\mathrm{Ad}\, U_3V_2V_1)(\mathscr{L}^{P_3}) = \mathscr{L}^{Q_3}$. Then $\imath_{\alpha_i}(A) - \alpha_i(A) \in \mathscr{K}$ for all $A \in \mathscr{I}(\mathscr{L})$, so $\imath_{\alpha_2}(\imath_{\alpha_1}(A)) - \alpha_2(\imath_{\alpha_1}(A)) \in \mathscr{K}$. But then

$$\begin{aligned}(\imath_{\alpha_2} \circ \imath_{\alpha_1})(A) - \alpha_3(A) &= \imath_{\alpha_2}(\imath_{\alpha_1}(A)) - \alpha_2(\imath_{\alpha_1}(A)) \\ &\quad + \alpha_2(\imath_{\alpha_1}(A)) - \alpha_2(\alpha_1(A)) \\ &= \imath_{\alpha_2}(\imath_{\alpha_1}(A)) - \alpha_2(\imath_{\alpha_1}(A)) \\ &\quad + V_2(\imath_{\alpha_1}(A) - \alpha_1(A))V_2^* \in \mathscr{K},\end{aligned}$$

and it ıollows that $\imath_{\alpha_3} = \imath_{\alpha_2} \circ \imath_{\alpha_1}$.

Now let $\mathscr{P}_i = \{A \in \mathscr{F}(\mathscr{L}) : A \leq P_i\}$ and $\mathscr{Q}_i = \{A \in \mathscr{F}(\mathscr{L}) : A \leq Q_i\}$, $i = 1, 2, 3$, so $\varphi_{\alpha_i} = [(\mathrm{Ad}\, U_iV_i)|_{\mathscr{F}(\mathscr{L})\setminus\mathscr{P}_i}]$ for $i = 1, 2$, and $\varphi_{\alpha_3} = [(\mathrm{Ad}\, U_3V_2V_1)|_{\mathscr{F}(\mathscr{L})\setminus\mathscr{P}_3}]$. If we define

$$\mathscr{P}_4 = \mathscr{P}_1 \cup \{(\mathrm{Ad}\, V_1^*U_1^*)(A) : A \in \mathscr{P}_2 \setminus \mathscr{Q}_1\}$$

and

$$\mathscr{Q}_4 = \mathscr{Q}_2 \cup \{(\mathrm{Ad}\, U_2V_2)(A) : A \in Q_1 \setminus P_2\},$$

then $\mathrm{Ad}\, U_2V_2U_1V_1$ is a bijection of $\mathscr{F}(\mathscr{L}) \setminus \mathscr{P}_4$ onto $\mathscr{F}(\mathscr{L}) \setminus \mathscr{Q}_4$, and $\varphi_{\alpha_2}\varphi_{\alpha_1} = [(\mathrm{Ad}\, U_2V_2U_1V_1)|_{\mathscr{F}(\mathscr{L})\setminus\mathscr{P}_4}]$ by definition of the product in a-$\mathrm{ISO}_{\mathscr{F}}/\sim$. Define $X = U_3V_2V_1$ and $Y = U_2V_2U_1V_1$, and let

$$\mathscr{R} = \mathscr{P}_3 \cup \{(\mathrm{Ad}\, X^*)(A) : A \in \mathscr{Q}_4 \setminus \mathscr{Q}_3\}$$

and

$$\mathscr{S} = \mathscr{P}_4 \cup \{(\mathrm{Ad}\, Y^*)(A) : A \in \mathscr{Q}_3 \setminus \mathscr{Q}_4\}.$$

Then $\mathrm{Ad}\, Y^*X$ is a bijection of $\mathscr{F}(\mathscr{L}) \setminus \mathscr{R}$ onto $\mathscr{F}(\mathscr{L}) \setminus \mathscr{S}$ and $Y^*X = I +$ compact, so Lemma 14, with $W = Y^*X$, implies that $(\mathrm{Ad}\, X)(A) = (\mathrm{Ad}\, Y)(A)$ for all $A \in \mathscr{F}(\mathscr{L}) \setminus \widetilde{\mathscr{R}}$, where $\widetilde{\mathscr{R}}$ is some finite set containing $\mathscr{R}$. Therefore $(\mathrm{Ad}\, X)|_{\mathscr{F}(\mathscr{L})\setminus\mathscr{P}_3} \sim (\mathrm{Ad}\, Y)|_{\mathscr{F}(\mathscr{L})\setminus\mathscr{P}_4}$, i.e., $\varphi_{\alpha_3} = \varphi_{\alpha_2}\varphi_{\alpha_1}$. □

LEMMA 18. $\mathrm{Ker}(\Gamma) = \mathrm{Inn}(\mathscr{L}' + \mathscr{K})$.

Proof. First suppose $\theta \in \mathrm{Inn}(\mathscr{L}' + \mathscr{K})$, $\theta = \mathrm{Ad}\, V$. Let U be any unitary and $P, Q \in \mathscr{L}'$ be any σ-finite projections satisfying Lemma 8. Suppose $\imath_\theta(A) \neq A$ for some $A \in \mathscr{I}(\mathscr{L})$. Write $V = S + K$ with $S \in \mathscr{L}'$ and $K \in \mathscr{K}$. Then $A^\perp VAV^* = A^\perp(S + K)AV^* = A^\perp KAV^* \in \mathscr{K}$. Also, $\imath_\theta(A) - VAV^* = \imath_\theta(A) - \theta(A) \in \mathscr{K}$. It follows that $\imath_\theta(A) = A^\perp(\imath_\theta(A) - A) = A^\perp(\imath_\theta(A) - VAV^*) + A^\perp(VAV^* - A) \in \mathscr{K}$, a contradiction. Therefore, $\imath_\theta = \mathrm{id} \in \mathrm{Sym}(\mathscr{I}(\mathscr{L}))$.

Now $\operatorname{Ad} UV$ is a bijection of $\mathscr{F}(\mathscr{L})\backslash\mathscr{P}$ onto $\mathscr{F}(\mathscr{L})\backslash\mathscr{Q}$ for some finite sets $\mathscr{P}$, $\mathscr{Q} \subseteq \mathscr{F}(\mathscr{L})$. $UV \in \mathscr{L}'+\mathscr{K}$ since $U-I \in \mathscr{K}$, so Lemma 14 implies that there is a finite set $\widetilde{\mathscr{P}} \supseteq \mathscr{P}$ such that $(\operatorname{Ad} UV)(A) = A$ for all $A \in \mathscr{F}(\mathscr{L}) \setminus \widetilde{\mathscr{P}}$. It follows that $\varphi_\theta = [\mathrm{id}]$, and therefore $\Gamma(\theta) = (\mathrm{id}, [\mathrm{id}])$.

Suppose on the other hand that $\Gamma(\theta) = (\mathrm{id}, [\mathrm{id}])$. We want to show that $\theta = \operatorname{Ad} V \in \operatorname{Inn}(\mathscr{L}' + \mathscr{K})$. Let U be any unitary and $P, Q \in \mathscr{L}'$ be any σ-finite projections satisfying Lemma 8. Let $\mathscr{P} = \{A \in \mathscr{F}(\mathscr{L})\colon A \le P\}$ and $\mathscr{Q} = \{A \in \mathscr{F}(\mathscr{L})\colon A \le Q\}$. Then $\operatorname{Ad} UV$ is a bijection of $\mathscr{F}(\mathscr{L})\backslash\mathscr{P}$ onto $\mathscr{F}(\mathscr{L})\backslash\mathscr{Q}$, and $(\operatorname{Ad} UV)|_{\mathscr{F}(\mathscr{L})\backslash\mathscr{P}} \sim \mathrm{id}$ by the definition of φ_θ. It follows that there is a finite set $\widetilde{\mathscr{P}} \supseteq \mathscr{P}$ such that $(\operatorname{Ad} UV)|_{\mathscr{F}(\mathscr{L})\backslash\widetilde{\mathscr{P}}} = \mathrm{id}|_{\mathscr{F}(\mathscr{L})\backslash\widetilde{\mathscr{P}}}$. Also, $\operatorname{Ad} UV$ is a bijection of $\mathscr{I}(\mathscr{L}^P)$ onto $\mathscr{I}(\mathscr{L}^Q)$ such that $AQ^\perp = (\operatorname{Ad} UV)(AP^\perp)$ for all $A \in \mathscr{I}(\mathscr{L})$. Let $\mathscr{R}$ be the finite set $\widetilde{\mathscr{P}} \cup \{A \in \mathscr{I}(\mathscr{L})\colon AP \neq 0$ or $AQ \neq 0\}$, and define $\widetilde{P} = \sum_{A\in\widetilde{\mathscr{P}}} A$ and $R = \sum_{A\in\mathscr{R}} A$. Then for $T \in \mathscr{L}''$,

$$\begin{aligned} UVTV^*U^* &= UVTR^\perp V^*U^* + UVTRV^*U^* \\ &= TR^\perp + UVTR\widetilde{P}^\perp P^\perp V^*U^* + UVTR\widetilde{P}P^\perp V^*U^* \\ &\quad + UVTRPV^*U^* \\ &= TR^\perp + TR\widetilde{P}^\perp Q^\perp + UVT\widetilde{P}P^\perp V^*U^* + UVTPV^*U^* \\ &= T + \text{compact}. \end{aligned}$$

Equivalently, $(UV)T - T(UV) \in \mathscr{K}$ for all $T \in \mathscr{L}''$. Therefore, $UV \in \mathscr{L}' + \mathscr{K}$ by [**JP**, Theorem 2.1], so $V \in \mathscr{L}' + \mathscr{K}$ also since $U - I \in \mathscr{K}$. □

Proof of Theorem 13. Apply Lemmas 15–18. □

THEOREM 19. *If $\mathscr{L}$ is a nonatomic CSL, then*

$$\operatorname{Out}(\mathscr{L}' + \mathscr{K}) \cong s\text{-}\operatorname{Aut}(\mathscr{L}'').$$

Proof. As before, we can assume $\mathscr{L}$ is complemented. Let $\alpha = \operatorname{Ad} V \in \operatorname{Aut}(\mathscr{L}' + \mathscr{K})$, and let U, P, and Q be given by Lemma 8. Then $P = Q = 0$ by Lemma 1, so $(\operatorname{Ad} UV)(\mathscr{L}) = \mathscr{L}$, and therefore $\operatorname{Ad} UV$ also implements an automorphism of $\mathscr{L}''$. By the same reasoning used in the proof of Theorem 13, another choice for V has no effect on $\operatorname{Ad} UV$. If U', P', and Q' satisfy the same properties as U, P, and Q, then $(\operatorname{Ad} UV)(E) - (\operatorname{Ad} U'V)(E) \in \mathscr{K}$ for every projection $E \in \mathscr{L}$, and this implies that $(\operatorname{Ad} UV)(E) = (\operatorname{Ad} U'V)(E)$

since E is infinite. Thus, the map $\gamma\colon \operatorname{Aut}(\mathscr{L}'+\mathscr{K}) \to s\text{-}\operatorname{Aut}(\mathscr{L}'')$ by $\gamma(\alpha) = \operatorname{Ad} UV$ is well-defined, i.e., every possible choice of V, U, P, and Q yields the same automorphism of $\mathscr{L}''$.

Now if $\operatorname{Ad} W \in s\text{-}\operatorname{Aut}(\mathscr{L}'')$, then $\operatorname{Ad} W \in \operatorname{Aut}(\mathscr{L}' + \mathscr{K})$ also. $U = I$ and $P = Q = 0$ satisfy Lemma 8, so $\gamma(\operatorname{Ad} W) = \operatorname{Ad} W$. This shows that γ is surjective.

Suppose α, $\beta \in \operatorname{Aut}(\mathscr{L}'+\mathscr{K})$ with $\alpha = \operatorname{Ad} V_1$ and $\beta = \operatorname{Ad} V_2$. Let U_1, U_2, and U_3 be unitaries with $U_i - I \in \mathscr{K}$ such that $\operatorname{Ad} U_1V_1$, $\operatorname{Ad} U_2V_2$, $\operatorname{Ad} U_3V_2V_1 \in \operatorname{Aut}(\mathscr{L}'')$, i.e., $\gamma(\alpha) = \operatorname{Ad} U_1V_1$, $\gamma(\beta) = \operatorname{Ad} U_2V_2$, and $\gamma(\beta \circ \alpha) = \operatorname{Ad} U_3V_2V_1$. But then

$$(\operatorname{Ad} U_2V_2 \circ \operatorname{Ad} U_1V_1)(E) - (\operatorname{Ad} U_3V_2V_1)(E) \in \mathscr{K}$$

for every projection $E \in \mathscr{L}$, and therefore

$$(\operatorname{Ad} U_2V_2 \circ \operatorname{Ad} U_1V_1)(E) = (\operatorname{Ad} U_3V_2V_1)(E).$$

Thus $\gamma(\beta) \circ \gamma(\alpha) = \gamma(\beta \circ \alpha)$, and γ is a group homomorphism.

Now suppose $\alpha = \operatorname{Ad} V \in \operatorname{Inn}(\mathscr{L}' + \mathscr{K})$. Let $\gamma(\alpha) = \operatorname{Ad} UV$, and write $U = I + K$, $V = S + L$ with $K, L \in \mathscr{K}$ and $S \in \mathscr{L}'$. Then for each projection $E \in \mathscr{L}$, $(\operatorname{Ad} UV)(E) = (I+K)(S+L)EV^*U^* = ESV^*U^* + M$ for some $M \in \mathscr{K}$. Therefore, $E^\perp(\operatorname{Ad} UV)(E) \in \mathscr{K}$, a contradiction unless $E^\perp(\operatorname{Ad} UV)(E) = 0$. But the same argument applied to $E^\perp$ shows that $E(\operatorname{Ad} UV)(E^\perp) = 0$, and these two identities imply that $(\operatorname{Ad} UV)(E) = E$ for all $E \in \mathscr{L}$. Therefore, $\gamma(\alpha) = \mathrm{id}$.

Finally, suppose $\alpha = \operatorname{Ad} V \in \operatorname{Aut}(\mathscr{L}' + \mathscr{K})$ and $\gamma(\alpha) = \operatorname{Ad} UV = \mathrm{id} \in s\text{-}\operatorname{Aut}(\mathscr{L}'')$. Then $UV \in \mathscr{L}'$, so $V \in \mathscr{L}' + \mathscr{K}$, i.e., $\alpha \in \operatorname{Inn}(\mathscr{L}' + \mathscr{K})$. Therefore, $\operatorname{Ker}(\gamma) = \operatorname{Inn}(\mathscr{L}' + \mathscr{K})$, and the proof is complete. □

COROLLARY 20. *If $\mathscr{L}$ is a CSL, then*

$$\operatorname{Out}(\mathscr{L}' + \mathscr{K}) \cong s\text{-}\operatorname{Aut}(\mathscr{L}_c'') \times \operatorname{Sym}(\mathscr{I}(\mathscr{L})) \times a\text{-}ISO_{\mathscr{F}}(\mathscr{L}).$$

Proof. Again, we can assume without loss of generality that $\mathscr{L}$ is complemented. Let $\alpha = \operatorname{Ad} V \in \operatorname{Aut}(\mathscr{L}' + \mathscr{K})$, and let U, P, and Q be given by Lemma 8. Let $P_a = \sum_{A \in \mathscr{A}(\mathscr{L})} A$ and $P_c = I - P_a$. Then $(\operatorname{Ad} UV)(\mathscr{L}_c) = \mathscr{L}_c$ and $(\operatorname{Ad} UV)(\mathscr{L}_a^P) = \mathscr{L}_a^Q$, so $(\operatorname{Ad} UV)|_{P_c\mathscr{H}}$ implements an automorphism α_c of $\mathscr{L}_c' + \mathscr{K}(P_c\mathscr{H})$ and $(\operatorname{Ad} UV)|_{P_a\mathscr{H}}$ implements an automorphism α_a of $\mathscr{L}_a' + \mathscr{K}(P_a\mathscr{H})$. In addition, $(\alpha_c \oplus \alpha_a) \circ \alpha^{-1} \in \operatorname{Inn}(\mathscr{L}' + \mathscr{K})$ since $U - I \in \mathscr{K}$. Thus, by defining

$\widetilde{\Gamma}(\alpha) = (\gamma(\alpha_c), \Gamma(\alpha_a)) = (\gamma(\alpha_c), \iota_{\alpha_a}, \varphi_{\alpha_a})$, it follows from Theorems 13 and 19 that $\widetilde{\Gamma}$ induces the desired isomorphism. □

Since any unitary which implements an automorphism of $\mathscr{L}_c''$ must map $\mathscr{C}(\mathscr{L}_c)$ onto itself, Corollary 20 thus gives a characterization of $\mathrm{Out}(\mathscr{L}' + \mathscr{K})$ in terms of certain maps on $\mathscr{C}(\mathscr{L})$. Of course, $\mathrm{Sym}(\mathscr{I}(\mathscr{L}))$ is isomorphic to the symmetric group on n letters if $i(\mathscr{L}) = n$, and $\mathrm{Sym}(\mathscr{I}(\mathscr{L})) = \mathrm{Sym}(\mathbb{N})$ if $i(\mathscr{L}) = \infty$. In addition, $\mathrm{Out}(\mathscr{L}_c')$ provides an alternate description of $s\text{-}\mathrm{Aut}(\mathscr{L}_c'')$. To see this, first note that every element of $s\text{-}\mathrm{Aut}(\mathscr{L}_c'')$ extends to an automorphism of $\mathscr{L}_c'$. On the other hand, every automorphism of $\mathscr{L}_c'$ is spatial by [**Di**, §III.3.2, Corollary 1], and thus defines an element of $s\text{-}\mathrm{Aut}(\mathscr{L}_c'')$ by restriction. Moreover, if $\mathrm{Ad}\, U \in \mathrm{Aut}(\mathscr{L}_c')$, then $\mathrm{Ad}\, U = \mathrm{id}$ on $\mathscr{L}_c''$ if and only if $U \in \mathscr{L}_c'$, i.e., $\mathrm{Ad}\, U \in \mathrm{Inn}(\mathscr{L}_c')$. It follows that $\mathrm{Out}(\mathscr{L}' + \mathscr{K})$ and $\mathrm{Out}(\mathscr{L}')$ differ only in the factor $a\text{-}\mathrm{ISO}_{\mathscr{F}}(\mathscr{L})$. $s\text{-}\mathrm{Aut}(\mathscr{L}_c'')$ can also be described in another way. As noted earlier, $\mathscr{L}_c'' \cong \sum_{m\in\mathscr{J}}^{\oplus} \mathscr{Z}^{(m)}$, where $\mathscr{J} \subseteq \{1, 2, \dots, \infty\}$. However, it follows from multiplicity theory that any unitary U which implements an automorphism of $\sum_{m\in\mathscr{J}}^{\oplus} \mathscr{Z}^{(m)}$ can be decomposed as a direct sum $\sum_{m\in\mathscr{J}}^{\oplus} U_m$ of unitaries U_m which implement automorphisms of $\mathscr{Z}^{(m)}$. Consequently, $s\text{-}\mathrm{Aut}(\mathscr{L}_c'') \cong \sum_{m\in\mathscr{J}}^{\oplus} s\text{-}\mathrm{Aut}(\mathscr{Z}^{(m)})$.

We turn now to derivations of $\mathscr{L}' + \mathscr{K}$. A *derivation* of a Banach algebra $\mathfrak{A} \subseteq \mathscr{B}(\mathscr{H})$ is a linear operator $\delta: \mathfrak{A} \to \mathfrak{A}$ which satisfies the property $\delta(ST) = \delta(S)T + S\delta(T)$ for all $S, T \in \mathfrak{A}$. δ is denoted by $\mathrm{ad}\, X$ if $\delta(S) = XS - SX$ for some $X \in \mathscr{B}(\mathscr{H})$, and δ is *inner* if $\delta = \mathrm{ad}\, X$ for some $X \in \mathfrak{A}$. If δ is a continuous derivation, then δ is the infinitesimal generator of the uniformly continuous one-parameter automorphism group $\{\exp(t\delta): t \in \mathbb{R}\}$, and if $\delta = \mathrm{ad}\, X$, then $\exp(t\delta) = \mathrm{Ad}(\exp(tX))$. The approach taken here for derivations of essential commutants of abelian von Neumann algebras is similar to the one used in [**DW**, 3.11–3.13] for quasitriangular algebras (also see [**W1**] and [**W2**]). In the following, $\|T\|_e$ denotes the essential norm of T, i.e., $\|T\|_e = \|q(A)\|$, where q is the canonical projection of $\mathscr{B}(\mathscr{H})$ onto the Calkin algebra $\mathscr{B}(\mathscr{H})/\mathscr{K}(\mathscr{H})$.

LEMMA 21. *Suppose $\mathscr{L}$ is a CSL. If $\mathrm{Ad}\, V \in \mathrm{Aut}(\mathscr{L}' + \mathscr{K})$ and $\|V - I\|_e < \frac{1}{2}$, then $\mathrm{Ad}\, V$ is inner.*

Proof. Without loss of generality, assume $\mathscr{L}$ is complemented, and apply Lemma 8 to obtain operators U, P, and Q. Let $\alpha = \mathrm{Ad}\, V$, and define $\widetilde{\Gamma}$ as in the proof of Corollary 20. Since $U - I \in \mathscr{K}$, we have $\|UV - I\|_e < \frac{1}{2}$ also, and it follows that

$$\|UVTV^*U^* - T\|_e \leq \|UVTV^*U^* - TV^*U^*\|_e + \|TV^*U^* - T\|_e < \|T\|_e$$

for all $T \in \mathcal{B}(\mathcal{H})$. Now $(\operatorname{Ad} UV)(\mathcal{L}_c) = \mathcal{L}_c$ and $(\operatorname{Ad} UV)(\mathcal{L}_a^P) = \mathcal{L}_a^Q$ with $(\operatorname{Ad} UV)(P) = Q$. Therefore, $(\operatorname{Ad} UV)(A) = A$ for all $A \in \mathcal{L}_c$ since $UVAV^*U^* - A$ is the difference of two infinite commuting projections. Likewise, $(\operatorname{Ad} UV)(AP^\perp) = AQ^\perp$ for all $A \in \mathcal{F}(\mathcal{L})$. Thus, $\gamma(\alpha) = \mathrm{id}$ and $\iota_\alpha = \mathrm{id}$.

Let $\mathcal{P} = \{A \in \mathcal{F}(\mathcal{L}) \colon A \leq P\}$ and $\mathcal{Q} = \{A \in \mathcal{F}(\mathcal{L}) \colon A \leq Q\}$. Then $\operatorname{Ad} UV$ is a bijection of $\mathcal{F}(\mathcal{L}) \setminus \mathcal{P}$ onto $\mathcal{F}(\mathcal{L}) \setminus \mathcal{Q}$. Suppose $(\operatorname{Ad} UV)(E_i) \neq E_i$ for an infinite set $\{E_i \colon 1 \leq i < \infty\} \subseteq \mathcal{F}(\mathcal{L}) \setminus \mathcal{P}$. Let K be a compact operator such that $\|UV - I - K\| < \frac{1}{2}$, and let $e_i \in E_i\mathcal{H}$ with $\|e_i\| = 1$. Then

$$\|(UV - I - K)(e_i)\| \geq \|UVe_i - e_i\| - \|Ke_i\| \geq \sqrt{2} - \|Ke_i\| > \frac{1}{2}$$

for i large enough, since $\|Ke_i\| \to 0$. This contradiction shows that there is a finite set $\widetilde{\mathcal{P}} \supseteq \mathcal{P}$ such that $(\operatorname{Ad} UV)(A) = A$ for all $A \in \mathcal{F}(\mathcal{L}) \setminus \widetilde{\mathcal{P}}$, and therefore, $\varphi_\alpha = [\mathrm{id}]$. The result now follows from Corollary 20 and the definition of $\widetilde{\Gamma}$. □

THEOREM 22. *If $\mathcal{L}$ is a CSL, then every derivation $\delta \colon \mathcal{L}' + \mathcal{K} \to \mathcal{L}' + \mathcal{K}$ is inner.*

Proof. The proof is essentially identical to [**DW**, Theorem 3.13]. Again, we can assume $\mathcal{L}$ is complemented. By [**DW**, Lemma 3.12], $\delta = \operatorname{ad} X$ for some $X \in \mathcal{B}(\mathcal{H})$. Then $\{\alpha_t = \exp(t\delta) = \operatorname{Ad}(e^{tX}) \colon t \in \mathbb{R}\}$ is a uniformly continuous automorphism group of $\mathcal{L}' + \mathcal{K}$, and $\|e^{tX} - I\|_e < \frac{1}{2}$ for t sufficiently small. Thus, by Lemma 21, α_t is inner for t small. It follows that $X = \lim_{t\to 0} t^{-1}(e^{tX} - I) \in \mathcal{L}' + \mathcal{K}$, and therefore δ is inner. □

Finally, we can make the following improvements to Lemma 21.

COROLLARY 23. *Suppose $\mathcal{L}$ is a CSL and $\alpha \in \operatorname{Aut}(\mathcal{L}' + \mathcal{K})$.*

(i) *If the spectrum of α is contained in $\Omega = \{z \in \mathbb{C} \colon \operatorname{Re} z > 0\}$, then α is inner.*

(ii) *If $\|\alpha - \mathrm{id}\| < 1$, then α is inner.*

Proof. (i) follows from Theorem 22, using the argument given in [**W2**, Corollary 2.4]. (ii) follows from (i) since $\|\alpha - \mathrm{id}\| < 1$ implies that the spectrum of α is contained in Ω. □

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